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Factorization of the Shoenfield-like Bounded Functional Interpretation

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Abstract We adapt Streicher and Kohlenbach's proof of the factorization $S = KD$ of the Shoenfield translation S in terms of Krivine's negative translation K and the Gödel functional interpretation D, obtaining a proof of the factorization $U = KB$ of Ferreira's Shoenfield-like bounded functional interpretation U in terms of K and Ferreira and Oliva's bounded functional interpretation B.

1 Introduction

In 1958, Gödel [5] presented a functional interpretation D of Heyting arithmetic HA^ω into itself (actually, into a quantifier-free theory, for foundational reasons). When composed with a negative translation N of Peano arithmetic PA^ω into HA^ω (Gödel [4]), it results in a two-step functional interpretation ND of PA^ω into HA^ω [5]. Nine years later, Shoenfield [9] presented a one-step functional interpretation S of PA^ω into HA^ω.

In 2007, Streicher and Kohlenbach [11], and independently Avigad [1], proved the factorization $S = KD$ of S in terms of D and a negative translation K due to Streicher and Reus [10], inspired by Krivine [8].

$$\begin{tikzcd} \mathsf{PA}^\omega \arrow[r, "K"] \arrow[rr, bend right, "S"'] & \mathsf{HA}^\omega \arrow[r, "D"] & \mathsf{HA}^\omega \end{tikzcd}$$

In 2005, Ferreira and Oliva [3] presented a functional interpretation B of Heyting arithmetic with majorizability $\mathsf{HA}^\omega_\trianglelefteq$ into itself. Like D, when composed with a negative translation N of Peano arithmetic with majorizability $\mathsf{PA}^\omega_\trianglelefteq$ into $\mathsf{HA}^\omega_\trianglelefteq$, it results in a two-step functional interpretation NB of $\mathsf{PA}^\omega_\trianglelefteq$ into $\mathsf{HA}^\omega_\trianglelefteq$ [3]. Two years later, Ferreira [2] presented a one-step functional interpretation U of $\mathsf{PA}^\omega_\trianglelefteq$ into $\mathsf{HA}^\omega_\trianglelefteq$.

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By adapting Streicher and Kohlenbach's proof, we obtain the factorization $U = KB$.

$$\begin{tikzcd} \mathsf{PA}^\omega_{\trianglelefteq} \arrow[r, "K"] \arrow[rr, bend right, "U"'] & \mathsf{HA}^\omega_{\trianglelefteq} \arrow[r, "B"] & \mathsf{HA}^\omega_{\trianglelefteq} \end{tikzcd}$$

2 Framework

Definition 2.1 ([3], [12]) The *Heyting arithmetic* HA^ω that we consider is the usual Heyting arithmetic in all finite types but with a minimal treatment of equality and no extensionality, following Troelstra [12].

The *Heyting arithmetic with majorizability* $\mathsf{HA}^\omega_{\trianglelefteq}$ is obtained from HA^ω by

1. adding new atomic formulas $t \trianglelefteq_\rho q$ for all finite types ρ (where t and q are terms of type ρ);
2. adding syntactically new *bounded quantifications* $\forall x \trianglelefteq_\rho tA$ and $\exists x \trianglelefteq_\rho tA$ (where A is a formula and the variable x does not occur in the term t);
3. adding the axioms
$$\forall x \trianglelefteq tA \leftrightarrow \forall x(x \trianglelefteq t \to A), \qquad \exists x \trianglelefteq tA \leftrightarrow \exists x(x \trianglelefteq t \wedge A)$$
governing the bounded quantifications;
4. adding the axioms and rule
$$x \trianglelefteq_0 y \leftrightarrow x \leq_0 y, \qquad x \trianglelefteq y \to \forall u \trianglelefteq v(xu \trianglelefteq yv \wedge yu \trianglelefteq yv),$$
$$\frac{A_b \wedge u \trianglelefteq v \to tu \trianglelefteq qv \wedge qu \trianglelefteq qv}{A_b \to t \trianglelefteq q}$$
governing the *majorizability* symbol $\trianglelefteq$ (where $\leq_0$ is the usual inequality between terms of type 0, A_b is a *bounded formula*, that is, a formula with all quantifications bounded, and in the rule the variables u and v do not occur free in the formula A_b neither in the terms t and q);
5. extending the induction axiom to the new formulas.

This system is presented in detail in [3].

We will need the following notation.

Notation 2.2 ([3]) An underlined letter $\underline{t}$ means a tuple (possibly empty) of terms $t_1, \ldots, t_n$. We use the abbreviations

$$\underline{t} \trianglelefteq \underline{t} :\equiv t_1 \trianglelefteq t_1 \wedge \cdots \wedge t_n \trianglelefteq t_n,$$
$$\forall \underline{x} A :\equiv \forall x_1 \cdots \forall x_n A, \qquad \exists \underline{x} A :\equiv \exists x_1 \cdots \exists x_n A,$$
$$\forall \underline{x} \trianglelefteq \underline{t} A :\equiv \forall x_1 \trianglelefteq t_1 \cdots \forall x_n \trianglelefteq t_n A, \qquad \exists \underline{x} \trianglelefteq \underline{t} A :\equiv \exists x_1 \trianglelefteq t_1 \cdots \exists x_n \trianglelefteq t_n A,$$
$$\tilde{\forall} \underline{x} A :\equiv \forall \underline{x}(\underline{x} \trianglelefteq \underline{x} \to A), \qquad \tilde{\exists} \underline{x} A :\equiv \exists \underline{x}(\underline{x} \trianglelefteq \underline{x} \wedge A),$$
$$\tilde{\forall} \underline{x} \trianglelefteq \underline{t} A :\equiv \forall \underline{x} \trianglelefteq \underline{t}(\underline{x} \trianglelefteq \underline{x} \to A), \qquad \tilde{\exists} \underline{x} \trianglelefteq \underline{t} A :\equiv \exists \underline{x} \trianglelefteq \underline{t}(\underline{x} \trianglelefteq \underline{x} \wedge A).$$

We consider two logical principles.

Definition 2.3 The *law of excluded middle for bounded formulas* $\mathsf{B\text{-}LEM}$ is the principle
$$A_b \vee \neg A_b,$$
where A_b is a bounded formula.

Definition 2.4 ([2]) The *monotone bounded choice* B-mAC is the principle

$$\tilde{\forall}\underline{x}\tilde{\exists}\underline{y}A_b(\underline{x},\underline{y}) \to \tilde{\exists}\underline{Y}\tilde{\forall}\underline{x}\tilde{\exists}\underline{y} \trianglelefteq \underline{Y}\,\underline{x}A_b(\underline{x},\underline{y}),$$

where A_b is a bounded formula.

3 Negative Translation and Bounded Functional Interpretations

For the convenience of the reader, we recall the definitions of K, B, and U.

Definition 3.1 ([1], [8], [10], [11]) *Krivine's negative translation* (extended to arithmetic with majorizability)[1] A^K of a formula A of $\mathsf{PA}^{\omega}_{\trianglelefteq}$ based on $\neg, \vee, \forall\trianglelefteq, \forall$ is $A^K :\equiv \neg A_K$, where A_K is defined by induction on the complexity of formulas.

1. If A is an atomic formula, then $A_K :\equiv \neg A$;
2. $(\neg A)_K :\equiv \neg A_K$;
3. $(A \vee B)_K :\equiv A_K \wedge B_K$;
4. $(\forall x \trianglelefteq tA)_K :\equiv \exists x \trianglelefteq tA_K$;
5. $(\forall xA)_K :\equiv \exists xA_K$.

If we consider $\wedge$ a primitive symbol, then

6. $(A \wedge B)_K :\equiv A_K \vee B_K$.

Definition 3.2 ([3]) The *bounded functional interpretation* A^B of a formula A of $\mathsf{HA}^{\omega}_{\trianglelefteq}$ based on $\bot, \wedge, \vee, \to, \forall\trianglelefteq, \exists\trianglelefteq, \forall, \exists$ is defined by induction on the complexity of formulas.

1. If A is an atomic formula, then $A^B :\equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}A_B(\underline{x},\underline{y}) :\equiv A$, where $\underline{x}$ and $\underline{y}$ are empty tuples.

If $A^B \equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}A_B(\underline{x},\underline{y})$ and $B^B \equiv \tilde{\exists}\underline{x'}\tilde{\forall}\underline{y'}B_B(\underline{x'},\underline{y'})$, then

2. $(A \wedge B)^B :\equiv \tilde{\exists}\underline{x},\underline{x'}\tilde{\forall}\underline{y},\underline{y'}(A \wedge B)_B(\underline{x},\underline{x'},\underline{y},\underline{y'}) :\equiv$
 $\tilde{\exists}\underline{x},\underline{x'}\tilde{\forall}\underline{y},\underline{y'}[A_B(\underline{x},\underline{y}) \wedge B_B(\underline{x'},\underline{y'})]$;
3. $(A \vee B)^B :\equiv \tilde{\exists}\underline{x},\underline{x'}\tilde{\forall}\underline{y},\underline{y'}(A \vee B)_B(\underline{x},\underline{x'},\underline{y},\underline{y'}) :\equiv$
 $\tilde{\exists}\underline{x},\underline{x'}\tilde{\forall}\underline{y},\underline{y'}[\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}A_B(\underline{x},\underline{\tilde{y}}) \vee \tilde{\forall}\underline{\tilde{y}'} \trianglelefteq \underline{y'}B_B(\underline{x'},\underline{\tilde{y}'})]$;
4. $(A \to B)^B :\equiv \tilde{\exists}\underline{X'},\underline{Y}\tilde{\forall}\underline{x},\underline{y'}(A \to B)_B(\underline{X'},\underline{Y},\underline{x},\underline{y'}) :\equiv$
 $\tilde{\exists}\underline{X'},\underline{Y}\tilde{\forall}\underline{x},\underline{y'}[\tilde{\forall}\underline{y} \trianglelefteq \underline{Y}\,\underline{x}\,\underline{y'}A_B(\underline{x},\underline{y}) \to B_B(\underline{X'}\,\underline{x},\underline{y'})]$;
5. $(\forall z \trianglelefteq tA)^B :\equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}(\forall z \trianglelefteq tA)_B(\underline{x},\underline{y}) :\equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}\forall z \trianglelefteq tA_B(\underline{x},\underline{y})$;
6. $(\exists z \trianglelefteq tA)^B :\equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}(\exists z \trianglelefteq tA)_B(\underline{x},\underline{y}) :\equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}\exists z \trianglelefteq t\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}A_B(\underline{x},\underline{\tilde{y}})$;
7. $(\forall zA)^B :\equiv \tilde{\exists}\underline{X}\tilde{\forall}w,\underline{y}(\forall zA)_B(\underline{X},w,\underline{y}) :\equiv \tilde{\exists}\underline{X}\tilde{\forall}w,\underline{y}\forall z \trianglelefteq wA_B(\underline{X}w,\underline{y})$;
8. $(\exists zA)^B :\equiv \tilde{\exists}w,\underline{x}\tilde{\forall}\underline{y}(\exists zA)_B(w,\underline{x},\underline{y}) :\equiv \tilde{\exists}w,\underline{x}\tilde{\forall}\underline{y}\exists z \trianglelefteq w\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}A_B(\underline{x},\underline{\tilde{y}})$.

Remark 3.3 ([3]) From (1) and (4) we conclude that if $A^B \equiv \tilde{\exists}\underline{x}\tilde{\forall}\underline{y}A_B(\underline{x},\underline{y})$, then $(\neg A)^B \equiv \tilde{\exists}\underline{Y}\tilde{\forall}\underline{x}(\neg A)_B(\underline{Y},\underline{x}) \equiv \tilde{\exists}\underline{Y}\tilde{\forall}\underline{x}\neg\tilde{\forall}\underline{y} \trianglelefteq \underline{Y}\,\underline{x}A_B(\underline{x},\underline{y})$.

Remark 3.4 ([3]) We can prove by induction on the complexity of formulas that $A_B(\underline{x},\underline{y})$ is a bounded formula.

Definition 3.5 ([2]) The *Shoenfield-like bounded functional interpretation* A^U of a formula A of $\mathsf{PA}^\omega_\trianglelefteq$ based on $\neg$, $\vee$, $\forall\trianglelefteq$, $\forall$ is defined by induction on the complexity of formulas.

1. If A is an atomic formula, then $A^U :\equiv \tilde\forall \underline{x} \tilde\exists \underline{y} A_U(\underline{x}, \underline{y}) :\equiv A$, where $\underline{x}$ and $\underline{y}$ are empty tuples.

If $A^U \equiv \tilde\forall \underline{x} \tilde\exists \underline{y} A_U(\underline{x}, \underline{y})$ e $B^U \equiv \tilde\forall \underline{x'} \tilde\exists \underline{y'} B_U(\underline{x'}, \underline{y'})$, then

2. $(\neg A)^U :\equiv \tilde\forall \underline{Y} \tilde\exists \underline{x} (\neg A)_U(\underline{Y}, \underline{x}) :\equiv \tilde\forall \underline{Y} \tilde\exists \underline{x} \tilde\exists \underline{\tilde x} \trianglelefteq \underline{x} \neg A_U(\underline{\tilde x}, \underline{Y}\,\underline{\tilde x})$;
3. $(A \vee B)^U :\equiv \tilde\forall \underline{x}, \underline{x'} \tilde\exists \underline{y}, \underline{y'} (A \vee B)_U(\underline{x}, \underline{x'}, \underline{y}, \underline{y'}) :\equiv$ $\tilde\forall \underline{x}, \underline{x'} \tilde\exists \underline{y}, \underline{y'} [A_U(\underline{x}, \underline{y}) \vee B_U(\underline{x'}, \underline{y'})]$;
4. $(\forall z \trianglelefteq t A)^U :\equiv \tilde\forall \underline{x} \tilde\exists \underline{y} (\forall z \trianglelefteq t A)_U(\underline{x}, \underline{y}) :\equiv \tilde\forall \underline{x} \tilde\exists \underline{y} \forall z \trianglelefteq t A_U(\underline{x}, \underline{y})$;
5. $(\forall z A)^U :\equiv \tilde\forall w, \underline{x} \tilde\exists \underline{y} (\forall z A)_U(w, \underline{x}, \underline{y}) :\equiv \tilde\forall w, \underline{x} \tilde\exists \underline{y} \forall z \trianglelefteq w A_U(\underline{x}, \underline{y})$.

If we consider $\wedge$ a primitive symbol, then

6. $(A \wedge B)^U :\equiv \tilde\forall \underline{x}, \underline{x'} \tilde\exists \underline{y}, \underline{y'} (A \wedge B)_U(\underline{x}, \underline{x'}, \underline{y}, \underline{y'}) :\equiv$ $\tilde\forall \underline{x}, \underline{x'} \tilde\exists \underline{y}, \underline{y'} [A_U(\underline{x}, \underline{y}) \wedge B_U(\underline{x'}, \underline{y'})]$.

Remark 3.6 ([2]) We can also prove by induction on the complexity of formulas that $A_U(\underline{x}, \underline{y})$ is a bounded formula.

U is monotone on the second tuple of the variables in the following sense.

Lemma 3.7 (monotonicity of U [2]) $\mathsf{HA}^\omega_\trianglelefteq \vdash \forall \underline{x} \forall \underline{y} \forall \underline{\tilde y} \trianglelefteq \underline{y} [A_U(\underline{x}, \underline{\tilde y}) \to A_U(\underline{x}, \underline{y})]$.

4 Factorization

We want to prove $A^U \leftrightarrow (A^K)^B$ by induction on the complexity of formulas. Because it isn't A^K but A_K that is defined by induction on the complexity of formulas, it would be better to write $A^U \leftrightarrow (\neg A_K)^B$. If $A^U \equiv \tilde\forall \underline{x} \tilde\exists \underline{y} A_U(\underline{x}, \underline{y})$ and $(A_K)^B \equiv \tilde\exists \underline{x'} \tilde\forall \underline{y'} (A_K)_B(\underline{x'}, \underline{y'})$, then using B-mAC in the first equivalence and the monotonicity of U in the second equivalence, we have

$$\begin{aligned} A^U &\equiv \tilde\forall \underline{x} \tilde\exists \underline{y} A_U(\underline{x}, \underline{y}) \\ &\leftrightarrow \tilde\exists \underline{Y} \tilde\forall \underline{x} \tilde\exists \underline{y} \trianglelefteq \underline{Y}\,\underline{x} A_U(\underline{x}, \underline{y}) \\ &\leftrightarrow \tilde\exists \underline{Y} \tilde\forall \underline{x} A_U(\underline{x}, \underline{Y}\,\underline{x}), \qquad (1) \\ (\neg A_K)^B &\equiv \tilde\exists \underline{Y'} \tilde\forall \underline{x'} \neg \tilde\forall \underline{y'} \trianglelefteq \underline{Y'}\,\underline{x'} (A_K)_B(\underline{x'}, \underline{y'}). \qquad (2) \end{aligned}$$

The comparison of formulas (1) and (2) suggests that we first prove $A_U(\underline{x}, \underline{Y}\,\underline{x}) \leftrightarrow \neg \tilde\forall \underline{y} \trianglelefteq \underline{Y}\,\underline{x} (A_K)_B(\underline{x}, \underline{y})$, or even better, $A_U(\underline{x}, \underline{y}) \leftrightarrow \neg \tilde\forall \underline{\tilde y} \trianglelefteq \underline{y} (A_K)_B(\underline{x}, \underline{\tilde y})$. Then, by the above argument, we would have $A^U \leftrightarrow (A^K)^B$.

The factorization proof is almost the straightforward adaptation of Streicher and Kohlenbach's proof but with two tweaks.

1. Instead of proving $A_U(\underline{x}, \underline{y}) \leftrightarrow \neg (A_K)_B(\underline{x}, \underline{y})$, along the lines of Streicher and Kohlenbach's proof, we prove $A_U(\underline{x}, \underline{y}) \leftrightarrow \neg \tilde\forall \underline{\tilde y} \trianglelefteq \underline{y} (A_K)_B(\underline{x}, \underline{\tilde y})$, where the appearance of the quantification $\tilde\forall \underline{\tilde y} \trianglelefteq \underline{y}$ is explained by the above argument.

2. In proving $A_U(\underline{x}, \underline{y}) \leftrightarrow \neg\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}(A_K)_B(\underline{x}, \underline{\tilde{y}})$ we need the hypothesis $\underline{x} \trianglelefteq \underline{x} \wedge \underline{y} \trianglelefteq \underline{y}$ for technical reasons explained in notes.

Theorem 4.1 (factorization $U = KB$) *We have*

$$\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall}\underline{Y}, \underline{x}[A_U(\underline{x}, \underline{Y}\,\underline{x}) \leftrightarrow (A^K)_B(\underline{Y}, \underline{x})], \tag{3}$$

$$\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{B\text{-}LEM} + \mathsf{B\text{-}mAC} \vdash A^U \leftrightarrow (A^K)^B. \tag{4}$$

Proof

Step 1 First we prove

$$\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall}\underline{x}, \underline{y}[A_U(\underline{x}, \underline{y}) \leftrightarrow \neg\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}(A_K)_B(\underline{x}, \underline{\tilde{y}})] \tag{5}$$

by induction on the complexity of formulas.

Let us consider the case of atomic formulas A. Using $\mathsf{B\text{-}LEM}$ in the equivalence, we have

$$\begin{aligned} A_U &\equiv A \\ &\leftrightarrow \neg\neg A \\ &\equiv \neg(A_K)_B. \end{aligned}$$

Let us now consider the case of negation $\neg A$. Assume $\underline{Y} \trianglelefteq \underline{Y}$ and $\underline{x} \trianglelefteq \underline{x}$. Using the induction hypothesis in the first equivalence and $\mathsf{B\text{-}LEM}$ in the second equivalence, we have

$$\begin{aligned} (\neg A)_U(\underline{Y}, \underline{x}) &\equiv \tilde{\exists}\underline{\tilde{x}} \trianglelefteq \underline{x}\neg A_U(\underline{\tilde{x}}, \underline{Y}\,\underline{\tilde{x}}) \\ &\leftrightarrow \tilde{\exists}\underline{\tilde{x}} \trianglelefteq \underline{x}\neg\neg\tilde{\forall}\underline{y} \trianglelefteq \underline{Y}\,\underline{\tilde{x}}(A_K)_B(\underline{\tilde{x}}, \underline{y}) \\ &\leftrightarrow \neg\tilde{\forall}\underline{\tilde{x}} \trianglelefteq \underline{x}\neg\tilde{\forall}\underline{y} \trianglelefteq \underline{Y}\,\underline{\tilde{x}}(A_K)_B(\underline{\tilde{x}}, \underline{y}) \\ &\equiv \neg\tilde{\forall}\underline{\tilde{x}} \trianglelefteq \underline{x}[(\neg A)_K]_B(\underline{Y}, \underline{\tilde{x}}). \end{aligned}$$

Let us now consider the case of disjunction $A \vee B$. Assume $\underline{x} \trianglelefteq \underline{x}$, $\underline{x}' \trianglelefteq \underline{x}'$, $\underline{y} \trianglelefteq \underline{y}$, and $\underline{y}' \trianglelefteq \underline{y}'$. Using the induction hypothesis in the first equivalence, $\mathsf{B\text{-}LEM}$ in the second equivalence, and intuitionistic logic in the third equivalence,[2] we have

$$\begin{aligned} (A \vee B)_U(\underline{x}, \underline{x}', \underline{y}, \underline{y}') &\equiv A_U(\underline{x}, \underline{y}) \vee B_U(\underline{x}', \underline{y}') \\ &\leftrightarrow \neg\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}(A_K)_B(\underline{x}, \underline{\tilde{y}}) \vee \neg\tilde{\forall}\underline{\tilde{y}}' \trianglelefteq \underline{y}'(B_K)_B(\underline{x}', \underline{\tilde{y}}') \\ &\leftrightarrow \neg[\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}(A_K)_B(\underline{x}, \underline{\tilde{y}}) \wedge \tilde{\forall}\underline{\tilde{y}}' \trianglelefteq \underline{y}'(B_K)_B(\underline{x}', \underline{\tilde{y}}')] \\ &\leftrightarrow \neg\tilde{\forall}\underline{\tilde{y}}, \underline{\tilde{y}}' \trianglelefteq \underline{y}, \underline{y}'[(A_K)_B(\underline{x}, \underline{\tilde{y}}) \wedge (B_K)_B(\underline{x}', \underline{\tilde{y}}')] \\ &\equiv \neg\tilde{\forall}\underline{\tilde{y}}, \underline{\tilde{y}}' \trianglelefteq \underline{y}, \underline{y}'[(A \vee B)_K]_B(\underline{x}, \underline{x}', \underline{\tilde{y}}, \underline{\tilde{y}}'). \end{aligned}$$

Let us now consider the case of bounded universal quantification $\forall z \trianglelefteq tA$. Assume $\underline{x} \trianglelefteq \underline{x}$ and $\underline{y} \trianglelefteq \underline{y}$. Using the induction hypothesis in the first equivalence and intuitionistic logic in the second and third[3] equivalences, we have

$$
\begin{aligned}
(\forall z \trianglelefteq t A)_U(\underline{x}, \underline{y}) &\equiv \forall z \trianglelefteq t A_U(\underline{x}, \underline{y}) \\
&\leftrightarrow \forall z \trianglelefteq t \neg \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{y} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \\
&\leftrightarrow \neg \exists z \trianglelefteq t \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{y} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \\
&\leftrightarrow \neg \tilde{\forall} \underline{\hat{y}} \,\underline{\trianglelefteq}\, \underline{y} \exists z \trianglelefteq t \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{\hat{y}} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \\
&\equiv \neg \tilde{\forall} \underline{\hat{y}} \,\underline{\trianglelefteq}\, \underline{y} [(\forall z \trianglelefteq t A)_K]_B(\underline{x}, \underline{\hat{y}}).
\end{aligned}
$$

Finally, let us consider the case of unbounded universal quantification $\forall z A$. Assume $w \,\underline{\trianglelefteq}\, w$, $\underline{x} \,\underline{\trianglelefteq}\, \underline{x}$ and $\underline{y} \,\underline{\trianglelefteq}\, \underline{y}$. Using the induction hypothesis in the first equivalence and intuitionistic logic in the second and third equivalences, we have

$$
\begin{aligned}
(\forall z A)_U(w, \underline{x}, \underline{y}) &\equiv \forall z \trianglelefteq w A_U(\underline{x}, \underline{y}) \\
&\leftrightarrow \forall z \trianglelefteq w \neg \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{y} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \\
&\leftrightarrow \neg \exists z \trianglelefteq w \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{y} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \\
&\leftrightarrow \neg \tilde{\forall} \underline{\hat{y}} \,\underline{\trianglelefteq}\, \underline{y} \exists z \trianglelefteq w \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{\hat{y}} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \\
&\equiv \neg \tilde{\forall} \underline{\hat{y}} \,\underline{\trianglelefteq}\, \underline{y} [(\forall z A)_K]_B(w, \underline{x}, \underline{\hat{y}}).
\end{aligned}
$$

In case we consider $\wedge$ a primitive symbol, let us now see the case of conjunction $A \wedge B$. Assume $\underline{x} \,\underline{\trianglelefteq}\, \underline{x}$, $\underline{x'} \,\underline{\trianglelefteq}\, \underline{x'}$, $\underline{y} \,\underline{\trianglelefteq}\, \underline{y}$, and $\underline{y'} \,\underline{\trianglelefteq}\, \underline{y'}$. Using the induction hypothesis in the first equivalence and intuitionistic logic in the second and third equivalences, we have

$$
\begin{aligned}
(A \wedge B)_U(\underline{x}, \underline{x'}, \underline{y}, \underline{y'}) &\equiv A_U(\underline{x}, \underline{y}) \wedge B_U(\underline{x'}, \underline{y'}) \\
&\leftrightarrow \neg \tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{y} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \wedge \neg \tilde{\forall} \underline{\tilde{y}'} \,\underline{\trianglelefteq}\, \underline{y'} (B_K)_B(\underline{x'}, \underline{\tilde{y}'}) \\
&\leftrightarrow \neg [\tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{y} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \vee \tilde{\forall} \underline{\tilde{y}'} \,\underline{\trianglelefteq}\, \underline{y'} (B_K)_B(\underline{x'}, \underline{\tilde{y}'})] \\
&\leftrightarrow \neg \tilde{\forall} \underline{\hat{y}}, \underline{\hat{y}'} \,\underline{\trianglelefteq}\, \underline{y}, \underline{y'} [\tilde{\forall} \underline{\tilde{y}} \,\underline{\trianglelefteq}\, \underline{\hat{y}} (A_K)_B(\underline{x}, \underline{\tilde{y}}) \vee \\
&\qquad \tilde{\forall} \underline{\tilde{y}'} \,\underline{\trianglelefteq}\, \underline{\hat{y}'} (B_K)_B(\underline{x'}, \underline{\tilde{y}'})] \\
&\equiv \neg \tilde{\forall} \underline{\hat{y}}, \underline{\hat{y}'} \,\underline{\trianglelefteq}\, \underline{y}, \underline{y'} [(A \wedge B)_K]_B(\underline{x}, \underline{x'}, \underline{\hat{y}}, \underline{\hat{y}'}).
\end{aligned}
$$

Step 2 Now we prove (3). Assume $\underline{Y} \,\underline{\trianglelefteq}\, \underline{Y}$ and $\underline{x} \,\underline{\trianglelefteq}\, \underline{x}$. Using (5) in the equivalence, we have

$$
\begin{aligned}
A_U(\underline{x}, \underline{Y}\,\underline{x}) &\leftrightarrow \neg \tilde{\forall} \underline{y} \,\underline{\trianglelefteq}\, \underline{Y}\,\underline{x} (A_K)_B(\underline{x}, \underline{y}) \\
&\equiv (\neg A_K)_B(\underline{Y}, \underline{x}) \\
&\equiv (A^K)_B(\underline{Y}, \underline{x}).
\end{aligned}
$$

Step 3 Finally, we prove (4). Using $\mathsf{B\text{-}mAC}$ in the first equivalence, the monotonicity of U in the second equivalence, and (3) in the third equivalence, we have

$$\begin{aligned} A^U &\equiv \tilde{\forall}\underline{x}\tilde{\exists}\underline{y}A_U(\underline{x}, \underline{y}) \\ &\leftrightarrow \tilde{\exists}\underline{Y}\tilde{\forall}\underline{x}\tilde{\exists}\underline{y} \trianglelefteq \underline{Y}\,\underline{x}A_U(\underline{x}, \underline{y}) \\ &\leftrightarrow \tilde{\exists}\underline{Y}\tilde{\forall}\underline{x}A_U(\underline{x}, \underline{Y}\,\underline{x}) \\ &\leftrightarrow \tilde{\exists}\underline{Y}\tilde{\forall}\underline{x}(A^K)_B(\underline{Y}, \underline{x}) \\ &\equiv (A^K)^B. \end{aligned}$$

□

Notes

1. It still holds a soundness theorem $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash A \Rightarrow \mathsf{HA}^{\omega}_{\trianglelefteq} \vdash A^K$ and a characterization theorem $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash A \leftrightarrow A^K$.

2. The rule for conversion to prenex normal form $\forall u \trianglelefteq v(C \wedge D) \to \forall u \trianglelefteq vC \wedge D$ (where the variable u does not occur free in the formula D), despite its innocuous look, does not hold without the hypothesis $v \trianglelefteq v$. So we need to use the hypothesis $\underline{x} \trianglelefteq \underline{x} \wedge \underline{y} \trianglelefteq \underline{y}$ in the proof.

3. Probably the easiest way to prove the third equivalence is to prove
$$\exists z \trianglelefteq t\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{y}(A_K)_B(\underline{x}, \underline{\tilde{y}}) \leftrightarrow \tilde{\forall}\underline{\hat{y}} \trianglelefteq \underline{y}\exists z \trianglelefteq t\tilde{\forall}\underline{\tilde{y}} \trianglelefteq \underline{\hat{y}}(A_K)_B(\underline{x}, \underline{\tilde{y}}).$$
To prove the right-to-left implication, we just take $\underline{\hat{y}} = \underline{y}$, which we can do because $\underline{y} \trianglelefteq \underline{y}$. So here again we need to use the hypothesis $\underline{x} \trianglelefteq \underline{x} \wedge \underline{y} \trianglelefteq \underline{y}$.

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