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Nonstandardness and the bounded functional interpretation

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ABSTRACT

The bounded functional interpretation of arithmetic in all finite types is able to interpret principles like weak König's lemma without the need of any form of bar recursion. This interpretation requires the use of *intensional* (rule-governed) majorizability relations. This is a somewhat unusual feature. The main purpose of this paper is to show that if the base domain of the natural numbers is extended with nonstandard elements, then the bounded functional interpretation can be seen as falling out from a functional interpretation of nonstandard number theory without intensional notions. The original bounded functional interpretation can be seen as the trace left behind by the new interpretation when one sees it restricted to the standard number theoretical setting.

We also answer an open question regarding the conservativity of the transfer principle *vis-à-vis* functional interpretations of nonstandard arithmetic.

1. Introduction

The bounded functional interpretation was introduced in [5]. In its workings and definition, it relies on a systematic use of the Howard/Bezem (strong) majorizability notion. A somewhat unusual feature is the presence of rule-governed (as opposed to axiomatic-governed) primitive relations: the so-called *intensional* majorizability relations. This permits to show that bounded domains (in the sense of being bounded with respect to the intensional majorizability notion) enjoy some "compactness" properties, the paradigmatic example being the bounded domain of the Cantor space (thereby obtaining weak König's lemma). Of course, the presence of rules in the deductive apparatus obfuscates a clear semantic picture. In this paper we show that if the number theory is allowed to have nonstandard elements, then we can define a new bounded functional interpretation, this time without intensional notions, and recover from it the original bounded

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functional interpretation. In a sense, the nonstandard numbers provide a kind of "compactification" of the natural numbers, making them behave like a bounded domain.

The starting point for this paper was the recent functional interpretations of nonstandard arithmetic due to Benno van den Berg, Eyvind Briseid and Pavol Safarik [15]. Even a cursory look at the paper shows many similarities between their interpretation and the bounded functional interpretation (this is particularly striking in the intuitionistic case). As they say in the paper, their interpretation is inspired (in the classical case) by the similarities between Shoenfield's functional interpretation [13] and the "reduction algorithm" of Edward Nelson for converting proofs in IST (Internal Set Theory) into proofs in ZFC (see [12]). Neither the interpretation of Berg et al. nor Nelson's "reduction algorithm" is based on majorizability considerations. They are rather based on finiteness considerations. The ultimate goal of Berg et al. is to extract computational information – in the form of appropriate term witnesses – from proofs in the nonstandard systems. Their goal is feasible, but it comes with some costs. For instance, the transfer principle – a cornerstone of Nelson's interpretation – has a trivial "reduction" in Nelson's setting but, as pointed by Berg et al., does not have a term witnessing functional. (They conjecture that the transfer principle is nevertheless conservative over the base standard setting, but we show in Appendix A of this paper that this is not the case.) Given that they use term witnessing functionals, in the case of Berg et al. the road is open for replacing finiteness by majorizability (because majorizability arguments rely upon an appropriate theorem concerning the majorizability of closed terms, as in [8]). But why try making this replacement? Apart from the main objective of this paper, viz. to show that the bounded functional interpretation can be recast (without intensional notions) by considering the wider nonstandard setting, mere finiteness conditions are not as surprising as using majorizability notions because the interpretations based on the latter are able to validate so-called *uniform boundedness principles* (introduced in [9] and conveniently discussed in [10]), of which weak König's lemma is a consequence.

For the sake of brevity, this paper only studies classical theories. In the next section, inspired by (but not following) Berg et al., we introduce the finite type system $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ of nonstandard arithmetic and describe some pertinent principles. In Section 3, we define the new majorizability interpretation and prove a corresponding soundness theorem. Section 4 discusses the sense in which the bounded functional interpretation of [5] can be recovered from the interpretation of the nonstandard system. We also include a small Appendix A where we discuss the transfer principle, both in the new interpretation of this paper and in the interpretation of Berg et al.

2. Basic framework

Let $\mathsf{E\text{-}PA}^{\omega}$ be the theory of extensional Peano arithmetic in all finite types. We follow the treatment of [10] where there is only an equality for the base type 0. Equality at other types is defined extensionally and a pertinent axiom of extensionality is uphold. The main purpose of this section is to introduce an extension $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ of $\mathsf{E\text{-}PA}^{\omega}$. The language of this extension extends the language of $\mathsf{E\text{-}PA}^{\omega}$ by having unary predicates st^{σ} for each finite type σ (the predicates for *standardness*). Note that the terms of both languages remain the same. Before we proceed, let us give a word of caution: our theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ below differs from the theory $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ of Berg et al. not only by not having types for finite sequences but, more importantly, because it has different axioms concerning the new predicates st^{σ} (note the second standardness axiom below).

The axioms of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ are those of $\mathsf{E\text{-}PA}^{\omega}$ together with the *standardness axioms* and the *external induction rule*. Let us introduce some notations and make some observations. First of all, since we are working in classical logic, we adopt as our logical primitives $\vee$ (disjunction), $\neg$ (negation) and the universal quantifiers $\forall x^{\sigma}$. The other logical connectives are understood as being defined in the usual manner. We also adopt the complete deduction system for classical logic exposed in [13]. The Howard/Bezem notion of strong majorizability (introduced in [8] and [2]) is defined by induction on the finite type:

$x \leq_0^* y$ is $x \leq y$
$x \leq_{\rho\to\sigma}^* y$ is $\forall v \forall u \leq_\rho^* v\, (xu \leq_\sigma^* yv \wedge yu \leq_\sigma^* yv)$

This notion is fully studied in [10] under the notation y *s-maj*$_\sigma$ x instead of our $x \leq_\sigma^*$. Strong majorizability is transitive. It is not in general reflexive (except for the base type 0). We say that an element x^σ is *monotone* if $x \leq_\sigma^* x$. It can be proved that if x majorizes some element, then x is monotone. Apart from types of degree 0 and 1, it is not true (set-theoretically) that every element is majorizable. However, an important theorem, ultimately due to Howard in [8], says that for every *closed* term t^σ of the language there is a closed term q^σ such that $t \leq_\sigma^* q$ (provably so in, for instance, $\mathsf{E\text{-}PA}^\omega$). This result – which we call *Howard's majorizability theorem* – plays a pivotal role in the bounded functional interpretation (and in the new interpretation of this paper).

A formula is called *internal* if it is part of the original language of $\mathsf{E\text{-}PA}^\omega$ (i.e., the standard predicates st do not occur in the formula). Otherwise it is called *external*. We follow the convention of Nelson in reserving small Greek letters for denoting internal formulas, whereas capital Greek letters may denote any formula whatsoever (internal or external). Therefore, the axioms of $\mathsf{E\text{-}PA}^\omega$ are only constituted by internal formulas. Note, also, that the equality and majorizability relations are given by internal formulas. The universal quantifiers $\forall^{\mathrm{st}} x^\sigma$, $\tilde{\forall} x^\sigma$ and $\tilde{\forall}^{\mathrm{st}} x^\sigma$ are abbreviations of the universal quantifier relativized to the standard elements, to the monotone elements and, simultaneously, to the standard and monotone elements (respectively). We use similar abbreviations for the existential quantifier. Bounded quantifications of the form $\forall x \leq_\sigma^* t\, (\ldots)$ are defined in the usual way and also come in three varieties.

We are now ready to state the axioms of $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}}$ that involve external formulas. The *standardness axioms* are:

- $x =_\sigma y \to (\mathrm{st}^\sigma(x) \to \mathrm{st}^\sigma(y))$;
- $\mathrm{st}^\sigma(y) \to (x \leq_\sigma^* y \to \mathrm{st}^\sigma(x))$;
- $\mathrm{st}^\sigma(t)$, for each closed term t of type σ;
- $\mathrm{st}^{\sigma\to\tau}(z) \to (\mathrm{st}^\sigma(x) \to \mathrm{st}^\tau(zx))$;

where the types σ and τ are arbitrary. The *external induction rule* is

- From $\Phi(0)$ and $\forall^{\mathrm{st}} n^0 (\Phi(n) \to \Phi(n+1))$, infer $\forall^{\mathrm{st}} n^0 \Phi(n)$.

(External induction is formulated as a rule just as a matter of convenience. Since there is no restriction in the formulas Φ, the rule is equivalent to the corresponding axiom scheme.)

There are three principles which play an important role in the sequel. The proper formulation of the first two principles should be with tuples of variables. To ease readability, we formulate them with single variables.

I. *Monotone Choice* $\mathsf{mAC}^\omega_{\mathrm{st}}$: $\tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y\, \phi(x,y) \to \tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^* fx\, \phi(x,y)$.
II. *Realization* R^ω: $\forall x \exists^{\mathrm{st}} y\, \phi(x,y) \to \tilde{\exists}^{\mathrm{st}} z \forall x \exists y \leq^* z\, \phi(x,y)$.
III. *Majorizability Axioms* $\mathsf{MAJ}^\omega_{\mathrm{st}}$: $\forall^{\mathrm{st}} x \exists^{\mathrm{st}} y\, (x \leq^* y)$.

The first principle is a monotone form of restricted standardization, in Nelson's terminology. In the terminology of Berg et al., it is a herbrandized form of the axiom of choice (restricted to internal formulas). *Realization* is a kind of uniform boundedness principle in the sense of Kohlenbach. By passing to the contrapositive, the realization principle can be rewritten in *idealization* form:

II'. *Idealization* I^ω: $\tilde{\forall}^{\mathrm{st}} z \exists x \forall y \leq^* z\, \phi(x,y) \to \exists x \forall^{\mathrm{st}} y\, \phi(x,y)$.

A simple argument shows that I^ω entails the existence of nonstandard elements of type 0. In other words, it entails the existence of nonstandard natural numbers. To see this, just take $\phi(x,y)$ to be the formula $x \neq y$. The argument relies on the fact that $z+1 \not\leq z$. Let us see that a similar fact holds for each finite type. By recursion on the type, define $x^{\sigma\to\tau}+1$ as $\lambda w^\sigma.((xw)^\tau+1)$. We claim that, for each type ρ, $\forall z^\rho(z+1 \not\leq^*_\rho z)$. The proof is by induction on the type ρ. Suppose that $z^{\sigma\to\tau}+1 \leq^*_{\sigma\to\tau} z$. Then, $(z^{\sigma\to\tau}+1)(0^\sigma) \leq^*_\tau z(0^\sigma)$ (here 0^σ is the "zero" of type σ as defined, for instance, in [10]; it is easy to see that 0^σ is monotone). Hence, by definition of $+1$, we get $z(0^\sigma)+1 \leq^*_\tau z(0^\sigma)$, contradicting the induction hypothesis. The point of this discussion is that idealization also entails the existence of nonstandard elements in each finite type.

Finally, the majorizability axioms are peculiar to our interpretation and have no counterpart in Nelson, nor in Berg et al.

3. The interpretation and its soundness

We are now ready to define the new interpretation of this paper. The interpretation has many similarities, and this is not a coincidence, with the Shoenfield-like bounded functional interpretation of [4]. As before, we indulge in some lack of precision and, for ease of reading, we formulate the interpretation with single variables (the official definition should have tuples of variables in appropriate places).

Definition 1. To each formula Φ we assign formulas $\Phi^{\mathrm{U_{st}}}$ and $\Phi_{\mathrm{U_{st}}}$ so that $\Phi^{\mathrm{U_{st}}}$ is of the form $\tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, \Phi_{\mathrm{U_{st}}}(b,c)$, with $\Phi_{\mathrm{U_{st}}}(b,c)$ an *internal* formula, according to the following clauses:

1. $\Phi^{\mathrm{U_{st}}}$ and $\Phi_{\mathrm{U_{st}}}$ are simply Φ, for internal formulas Φ,
2. $\mathrm{st}(t)^{\mathrm{U_{st}}}$ is $\tilde{\exists}^{\mathrm{st}} c\, [t \leq^* c]$.

For the *remaining* cases, if we have already interpretations for Φ and Ψ given (respectively) by $\tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, \Phi_{\mathrm{U_{st}}}(b,c)$ and $\tilde{\forall}^{\mathrm{st}} d \tilde{\exists}^{\mathrm{st}} e\, \Psi_{\mathrm{U_{st}}}(d,e)$ then we define:

3. $(\Phi \vee \Psi)^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} b, d \tilde{\exists}^{\mathrm{st}} c, e\, [\Phi_{\mathrm{U_{st}}}(b,c) \vee \Psi_{\mathrm{U_{st}}}(d,e)]$,
4. $(\neg\Phi)^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} f \tilde{\exists}^{\mathrm{st}} b\, [\tilde{\exists} b' \leq^* b\, \neg\Phi_{\mathrm{U_{st}}}(b', fb')]$,
5. $(\forall x \Phi(x))^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, [\forall x\, \Phi_{\mathrm{U_{st}}}(x,b,c)]$,

where the internal formulas between square brackets are the corresponding *lower* $\mathrm{U_{st}}$-formulas.

As in [4], the explanation for the complication in clause (4) lies in the fact that one needs to ensure that $(\neg\Phi)_{\mathrm{U_{st}}}$ is monotone in the b variable. It will be instrumental to use of the fact that the theory $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}}$ proves the following monotonicity property:

$$\forall b, c \forall c' \leq^* c\, (\Phi_{\mathrm{st}}(b,c') \to \Phi_{\mathrm{st}}(b,c)),$$

for every formula Φ of the language.

It is illustrative to work out explicitly the interpretations of the conditional and the existential quantifier. They are, modulo classical logic, as follows:

6. $(\Phi \to \Psi)^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} f, d \tilde{\exists}^{\mathrm{st}} b, e\, [\tilde{\forall} b' \leq^* b\, \Phi_{\mathrm{U_{st}}}(b', fb') \to \Psi_{\mathrm{U_{st}}}(d,e)]$,
7. $(\exists x \Phi(x))^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} F \tilde{\exists}^{\mathrm{st}} f\, [\tilde{\exists} f' \leq^* f \exists x \tilde{\forall} b' \leq^* F f' \Phi_{\mathrm{U_{st}}}(x, b', f'b')]$.

We skipped the interpretation of conjunction. The interested reader should consult the last paragraph of Section 5 of [4]. The phenomenon described there also applies to the present setting.

Theorem 1 *(Soundness). Suppose that*

$$\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega}_{\mathrm{st}} + \mathsf{R}^{\omega} + \mathsf{MAJ}^{\omega}_{\mathrm{st}} \vdash \Phi,$$

where Φ *is an arbitrary formula (it may have free variables). Then there are* closed *monotone terms* t *of appropriate types such that*

$$\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{U_{st}}}(b, tb).$$

Proof. We need to check two things: (1) that if Φ is an axiom of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega}_{\mathrm{st}} + \mathsf{R}^{\omega} + \mathsf{MAJ}^{\omega}_{\mathrm{st}}$, then there are closed monotone terms t of appropriate types such that $\tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{U_{st}}}(b, tb)$ is a theorem of the target theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$; (2) that the property of the theorem is preserved by the logical rules and the external induction rule. The verification of the external induction rule is straightforward: it uses the recursors and an inductive argument within the target theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$. The verifications of the logical rules are mostly analogous to those of the mentioned Shoenfield-like bounded functional interpretation of [4]. One that is not analogous (but is very simple) is the logical rule that infers $\forall x \Phi \vee \Psi$ from $\Phi \vee \Psi$, where x is not free in Ψ. This rule is preserved because we have insisted that the witnessing terms are closed (note the difference with [4]). The reason why this is so is due to the fact that we treat unbounded quantifications as "computationally empty" (see clause (5) of the definition above). Ultimately, this is possible because the first-order domain has now nonstandard elements and can be seen as a "bounded" domain. Let us go through the (few) details. If $\Phi \vee \Psi$ is provable then, by induction hypothesis there are *closed* terms t and q such that the target theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ proves $\tilde{\forall}^{\mathrm{st}} b, d\, (\Phi_{\mathrm{U_{st}}}(b, tbd, x) \vee \Psi_{\mathrm{U_{st}}}(b, qbd))$. Then, clearly

$$\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} b, d\, (\forall x \Phi_{\mathrm{U_{st}}}(b, tbd, x) \vee \Psi(b, qbd)_{\mathrm{U_{st}}}),$$

and this shows that the same terms t and q witness the interpretation of $\forall x \Phi \vee \Psi$.

The verifications that deserve special attention are the ones pertaining to the standardness axioms and to the characteristic principles. Given that $x =_\sigma y$ and $x \leq^*_\sigma y$ express internal relations, it is easy to see that the interpretations of the first two standardness axioms are, respectively, $\tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, (x =_\sigma y \to (x \leq^*_\sigma b \to y \leq^*_\sigma c))$ and $\tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, (y \leq^*_\sigma b \to (x \leq^*_\sigma y \to x \leq^*_\sigma c))$. It is clear that the term $\lambda b.b$ does the job for both of these axioms. To check the third standardness axiom, take a closed term t. One must show that there is a closed term q such that the theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ proves $t \leq^* q$. Of course, this follows from Howard's majorizability theorem. Regarding the fourth standard axiom, notice that (modulo equivalence in $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$):

$$(\mathrm{st}(z) \to (\mathrm{st}(x) \to \mathrm{st}(zx)))^{\mathrm{U_{st}}} \text{ is } \tilde{\forall}^{\mathrm{st}} b, c \tilde{\exists}^{\mathrm{st}} d\, [z \leq^* b \to (x \leq^* c \to zx \leq^* d)].$$

We need to check that there is a closed monotone term t such that

$$\tilde{\forall}^{\mathrm{st}} b, c\, [z \leq^* b \to (x \leq^* c \to zx \leq^* tbc)]$$

is a theorem of the target theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$. Clearly, $t := \lambda b, c.bc$ is such a term.

Let us now consider monotone choice: $\tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y\, \phi(x, y) \to \tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^* fx\, \phi(x, y)$. Attentive computations show that, modulo equivalence in $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$, $(\neg \tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y\, \phi(x, y))^{\mathrm{U_{st}}}$ is

$$\tilde{\forall}^{\mathrm{st}} g \tilde{\exists}^{\mathrm{st}} b \tilde{\exists} b' \leq^* b \tilde{\exists} x \leq^* b' \tilde{\forall} c' \leq^* g b' \tilde{\forall} y \leq^* c' \, \neg\phi(x,y),$$

and that $(\tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^* fx \, \phi(x,y))^{\mathrm{U_{st}}}$ is

$$\tilde{\forall}^{\mathrm{st}} H \tilde{\exists}^{\mathrm{st}} g \tilde{\exists} g' \leq^* g \tilde{\exists} f \leq^* g' \tilde{\forall} d' \leq^* H g' \tilde{\forall} x \leq^* d' \tilde{\exists} y \leq^* fx \, \phi(x,y).$$

Therefore, we must show that there are closed monotone terms t, q of appropriate types such that

$$\begin{aligned} \tilde{\forall}^{\mathrm{st}} g, H \, [\tilde{\exists} b' \leq^* tgH \tilde{\exists} x \leq^* b' \tilde{\forall} c' \leq^* g b' \tilde{\forall} y \leq^* c' \, \neg\phi(x,y) \vee \\ \tilde{\exists} g' \leq^* qgH \tilde{\exists} f \leq^* g' \tilde{\forall} d' \leq^* H g' \tilde{\forall} x \leq^* d' \tilde{\exists} y \leq^* fx \, \phi(x,y)] \end{aligned}$$

is a theorem of the target theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$. Let t be $\lambda g, H.Hg$ and q be $\lambda g, H.g$, and take arbitrary monotone standard g, H. We need to check that the disjunction between

$(*)$ $\tilde{\exists} b' \leq^* Hg \tilde{\exists} x \leq^* b' \tilde{\forall} c' \leq^* g b' \tilde{\forall} y \leq^* c' \, \neg\phi(x,y)$ and

$(\dagger)$ $\tilde{\exists} g' \leq^* g \tilde{\exists} f \leq^* g' \tilde{\forall} d' \leq^* H g' \tilde{\forall} x \leq^* d' \tilde{\exists} y \leq^* fx \, \phi(x,y)$.

is a theorem of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$. Suppose that $\tilde{\exists} x' \leq^* Hg \tilde{\forall} y \leq^* g x' \, \neg\phi(x', y)$. Take such an x'. Then we get $(*)$ by putting b' and x as x'. Otherwise, $\tilde{\forall} x' \leq^* Hg \tilde{\exists} y \leq^* g x' \, \phi(x', y)$. In this case we have $(\dagger)$ by putting g' and f as g.

We now consider idealization: $\forall x \exists^{\mathrm{st}} y \, \phi(x,y) \to \tilde{\exists}^{\mathrm{st}} z \forall x \exists y \leq^* z \, \phi(x,y)$. Modulo equivalence in $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$, we have that

$(\neg \forall x \exists^{\mathrm{st}} y \, \phi(x,y))^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} b \neg \forall x \tilde{\exists} b' \leq^* b \exists y \leq^* b' \, \phi(x,y)$, and that
$(\tilde{\exists}^{\mathrm{st}} z \forall x \exists y \leq^* z \, \phi(x,y))^{\mathrm{U_{st}}}$ is $\tilde{\exists}^{\mathrm{st}} c \tilde{\exists} c' \leq^* c \tilde{\exists} z \leq^* c' \forall x \exists y \leq^* z \, \phi(x,y)$.

We need to check that there are closed monotone terms t of appropriate types such that

$$\tilde{\forall}^{\mathrm{st}} b \, [\neg \forall x \tilde{\exists} b' \leq^* b \tilde{\exists} y \leq^* b' \, \phi(x,y) \vee \tilde{\exists} c' \leq^* tb \tilde{\exists} z \leq^* c' \forall x \tilde{\exists} y \leq^* z \, \phi(x,y)]$$

is a theorem of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$. It is not difficult to show that the term $t := \lambda b.b$ works.

The majorizability axioms have straightforward interpretations. □

The *flattening* Φ^* of a formula Φ is, by definition, the formula obtained from Φ by replacing the standardness predicate $st^{\rho}(x)$ by a provably universal predicate like the identity $x =_{\rho} x$. Of course, internal formulas are unmoved by this translation. Note that quantifiers of the form $\forall^{\mathrm{st}} x$ and $\tilde{\forall}^{\mathrm{st}} x$ translate (up to logical equivalence) into $\forall x$ and $\tilde{\forall} x$, respectively. Clearly, if $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} \vdash \Phi$ then $\mathsf{E\text{-}PA}^{\omega} \vdash \Phi^*$ (because the flattenings of the axioms of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ are provable in $\mathsf{E\text{-}PA}^{\omega}$). Therefore, the conclusion of the above theorem can be replaced by $\mathsf{E\text{-}PA}^{\omega} \vdash \tilde{\forall} b \, \Phi_{\mathrm{U_{st}}}(b, tb)$. The following result is a particular case:

Corollary 1. *The theory* $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega}_{\mathrm{st}} + \mathsf{R}^{\omega} + \mathsf{MAJ}^{\omega}_{\mathrm{st}}$ *is conservative over* $\mathsf{E\text{-}PA}^{\omega}$.

In the terminology of nonstandard analysis, the translation described above goes by the name of *internalization* (the word 'flattening' comes from [4]). Before proceeding, the reader should be made aware (if he is not already) that the flattening of the theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega}_{\mathrm{st}} + \mathsf{R}^{\omega} + \mathsf{MAJ}^{\omega}_{\mathrm{st}}$ is (obviously) inconsistent.

4. Recasting the bounded functional interpretation

In this section, we assume familiarity with the intensional theory $\mathsf{PA}^{\omega}_{\trianglelefteq}$ and its bounded functional interpretation, as exposed in [4]. We draw attention to the fact that the language of this theory contains, for each finite type σ, a primitive binary symbol $\trianglelefteq_{\sigma}$. These majorizability symbols are governed by certain axioms and a *rule*. The axioms are $x \trianglelefteq_0 y \leftrightarrow x \leq y$ and $x \trianglelefteq_{\rho\to\sigma} y \to \forall u \trianglelefteq_{\rho} v\,(xu \trianglelefteq_{\sigma} yv \wedge yu \trianglelefteq_{\sigma} yv)$. The *rule* is

$$\frac{A \wedge u \trianglelefteq v \to su \trianglelefteq tv \wedge tu \trianglelefteq tv}{A \to s \trianglelefteq t}$$

where s and t are terms, A is a bounded formula (in the sense of [4]) and u and v are variables which do not occur free in the conclusion.

The remainder of this section explains how the bounded functional interpretation can be recast as a particular case of the interpretation of the previous section. For all it's worth, the rule of thumb is that the intensional majorizability relations of $\mathsf{PA}^{\omega}_{\trianglelefteq}$ should be viewed as "coming from" the ordinary majorizability relations of a nonstandard extension.

Let us consider a translation $A \rightsquigarrow A^{\diamond}$ between formulas of the language of $\mathsf{PA}^{\omega}_{\trianglelefteq}$ and formulas of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$. The translation is given by the following clauses:

- $(t =_0 q)^{\diamond}$ is $t =_0 q$;
- $(t \trianglelefteq_{\sigma} q)^{\diamond}$ is $t \leq^{*}_{\sigma} q$;
- $(A \vee B)^{\diamond}$ is $A^{\diamond} \vee B^{\diamond}$;
- $(\neg A)^{\diamond}$ is $\neg A^{\diamond}$;
- $(\forall x^{\sigma} A)^{\diamond}$ is $\forall^{\mathrm{st}} x^{\sigma} A^{\diamond}$;
- $(\forall x \trianglelefteq_{\sigma} t\, A)^{\diamond}$ is $\forall x \leq^{*}_{\sigma} t\, A^{\diamond}$.

Clearly, this translation commutes with substitution. The next results show that the bounded functional interpretation $A \rightsquigarrow A^{\mathrm{U}} \equiv \tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b,c)$ defined in [4] can be seen as falling out from the interpretation of the previous section. The monotone quantifiers $\tilde{\forall}$ and $\tilde{\exists}$ of the U-interpretation are, of course, understood as being monotone with respect to the *intensional* majorizability relation (as opposed to $\leq^{*}$). In order not to multiply notation, we (ab)use the same notation for the two different notions, but the context should be enough to effect a proper disambiguation.

Proposition 1. *For each formula A of the language of $\mathsf{PA}^{\omega}_{\trianglelefteq}$ (possibly with parameters), we have the following:*

$$\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} \vdash \tilde{\forall} b, c\,[(A_{\mathrm{U}}(b,c))^{\diamond} \leftrightarrow (A^{\diamond})_{\mathrm{U_{st}}}(b,c)]$$
$$\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} \vdash [(A^{\mathrm{U}})^{\diamond} \leftrightarrow (A^{\diamond})^{\mathrm{U_{st}}}]$$

where b and c are, respectively, the universal and existential variables of the pertinent functional interpretations.

Observation 1. *The following picture conveys the relationship between the translations given by the bounded functional interpretation* U *and the interpretation* $\mathrm{U_{st}}$ *of this paper.*

$$\begin{tikzcd}
\mathsf{PA}^{\omega}_{\trianglelefteq} \arrow[r, "\mathrm{U}"] \arrow[d, "\diamond"'] & \mathsf{PA}^{\omega}_{\trianglelefteq} \arrow[d, "\diamond"] \\
\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}} \arrow[r, "\mathrm{U_{st}}"'] & \mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}
\end{tikzcd}$$

Proof. First of all, if A is a bounded formula of the intensional language of $\mathsf{PA}^\omega_\trianglelefteq$, it is clear that its translation $A^\diamond$ is an internal formula. Hence the equivalences above hold trivially because the pertinent formulas are $A^\diamond$. This is the base case of a proof by induction on the complexity of A. The boolean cases are straightforward, as well as the case of the bounded quantifier. The interesting case is the unbounded quantifier.

Take the formula $((\forall x A(x))^\diamond)^{\mathrm{U_{st}}}$. This is $(\forall^{\mathrm{st}} x A(x)^\diamond)^{\mathrm{U_{st}}}$, i.e., $(\forall x(\mathrm{st}(x) \to A(x)^\diamond))^{\mathrm{U_{st}}}$. By definition, $(A(x)^\diamond)^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, (A(x)^\diamond)_{\mathrm{U_{st}}}(b,c)$. Therefore, $(\mathrm{st}(x) \to A(x)^\diamond)^{\mathrm{U_{st}}}$ is $\tilde{\forall}^{\mathrm{st}} a \tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c\, (x \leq^* a \to (A(x)^\diamond)_{\mathrm{U_{st}}}(b,c))$. By the universal clause of the $\mathrm{U^{st}}$-translation, the original formula is

$$\tilde{\forall}^{\mathrm{st}} a \tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c \forall x \leq^* a\, (A(x)^\diamond)_{\mathrm{U_{st}}}(b,c).$$

By induction hypothesis, this is equivalent to $\tilde{\forall}^{\mathrm{st}} a \tilde{\forall}^{\mathrm{st}} b \tilde{\exists}^{\mathrm{st}} c \forall x \leq^* a\, (A(x)_{\mathrm{U}}(b,c))^\diamond$ which, in turn, is $(\tilde{\forall} a \tilde{\forall} b \tilde{\exists} c \forall x \trianglelefteq a\, A(x)_{\mathrm{U}}(b,c))^\diamond$. This is $((\forall x A(x))^{\mathrm{U}})^\diamond$, as we wanted. The part for the lower transformations can be read from the previous calculations. □

The proof of the next result includes an argument showing how the intensional majorizability rule is dealt by the wider nonstandard setting. It is quite illuminating, we believe.

Theorem 2. *If* $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathsf{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega_{\mathsf{bd}} \vdash A(z)$, *where the free variables are as shown, then*

$$\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{mAC}^\omega_{\mathrm{st}} + \mathsf{R}^\omega + \mathsf{MAJ}^\omega_{\mathrm{st}} \vdash \mathrm{st}(z) \to A(z)^\diamond.$$

Observation 2. *The principles* $\mathsf{mAC}^\omega_{\mathsf{bd}}$, bC^ω *and* $\mathsf{MAJ}^\omega_{\mathsf{bd}}$ *are the characteristic principles of the bounded functional interpretation of [4].*

Proof. The diamond translation is a relativization to the standard predicate st (modulo the replacement of $\trianglelefteq$ by $\leq^*$). This relativization is in good standing since the closed terms are standard, standardness is preserved by term application and an element majorizable by a standard element is itself standard (this is the content of the last three standardness axioms). Therefore, the property of the theorem is preserved by logical consequence. We need to check two things: (1) that if $A(z)$ is an axiom of $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathsf{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega_{\mathsf{bd}}$ (including the induction axioms), then $\mathrm{st}(z) \to A(z)^\diamond$ is a theorem of the target theory $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{mAC}^\omega_{\mathrm{st}} + \mathsf{R}^\omega + \mathsf{MAJ}^\omega_{\mathrm{st}}$; (2) that the property of the theorem is preserved by the rule governing the intensional majorizability relations.

Intensional bounded quantification is a primitive of the language of $\mathsf{PA}^\omega_\trianglelefteq$. It is regulated by the axiom scheme $\forall x \trianglelefteq t(z) A(z,x) \leftrightarrow \forall x(x \trianglelefteq t(z) \to A(z,x))$. Under the hypothesis $\mathrm{st}(z)$, the diamond translation of this axiom is a theorem of the target theory due (essentially) to the second standardness axiom. The equality axioms of $\mathsf{PA}^\omega_\trianglelefteq$ are treated minimally in the manner described in [14]. These are open axioms (without the intensional majorizability sign) and, hence, they are translated by themselves. Since the target theory has a stronger theory of equality (extensional equality regulated by the extensionality axiom), their translation is provable there. Apart from the scheme of induction, the arithmetical axioms are also open axioms (without the intensional sign) and, therefore, pose no trouble. The induction scheme is taken care by the external induction scheme of the target theory.

Clearly, the first characteristic principle $\mathsf{mAC}^\omega_{\mathsf{bd}}$ translates into cases of $\mathsf{mAC}^\omega_{\mathrm{st}}$ (note that bounded formulas of the intensional language translate into internal formulas of the language of the target theory). The translation of the third characteristic principle $\mathsf{MAJ}^\omega_{\mathsf{bd}}$ is, exactly, $\mathsf{MAJ}^\omega_{\mathrm{st}}$. The diamond translation of bC^ω is

$$\forall x \leq^* w \exists^{\mathrm{st}} y A(y,x)^\diamond \to \tilde{\exists}^{\mathrm{st}} z \forall x \leq^* w \exists y \leq^* z\, A(y,z)^\diamond,$$

where A is a bounded intensional formula (hence, $A(y,z)^\diamond$ is an internal formula). Let us assume $\forall x \leq^* w \exists^{\mathrm{st}} y A(y,z)^\diamond$. Therefore, $\forall x \exists^{\mathrm{st}} y\,(x \leq^* w \to A(y,z)^\diamond)$. Since the formula $x \leq^* w \to A(y,z)^\diamond$ is internal, we may apply the realization axiom R^ω to obtain $\tilde{\exists}^{\mathrm{st}} z \forall x \exists y \leq^* z\,(x \leq^* w \to A(y,z)^\diamond)$. The conclusion is now immediate.

It remains to check the two axioms and the rule of the intensional majorizability relations. Even though the axioms are very easy to deal with, we discuss the second axiom. We must show that the target theory proves

$$\mathrm{st}(x) \wedge \mathrm{st}(y) \wedge \mathrm{st}(v) \to (x \leq^* y \to \forall u(u \leq^* v \to xu \leq^* yv \wedge yu \leq^* yv)).$$

This is obvious. It is even true without the standardness restriction on v (and on x and y, but these are not important). Note that it is precisely this restriction on v that prevents the converse conditional (of the second majorizability axiom) from being interpreted. However, we will now see that the rule (which is a weakening of the converse conditional) preserves the property of the theorem. Suppose that

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathsf{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega_{\mathsf{bd}} \vdash A(z) \wedge u \trianglelefteq v \to s(z)u \trianglelefteq t(z)v \wedge t(z)u \trianglelefteq t(z)v,$$

where A is a bounded intensional formula and s and t are terms (the variable z stands for the parameters). By induction hypothesis, the target theory proves

$$\mathrm{st}(z) \wedge \mathrm{st}(u) \wedge \mathrm{st}(v) \to (A(z)^\diamond \wedge u \leq^* v \to s(z)u \leq^* t(z)v \wedge t(z)u \leq^* t(z)v).$$

By the soundness theorem of the previous section, it is easy to conclude that the theory $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}}$ proves

$$\tilde{\forall}^{\mathrm{st}} a, b, c \forall z \leq^* a \forall u \leq^* b \forall v \leq^* c\,(A(z)^\diamond \wedge u \leq^* v \to s(z)u \leq^* t(z)v \wedge t(z)u \leq^* t(z)v).$$

By flattening (note that A is a bounded intensional formula and, hence, $A^\diamond$ is already flattened), the theory $\mathsf{E\text{-}PA}^\omega$ proves

$$\tilde{\forall} a, b, c \forall z \leq^* a \forall u \leq^* b \forall v \leq^* c\,(A(z)^\diamond \wedge u \leq^* v \to s(z)u \leq^* t(z)v \wedge t(z)u \leq^* t(z)v).$$

Hence, $\mathsf{E\text{-}PA}^\omega$ proves

$$\tilde{\forall} a \forall z \leq^* a\,(A(z)^\diamond \to \tilde{\forall} b, c \forall u \leq^* b \forall v \leq^* c\,(u \leq^* v \to s(z)u \leq^* t(z)v \wedge t(z)u \leq^* t(z)v).$$

Using the fact that $u \leq^* v \to v \leq^* v$, we get

$$\mathsf{E\text{-}PA}^\omega \vdash \tilde{\forall} a \forall z \leq^* a\,(A(z)^\diamond \to \forall u \forall v\,(u \leq^* v \to s(z)u \leq^* t(z)v \wedge t(z)u \leq^* t(z)v).$$

By definition of majorizability, $\mathsf{E\text{-}PA}^\omega$ proves $\tilde{\forall} a \forall z \leq^* a\,(A(z)^\diamond \to s(z) \leq^* t(z))$. Since $\mathsf{E\text{-}PA}^\omega$ is a subtheory of $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}}$, the latter theory also proves this fact. It easily follows that $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{MAJ}^\omega_{\mathrm{st}} \vdash \mathrm{st}(z) \to (A(z)^\diamond \to s(z) \leq^* t(z))$. We are done. □

The above argument regarding the intensional majorizability rule reflects a particular case of the fact that the theory $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{mAC}^\omega_{\mathrm{st}} + \mathsf{R}^\omega + \mathsf{MAJ}^\omega_{\mathrm{st}}$ is closed under the following rule:

$$\frac{\forall^{\mathrm{st}} v\, \phi(v)}{\tilde{\forall} b \forall v \leq^* b\, \phi(v)}$$

where parameters are allowed and v (and the corresponding b) may stand for more than one variable. (This is a rule-version of the transfer principle for universal formulas. In Appendix A, we discuss axiom-versions of the transfer principle.) The argument for this general rule follows the blueprint of the particular case considered in the proof of the above theorem. Suppose that $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{mAC}^\omega_{\mathrm{st}} + \mathsf{R}^\omega + \mathsf{MAJ}^\omega_{\mathrm{st}} \vdash \forall^{\mathrm{st}} v\, \phi(v)$. By Theorem 1, it is easy to conclude that $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} b \forall v \leq^* b\, \phi(v)$. By flattening, $\tilde{\forall} b \forall v \leq^* b\, \phi(v)$ is provable in $\mathsf{E\text{-}PA}^\omega$ and, *a fortiori*, in $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{mAC}^\omega_{\mathrm{st}} + \mathsf{R}^\omega + \mathsf{MAJ}^\omega_{\mathrm{st}}$.

It is clear that the bounded functional interpretation of [4] is but a particular case of the wider interpretation of this paper. As a matter of fact, we can deduce (the flattened version of) the soundness theorem of [4] using the results of this section. To see this, suppose that $\mathsf{PA}^\omega_{\trianglelefteq} + \mathsf{mAC}^\omega_{\mathsf{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega_{\mathsf{bd}} \vdash A(z)$, where the free variables are as shown. By the above theorem, $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{mAC}^\omega_{\mathrm{st}} + \mathsf{R}^\omega + \mathsf{MAJ}^\omega_{\mathrm{st}} \vdash \mathrm{st}(z) \to A(z)^\diamond$. An easy computation shows that the U^{st}-interpretation of $\mathrm{st}(z) \to A(z)^\diamond$ is

$$\tilde{\forall}^{\mathrm{st}} a, b \tilde{\exists}^{\mathrm{st}} c \forall z \leq^* a\, (A^\diamond)_{\mathrm{U}_{\mathrm{st}}}(b, c, z).$$

Hence, by Theorem 1, there is a closed monotone term t such that

$$\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} a, b \forall z \leq^* a\, (A^\diamond)_{\mathrm{U}_{\mathrm{st}}}(b, tab, z).$$

By Proposition 1,

$$\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} a, b \forall z \leq^* a\, (A_{\mathrm{U}}(b, tab, z))^\diamond.$$

Since A_{U} is an intensional bounded formula, it is clear that its diamond translation is nothing but its flattening (in the sense given in [4]). Hence,

$$\mathsf{E\text{-}PA}^\omega \vdash \tilde{\forall} a, b \forall z \leq^* a\, (A_{\mathrm{U}}(b, tab, z))^*.$$

Well, this is the conclusion of the soundness theorem of [4] after flattening. (The attentive reader will have noticed that in [4] the verification of the interpretation – after flattening – was done within a theory with a minimal treatment of equality. The difference with the present setting is explained by the fact that we have worked with the theory $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}}$, a theory with full extensionality. Had we worked with a suitable version of this theory with the minimal treatment of equality, we would have obtained a perfect match.)

Appendix A

Both the interpretation of this paper and the interpretation of Berg et al. have characteristic principles which are forms of idealization and standardness, as introduced by Nelson in [11]. Conspicuous by its absence is the transfer principle (the 'T' of Nelson's IST). In our setting, it can be formulated as follows:

$$\textit{Transfer Principle}\ \mathsf{TP}_\forall:\ \forall^{\mathrm{st}} z\, (\forall^{\mathrm{st}} v\, \phi(v, z) \to \forall v\, \phi(v, z)),$$

where $\phi(v, z)$ has no undisplayed free variables (note that, according to our conventions, z and v may stand for more than one variable). This principle is incompatible with $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}} + \mathsf{R}^\omega$ and, therefore, it cannot be added to our theory $\mathsf{E\text{-}PA}^\omega_{\mathrm{st}}$ and its characteristic principles. To see this, consider the logical truth

$$\forall^{\mathrm{st}} z \leq^*_1 1 \exists^{\mathrm{st}} k^0 (\exists^{\mathrm{st}} n^0 (zn = 0) \to zk = 0).$$

It is a consequence of $\mathsf{TP}_\forall$ that $\exists^{\mathrm{st}} n(zn = 0)$ is equivalent to $\exists n(zn = 0)$ for standard z. Hence, we get the proposition $\forall z \leq^* 1\,(\exists n(zn = 0) \to \exists^{\mathrm{st}} k(zk = 0))$. Note that we are using the second standardness axiom to remove the standardness condition on z. Take r a nonstandard natural number and let z be $\lambda i^0.(\textbf{if } i \leq r \textbf{ then } 1, \textbf{otherwise } 0)$. Clearly, the proposition fails for this functional z. (The version of the transfer principle corresponding to the transfer rule discussed in Section 4 is refuted by the same example.)

In the case of the theory of Berg et al., we can argue that $\mathsf{E\text{-}PA}^{\omega *}_{\mathsf{st}} + \mathsf{I} + \mathsf{HAC}_{\mathsf{int}} + \mathsf{TP}_\forall$ (cf. [15] for the notation) is *not* a conservative extension of $\mathsf{E\text{-}PA}^{\omega *}_{\mathsf{st}}$. This answers negatively a conjecture of Berg *et al.* at the end of Section 7 of [15]. Let $\phi(n^0, k^0)$ be a bounded first-order formula such that $\exists k\, \phi(n, k)$ does not define a recursive set. By logic alone, $\forall^{\mathrm{st}} n \exists^{\mathrm{st}} k\, (\phi(n, k) \vee \forall^{\mathrm{st}} k \neg\phi(n, k))$. By $\mathsf{TP}_\forall$, we get $\forall^{\mathrm{st}} n \exists^{\mathrm{st}} k\, (\phi(n, k) \vee \forall k \neg\phi(n, k))$. Applying the herbrandized axiom of choice for internal formulas $\mathsf{HAC}_{\mathsf{int}}$, we may infer that $\exists^{\mathrm{st}} f \forall^{\mathrm{st}} n \exists k \leq fn\, (\phi(n, k) \vee \forall k \neg\phi(n, k))$. By $\mathsf{TP}_\forall$ again,

$$\exists^{\mathrm{st}} f \forall n \exists k \leq fn\, (\phi(n, k) \vee \forall k \neg\phi(n, k)).$$

A fortiori, $\exists f \forall n \exists k \leq fn\, (\phi(n, k) \vee \forall k \neg\phi(n, k))$. From this fact, it is easy to argue that $\exists g \forall n\, (gn = 0 \leftrightarrow \exists k \phi(n, k))$. Since this is an internal formula, by conservativity this would be provable in $\mathsf{E\text{-}PA}^\omega$. It is well-known that this is not the case because the above form of comprehension fails in the model *HEO* of the *hereditarily effective operations* (see, for instance, [1]).

Note that the second application of $\mathsf{TP}_\forall$ in the above argument is made with respect to the matrix $\exists k \leq fn\, (\phi(n, k) \vee \forall k \neg\phi(n, k))$. This formula has a type 1 parameter (the variable f). We do not know if the transfer principle restricted to first-order matrices can be added to $\mathsf{E\text{-}PA}^{\omega *}_{\mathsf{st}} + \mathsf{I} + \mathsf{HAC}_{\mathsf{int}}$ so that the extended theory is conservative over $\mathsf{E\text{-}PA}^\omega$.

If we do not worry about the complexity of the matrices ϕ of the transfer principle $\mathsf{TP}_\forall$, much stronger consequences can be proved. For instance, we can prove the existence of the non-continuous type 2 functional ${}^2\mathrm{E}$ satisfying the following property: ${}^2\mathrm{E}(z^1) = 0 \leftrightarrow \exists n^0(zn = 0)$. In effect, by $\mathsf{TP}_\forall$, $\forall^{\mathrm{st}} z\, (\exists n(zn = 0) \to \exists^{\mathrm{st}} n(zn = 0))$. Equivalently, $\forall^{\mathrm{st}} z \exists^{\mathrm{st}} n\, (\exists n(zn = 0) \to zn = 0)$. By $\mathsf{HAC}_{\mathsf{int}}$, we get

$$\exists^{\mathrm{st}} F \forall^{\mathrm{st}} z \exists n \leq Fz\, (\exists n(zn = 0) \to zn = 0).$$

By the transfer principle again we can remove the standardness condition on z. Therefore, $\exists^{\mathrm{st}} F \forall z \exists n \leq Fz\, (\exists n(zn = 0) \to zn = 0)$. Take such an F. We obtain

$$\forall z\, (\exists n(zn = 0) \leftrightarrow \exists n \leq Fz\, (zn = 0)).$$

The functional ${}^2\mathrm{E}$ can now easily be defined in terms of F.

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