Mathematical candies

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Resumo: Quando os alunos expressam os seus sentimentos pela matemática, os adjetivos usuais não são "interessante", "bela" e "agradável", mas "aborrecida", "feia" e "dolorosa". Esta predisposição contra a matemática é uma barreira que os professores de matemática têm de deitar abaixo para libertar o caminho para a aprendizagem da matemática. Como é que podemos fazê-lo? Deleitando os alunos com "doces matemáticos": pequenos pedaços de matemática que são "interessantes", "belos" e "agradáveis".

Abstract: When students express their feelings about mathematics, the usual adjectives are not "interesting", "beautiful" and "pleasant", but "boring", "ugly" and "painful". This predisposition against mathematics is a wall that teachers of mathematics need to tear down to free the way for the learning of mathematics. How do we do this? By delighting students with "mathematical candies": little pieces of mathematics that are "interesting", "beautiful" and "pleasant".

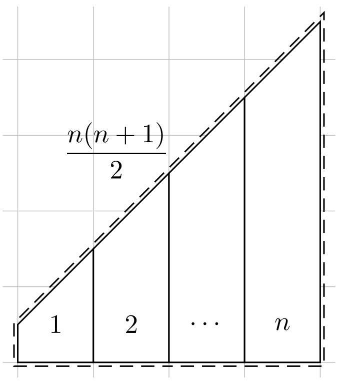
Palavras-chave: Doces matemáticos; número triangular; número irracional; demonstração (não) construtiva; classificação de triângulos.

Keywords: Mathematical candy; triangular/triangle number; irrational number; (non)constructive proof; classification of triangles.

1 Geometric proof of $1 + 2 + \cdots + n = n(n + 1)/2$

When Gauss's teacher asked him to add $1 + 2 + \cdots + 100$, he answered $50 \times 101 = 5050$ realising that $1 + 2 + \cdots + 100$ is the sum of the 50 terms $1 + 100, 2 + 99, \ldots, 50 + 51$ equal to 101. This generalises to $1 + 2 + \cdots + n = n(n + 1)/2$. We give a geometric proof of this formula [1].

Let us put together a first inner right trapezoid with bases of length 0.5 and 1.5, and area 1, a second inner right trapezoid with bases of length 1.5 and 2.5 and area 2, …, a nth inner right trapezoid with bases of length $n - 0.5$ and $n + 0.5$ and area n, to form an outer right trapezoid with bases of length 0.5 and $n + 0.5$. The sum of the areas of the inner trapezoids is $1 + 2 + \cdots + n$. The area of the outer trapezoid is $n(n+1)/2$.



2 At least half of the real numbers are irrational

If we ask students to mention some irrational numbers, we are already lucky if we hear $\sqrt{2}$, π and e. This is natural because almost all everyday numbers are rational, but deceiving because almost all real numbers are irrational. Can we show this to students? We give a elementary proof [2] that at least half (in an intuitive sense) of the real numbers are irrational.

Let us consider the function

$$f\colon [0, +\infty[\to \mathbb{R} \setminus \mathbb{Q}$$
$$x \mapsto \begin{cases} \sqrt{2} + x & \text{if } \sqrt{2} + x \in \mathbb{R} \setminus \mathbb{Q} \\ \sqrt{2} - x & \text{if } \sqrt{2} + x \notin \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(which is well defined: if $\sqrt{2} + x \notin \mathbb{R} \setminus \mathbb{Q}$, then $\sqrt{2} - x \in \mathbb{R} \setminus \mathbb{Q}$, otherwise $\sqrt{2} = \frac{(\sqrt{2}+x)+(\sqrt{2}-x)}{2} \in \mathbb{Q}$, which is false). The function f is injective: if $f(x) = f(y)$, then $x = |f(x) - \sqrt{2}| = |f(y) - \sqrt{2}| = y$. So for each different x in $[0, +\infty[$ we have a different irrational number $f(x)$, therefore there are as many irrational numbers as xs in $[0, +\infty[$, which is half of $\mathbb{R}$.

3 Constructive and nonconstructive proofs

Mathematicians can prove that an equation has a solution (1) constructively by giving a solution or (2) nonconstructively without giving a solution. Most mathematicians accept both proofs, but a minority only accepts constructive proofs. We give an illustrative example [3] of a simple equation with constructive and nonconstructive proofs.

Let us consider the equation

$$c^2x^2 - (c^2 + c)x + c = 0.$$

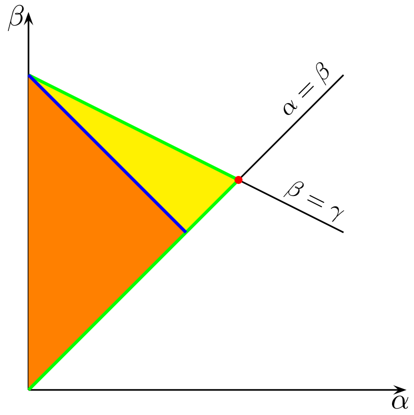
where c is a constant.

- First, we show that the equation has a solution nonconstructively: aiming at a contradiction, if there is no solution, then $x = 0$ is not a solution, so $c \neq 0$, therefore $x = 1/c$ exists and we can check that is a solution, contradicting the assumption that there is no solution.
- Second, we show that the equation has a solution constructively: $x = 1$ is a solution.

4 Visualising classes of triangles

Students are familiar with the classification of triangles as acute, right, obtuse, scalene, isosceles, equilateral, etc. But this "zoology" can easily become tedious. Can we fix this by doing something funny with it? We give a funny way of visualising the classes of triangles [4].

Each triangle is characterised (modulo similarity) by its internal angles α, β and γ. We can assume that we ordered them so that $0 \leq \alpha \leq \beta \leq \gamma$, and we can drop γ because we know $\gamma = 180° - \alpha - \beta$ (since the sum of the internal angles of a triangle is 180°). In conclusion, each triangle is characterised by its internal angles α and β such that $0 \leq \alpha \leq \beta \leq \gamma = 180° - \alpha - \beta$. The set $\Delta = \{(\alpha, \beta) \in \mathbb{R}^2 : 0 \leq \alpha \leq \beta \leq \gamma = 180° - \alpha - \beta\}$ of all "triangles" is itself a triangle. We can identify in Δ the "regions" corresponding to different classes of "triangles", for example, the "region" $\{(\alpha, \beta) \in \Delta : \alpha = \beta \vee \beta = \gamma\}$ corresponding to isosceles "triangles". We paint below these "regions".



isosceles triangles	$\alpha = \beta \vee \beta = \gamma$
equilateral triangles	$\alpha = \beta = \gamma$
right triangles	$\gamma = 90°$
acute triangles	$\gamma < 90°$
obtuse triangles	$\gamma > 90°$

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