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ON TAO'S "FINITARY" INFINITE PIGEONHOLE PRINCIPLE

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Abstract. In 2007, Terence Tao wrote on his blog an essay about soft analysis, hard analysis and the finitization of soft analysis statements into hard analysis statements. One of his main examples was a quasi-finitization of the infinite pigeonhole principle IPP, arriving at the "finitary" infinite pigeonhole principle FIPP_1. That turned out to not be the proper formulation and so we proposed an alternative version FIPP_2. Tao himself formulated yet another version FIPP_3 in a revised version of his essay.

We give a counterexample to FIPP_1 and discuss for both of the versions FIPP_2 and FIPP_3 the faithfulness of their respective finitization of IPP by studying the equivalences $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ and $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ in the context of reverse mathematics ([9]). In the process of doing this we also introduce a continuous uniform boundedness principle CUB as a formalization of Tao's notion of a correspondence principle and study the strength of this principle and various restrictions thereof in terms of reverse mathematics, i.e., in terms of the "big five" subsystems of second order arithmetic.

§1. Introduction.

In his article [10], T. Tao introduced the program of finitizing infinitary principles P in analysis. This is achieved by showing (using compactness and continuity arguments) the existence of a uniform bound on some existential number quantifier (in a suitably reformulated version of P, e.g., corresponding to its Herbrand normal form) that is independent from the infinitary input of the principle (typically an infinite sequence in some metric space). From this bound one then reads off that the new ("finitary") principle actually only refers to some finite part (e.g., a finite initial segment in the case of a sequence) of that infinitary input. As two of his prime examples he discusses the convergence principle for bounded monotone sequences of reals (PCM) and the infinitary pigeonhole principle (IPP). As observed in [8], the finitary version of PCM proposed by Tao directly follows from a well-studied proof-theoretic construction due to the second author, the so-called monotone Gödel functional interpretation of PCM. In [8] a similar case is made concerning IPP, i.e., it is shown that the monotone functional interpretation of IPP leads to a "finitary" version FIPP_0 similar, but not identical, to the one proposed by Tao in his first 2007 posting of [10] (FIPP_1). Like Tao, we use the prefix "finitary" here in quotation marks as neither of the finitizations of IPP is strictly finitary (in the sense the finitary form of PCM is) since non-finitary (in fact 2nd order) conditions on the Herbrand index function need to be imposed.

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One difference between FIPP_0 and FIPP_1 is that the former is formulated in a language of primitive recursive functionals whereas the latter is formulated in terms of sets and finitary set-functions. In closing the gap between the two formulations the second author reformulated FIPP_0 into a variant FIPP_2 in the same vocabulary as the latter. However, as it turns out, FIPP_2 has a slightly weaker conclusion than FIPP_1. Subsequently, the first author found a counterexample to FIPP_1 (see Section 4 below). In reaction to that counterexample, Tao modified FIPP_1 (in a revised posting of [10] from August 2008) to yet another version FIPP_3 which keeps the original conclusion of FIPP_1 but strengthens the premise of the latter principle. In order to compare the two finitizations FIPP_2 and FIPP_3 w.r.t. their faithfulness as finitizations of IPP we investigate in this paper the strength of the equivalences $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ and $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ in terms of the systems RCA_0, WKL_0 and ACA_0 from the program of reverse mathematics (see [9]). For FIPP_0 it follows from the reasoning given in [8] that it implies IPP over a system of functionals of finite type that is conservative over Kalmar elementary arithmetic and that the implication $\mathsf{IPP} \to \mathsf{FIPP}_0$ follows with an additional use of WKL (needed to show that continuous functionals $\Phi : 2^{\mathbb{N}} \to \mathbb{N}$ are bounded, see [6, 9]). This suggests that the version FIPP_2 that was prompted by FIPP_0 has a similar behavior: more precisely we show that RCA_0 proves $\mathsf{FIPP}_2 \to \mathsf{IPP}$ while WKL_0 proves $\mathsf{IPP} \to \mathsf{FIPP}_2$.

For FIPP_3 the direction $\mathsf{FIPP}_3 \to \mathsf{IPP}$ still follows in RCA_0. The implication $\mathsf{IPP} \to \mathsf{FIPP}_3$ can be established by an application of the Bolzano-Weierstraß property of the compact metric space $[n]^{\mathbb{N}}$ (with respect to the Baire metric) which in turn is provable in (and in fact equivalent to) ACA_0 (see [9]). So ACA_0 proves $\mathsf{IPP} \to \mathsf{FIPP}_3$. This, however, is unsatisfactory as ACA_0 is much stronger than IPP itself, whereas WKL_0 does not prove IPP by a result due to [3]. So it is natural to try to establish the implication $\mathsf{IPP} \to \mathsf{FIPP}_3$ by a WKL-type "Heine-Borel"-compactness argument rather than by using sequential compactness (requiring ACA_0). Towards this goal and aiming at a formalization of Tao's informal notion of "correspondence principle" from [11] we formulate a "continuous uniform boundedness principle" CUB that generalizes the usual FAN-uniform boundedness obtained from (the contraposite form of) WKL. In fact, CUB restricted to Σ^0_1 formulas, denoted by Σ^0_1-CUB, is equivalent to WKL_0 over RCA_0 and the proof that WKL_0 implies $\mathsf{IPP} \to \mathsf{FIPP}_2$ can nicely be recasted as an application of Σ^0_1-CUB as we will do below. Also $\mathsf{IPP} \to \mathsf{FIPP}_3$ can be established by an application of CUB. However, this time it seems that a Π^0_1-instance, i.e., a use of Π^0_1-CUB, is needed. Unfortunately, Π^0_1-CUB is no longer derivable in WKL_0 but, in fact, is equivalent to ACA_0 (over RCA_0) which shows that the strength of the correspondence principle as formalized by CUB crucially depends on the logical complexity of the instance involved. In fact, over RCA it turns out that the unrestricted CUB even is equivalent to full second order comprehension over numbers, i.e., to Z_2. While leaving open the question whether WKL_0 proves $\mathsf{IPP} \to \mathsf{FIPP}_3$, the results in this paper may suggest that the answer is negative and at the same time show that the logical structure of the formula to which a correspondence principle such as CUB is applied matters in determining how close a finitization of some infinitary principle stays to that principle.

The following diagram summarizes the picture established in this paper:

$$\begin{tikzcd}
\mathsf{Z}_2 \arrow[r, leftrightarrow, "\mathsf{RCA}"] & \mathsf{CUB} & \\
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[u, squiggly] & & \\
\mathsf{ATR}_0 \arrow[u, squiggly] & & \\
\mathsf{ACA}_0 \arrow[u, squiggly] \arrow[r, leftrightarrow, "\mathsf{RCA}_0"] & \Pi^0_1\text{-}\mathsf{CUB} \arrow[r, "\mathsf{RCA}_0"] & (\mathsf{IPP}\to\mathsf{FIPP}_3) \arrow[l, bend left, "?"] \\
\mathsf{WKL}_0 \arrow[u, squiggly] \arrow[r, leftrightarrow, "\mathsf{RCA}_0"] & \Sigma^0_0\text{-}\mathsf{CUB} \arrow[r, "\mathsf{RCA}_0"] & (\mathsf{IPP}\to\mathsf{FIPP}_2) \arrow[l, bend left, "?"] \\
\mathsf{RCA}_0 \arrow[u, squiggly] \arrow[rr] & & (\mathsf{FIPP}_{2/3}\to\mathsf{IPP})
\end{tikzcd}$$

§**2. Definitions.** In this paragraph we collect some notation and formulate the infinite pigeonhole principle IPP as well as the three "finitary" infinite pigeonhole principles FIPP_1, FIPP_2 and FIPP_3.

All the definitions take place in the context of the language of RCA_0, where RCA_0 is the base system used in reverse mathematics (see [9] for details). All undefined notations are to be understood in the sense of [9]. We need to be rather formal in our definitions working over the weak base system RCA_0. For example, in point 1 of Definition 5 we need to assume the existence of the union of a infinite sequence of sets, since RCA_0 in general doesn't prove that such a union exists. But over sufficiently strong systems such as ACA_0, the clause stating the existence of the union set is redundant.

Tao formulated his "finitary" infinite pigeonhole principle using *set functions*, i.e., functions that take as input a finite subset of $\mathbb{N}$ and return as output a natural number. Those are, however, objects of a higher type than those available in the language of RCA_0, so we had to reformulate Tao's principle using functions from $\mathbb{N}$ to $\mathbb{N}$ by identifying a finite subset of $\mathbb{N}$ with a natural number encoding it.

Definition 1. We denote by $[i]$ the set $\{j : j \leq i\}$ of the first $i+1$ natural numbers. If $i = 0$, then we make the convention that $[i-1]$ is the empty set $\emptyset$.

Definition 2. If $l \in \mathrm{Seq}$, then we define A_l to be the set *encoded* by the finite sequence with code l, i.e., $A_l := \{l(i) : i < \mathrm{lh}\, l\}$. We say that l is a *code* of a set A if $A = A_l$. One can also consider the minimal code which then is called *the* code of A.

Definition 3. If $f : \mathbb{N} \to \mathbb{N}$ is a function and $m > 0$ then we define $\bar{f}m$ to be the code of the finite sequence $\langle f(0), \dots, f(m-1)\rangle$. For $m = 0$ we make the convention that $\bar{f}m$ is the code of the empty sequence $\langle\rangle$. If $s \in \mathrm{Seq}$, then we denote the function that extends s by zeros by $s^\frown o$.

Definition 4. Let $f : X \to Y$ be a function between sets X, Y. We define $|A| = m$ to mean "exists an $f : [m-1] \to A$ one-to-one and onto". Then we define $|A| \geq m :\equiv \exists m' \, (|A| = m' \wedge m' \geq m)$, and analogously for $|A| > m$, $|A| \leq m$ and $|A| < m$.

DEFINITION 5. 1. A sequence (l_m) represents a *nested sequence with union* of finite subsets $\subset \mathbb{N}$ if and only if $\forall m\ (l_m \in \mathrm{Seq})$, $\forall m\ (A_{l_m} \subseteq A_{l_{m+1}})$ and $\bigcup_m A_{l_m}$ exists.

2. A sequence (l_m) *weakly converges* to an infinite set A if and only if $\forall m\ (l_m \in \mathrm{Seq})$ and for all finite sets B we have $\exists i\ \forall j \geq i\ (A_{l_j} \cap B = A \cap B)$. Then we say that (l_m) *weakly converges* if and only if it weakly converges to some infinite set.
3. A function $F : \mathbb{N} \to \mathbb{N}$ is *extensional* if and only if $\forall l, l' \in \mathrm{Seq}\ [A_l = A_{l'} \to F(l) = F(l')]$. Alternatively, one can always use the unique minimal code which allows one to drop the extensionality requirement.
4. A function $F : \mathbb{N} \to \mathbb{N}$ is *asymptotically stable*, denoted by $F \in AS$, if and only if it is extensional and for all nested sequences with union (l_m) we have $\exists i\ \forall j \geq i\ [F(l_i) = F(l_j)]$.
5. A function $F : \mathbb{N} \to \mathbb{N}$ is *asymptotically stable near infinite sets*, denoted by $F \in ASNIS$, if and only if it is extensional and for all weakly convergent sequences (l_m) we have $\exists i\ \forall j \geq i\ [F(l_i) = F(l_j)]$.

Remark 6. A nested sequence with finite union is never weakly convergent (otherwise it would converge to the finite union but a weakly convergent sequence must converge to an infinite set). Every nested sequence with infinite union is weakly convergent (to the infinite union), but there are weakly convergent sequences that are not nested (e.g., $A_{l_m} := [m] \cup \{m+2\}$ weakly converges to $\mathbb{N}$ but is not a chain).

We have $ASNIS \subsetneq AS$ (if $F \in ASNIS$, then F stabilizes over a nested chain with finite union because the chain itself stabilizes and F also stabilizes over a nested chain with infinite union because such a chain is weakly convergent, so $F \in AS$; the $F \in AS$ from the counterexample to FIPP_1 below is not in $ASNIS$ otherwise it would also be a counterexample to true FIPP_3).

DEFINITION 7. 1. The *infinite pigeonhole principle* IPP is the principle: every coloring f of $\mathbb{N}$ into $n+1$ colors has a infinite color class $f^{-1}(c)$. In symbols:

$$\forall n\ \forall f : \mathbb{N} \to [n]\ \exists c \in [n]\ [f^{-1}(c) \text{ infinite}],$$

where the set $f^{-1}(c)$ exists by Σ^0_0 comprehension: $\forall x\ [x \in f^{-1}(c) \leftrightarrow (x, c) \in f]$.

2. The *first "finitary" infinite pigeonhole principle* FIPP_1 is the principle: for all asymptotically stable functions F there exists a k such that every coloring f of $[k]$ into $n+1$ colors has a color class $A = f^{-1}(c)$ that is "big" in the sense of $|A| > F(A)$. In symbols:

$$\forall n\ \forall F \in AS\ \exists k\ \forall f : [k] \to [n]\ \exists l \in \mathrm{Seq}\ \exists c \in [n]\ [A_l = f^{-1}(c) \wedge |A_l| > F(l)].$$

3. The *second "finitary" infinite pigeonhole principle* FIPP_2 is the principle: for all asymptotically stable functions F there exists a k such that every coloring f of $[k]$ into $n+1$ colors has a monochromatic set A that is "big" in the sense of $|A| > F(A)$. In symbols:

$$\forall n\ \forall F \in AS\ \exists k\ \forall f : [k] \to [n]\ \exists l \in \mathrm{Seq}\ \left[A_l \subseteq [k] \wedge |A_l| > F(l) \wedge f|_{A_l} \text{ constant}\right].$$

4. The *third "finitary" infinite pigeonhole principle* FIPP_3 is analogous to FIPP_1 but with AS replaced by $ASNIS$.

Remark 8. IPP can also be formulated without reference to the set $f^{-1}(c)$ as

$$\forall n\, \forall f : \mathbb{N} \to [n]\, \exists c \in [n]\, [\forall m\, \exists k > m\, (f(k) = c)\,].$$

In the presence of Σ^0_0 comprehension (and hence over RCA_0) there is no difference between the two formulations.

By a well-known result due to J.L. Hirst [3], IPP is equivalent to the bounded collection principle for Σ^0_2 formulas (often called $B\Sigma^0_2$, though set parameters are allowed in the context of RCA_0) and is not provable in WKL_0 :

Proposition 9. [3] WKL_0 *does not prove* IPP.

Since $B\Sigma^0_2$ and hence IPP easily follows from Σ^0_2 induction we have that RCA proves IPP as well as ACA_0 proves IPP.

§3. Technical lemmas.

In this section we start by collecting in Lemma 10 some folklore properties about the cardinality of finite sets that we will need later. We first note that the formulas $\mathrm{lh}(s) = m$, $s(i) = m$, $s \subseteq t$ (expressing that the finite sequence encoded by s is an initial segment of the sequence encoded by t), etc. are all Σ^0_0 (see [9]).

At some point we will need to talk about (continuous) functionals $\phi : [n]^{\mathbb{N}} \to \mathbb{N}$ within RCA_0, and to do so we need to show the existence of a code (in the sense of [9]) for them. As shown in [6], the existence of such a code is equivalent to the existence of a so-called associate of ϕ in the sense of Kleene and Kreisel. In the cases at hand it turns out to be easier to construct an associate rather than to produce a code directly. For completeness we include Lemma 11 which shows that the existence of an associate implies the existence of a code. In the first point of Lemma 13 we show that every Σ^0_0 formula is provably "uniformly continuous" in RCA_0. In the second point we prove that every formula of the form $\forall f : \mathbb{N} \to [n]\, A(f)$ with $A \in \Sigma^0_0$ is (over RCA_0) equivalent to a Π^0_1 formula. Finally, in the first point of Lemma 14 we show that if $F \in \mathit{ASNIS}$ and A is a infinite set, then the stable value that F eventually attains on a sequence weakly converging to A doesn't depend on the sequence. In the second point we show that if $F \in \mathit{ASNIS}$, then F is "continuous" in the sense of the Baire space $\mathbb{N}^{\mathbb{N}}$ with the metric

$$d(f,g) := \begin{cases} 2^{-m} & \text{if exists } m = \min m'\, [f(m') \neq g(m')], \\ 0 & \text{otherwise,} \end{cases}$$

at points that are characteristic functions of infinite sets A.

Lemma 10. RCA_0 *proves the following.*

1. *A is a finite if and only if $\exists m\ (|A| = m)$ if and only if A has a code.*
2. *If $l \in Seq$, then the formulas $|A_l| = m$, $|A_l| < m$, $|A_l| > m$ are equivalent to Σ^0_0 formulas.*
3. *If A is infinite, then $\forall i\ \exists j\ (|A \cap [j-1]| = i)$.*
4. *If A and B are finite sets and $A \subseteq B$, then $|A| \leq |B|$.*

Lemma 11. [6] *If $\alpha : \mathbb{N} \to \mathbb{N}$ is an associate of $\phi : [n]^{\mathbb{N}} \to \mathbb{N}$, i.e.,*

1. $\forall \beta : \mathbb{N} \to [n]\, \exists m\, [\alpha(\bar{\beta}m) > 0]$;
2. $\forall \beta : \mathbb{N} \to [n]\, \forall m\, \big[\big(m = \min m'\, \alpha(\bar{\beta}m') > 0\big) \to \alpha(\bar{\beta}m) = \phi(\beta) + 1\big]$;

*then ϕ has a code as a continuous function $[n]^{\mathbb{N}} \to \mathbb{N}$ in the sense of the Definition II.*6.1 *in* [9].

PROOF. We may assume that α is a *neighborhood function*, i.e., $\forall i, j\ [i \subseteq j \wedge \alpha(i) > 0 \rightarrow \alpha(i) = \alpha(j)]$ for otherwise we would replace α by the associate of ϕ and neighborhood function

$$\alpha'(j) := \begin{cases} \alpha(i) & \text{if } j \in \mathrm{Seq} \text{ and exists the shortest } i \subseteq j \text{ such that } \alpha(i) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [9] (example II.5.5) a construction is given of a code A, d (where $A \subseteq \mathbb{N}$ and $d : A \times A \rightarrow \mathbb{R}$) for infinite product spaces of complete separable metric spaces. Particularizing the construction for $[n]^{\mathbb{N}} = \prod_{i=0}^{\infty}[n]$ we get

$$A = \{\langle a_i : i \leq j\rangle : j \in \mathbb{N} \wedge \forall i \leq j\ (a_i \in [n])\},$$
$$d(a, b) = \sum_{i=0}^{\infty} 2^{-i} \cdot \frac{|(a^\frown o)(i) - (b^\frown o)(i)|}{1 + |(a^\frown o)(i) - (b^\frown o)(i)|}.$$

One easily verifies that

1. $\forall a, b \in A\ [d(a, b) < 2^{-r} \rightarrow \overline{a^\frown o}\, r = \overline{b^\frown o}\, r]$;
2. $\forall a, b \in A\ [\overline{a^\frown o}\, r = \overline{b^\frown o}\, r \rightarrow d(a, b) < 2^{-(r-1)}]$.

Let $B(n, a, r, b, s)$ be a Σ^0_0 formula expressing

$$2^{-(n+1)} < r \leq 2^{-n} \wedge \alpha(\overline{a^\frown o}\, n) > 0 \wedge |\alpha(\overline{a^\frown o}\, n) - 1 - b| < s.$$

Let $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times \mathbb{N} \times \mathbb{Q}^+$ be defined by Σ^0_0 comprehension by $(n, a, r, b, s) \in \Phi \leftrightarrow B(n, a, r, b, s)$ and define $(a, r)\Phi(b, s) :\equiv \exists n\ [(n, a, r, b, s) \in \Phi]$.

It is straightforward (though tedious) to verify that Φ is indeed a code for the continuous function $\phi : [n]^{\mathbb{N}} \rightarrow \mathbb{N}$. $\dashv$

Remark 12. The lemma, as stated, doesn't fit the language of RCA_0 since it refers to the third order object ϕ. However, under the following interpretation it is provable in RCA_0: if α is an associate (i.e., it satisfies condition 1 of the lemma) and Φ is the code presented in the proof of the lemma, then for all $f : \mathbb{N} \rightarrow [n]$ the value on f extracted from α is equal to the value on f extracted from Φ.

LEMMA 13. *Let $A(f)$ be a Σ^0_0 formula, f be a set variable and $\underline{x}$ be a tuple of distinguished number variables in $A(f)$.*

1. RCA_0 *proves* $\forall z\ \exists y\ \forall f, g : \mathbb{N} \rightarrow [n]\ \left[\bar{f} y = \bar{g} y \rightarrow \forall \underline{x} \leq z\ \left(A(f) \leftrightarrow A(g)\right)\right]$.
2. *There exists a Σ^0_0 formula $B(m)$ such that* RCA_0 *proves* $\forall f : \mathbb{N} \rightarrow [n]\ A(f) \leftrightarrow \forall m\ B(m)$. *In particular,* $\forall f : \mathbb{N} \rightarrow [n]\ A(f)$ *is equivalent to a Π^0_1 formula.*
3. *There exists a Σ^0_0 formula $C(m)$ such that* RCA_0 *proves* $\forall f : \mathbb{N} \rightarrow [n]\ [A(f) \leftrightarrow \forall m\ C(\bar{f} m)]$.

PROOF. 1. The proof is by induction on the structure of formulas. If f doesn't occur in an atomic formula A, then the result is obvious. If it occurs, then A must be of the form $t(\underline{x}) \in f$, i.e., (abbreviating $t(\underline{x})$ by t) $\exists i, j \leq t\ [t = (i, j) \wedge f(i) = j]$. We prove by induction on the structure of the number term $t(\geq i)$ that $\exists w\ \forall \underline{x} \leq z\ (t \leq w)$ (for example, if $t \equiv t_1 \cdot t_2$ and by induction hypothesis we have $\exists w_1\ \forall \underline{x} \leq z\ (t_1 \leq w_1)$ and $\exists w_2\ \forall \underline{x} \leq z\ (t_2 \leq w_2)$, then $w := w_1 \cdot w_2$ is such that $\forall \underline{x} \leq z\ (t \leq w)$). Then $y := w + 1$ works.

For the negation $\neg A$ of A we take the same y that by induction hypothesis works for A. For conjunction $A \wedge B$ we take the maximum of the y's working for A and B, and analogously for disjunction, implication and equivalence. For the bounded universal quantifier $\forall i < t\ A(i)$, by induction hypothesis we have $\forall z\ \exists y\ \forall f, g : \mathbb{N} \to [n]\ \big[\bar{f} y = \bar{g} y \to \forall \underline{x}, i \le z\ \big(A(f, i) \leftrightarrow A(g, i)\big)\big]$. Thus taking $z' = \max(z, t)$ we get an y such that for all $f, g : \mathbb{N} \to [n]$, if $\bar{f} y = \bar{g} y$, then for all $\underline{x} \le z$ we have $\forall i < t\ [A(f, i) \leftrightarrow A(g, i)]$. Hence $\forall i < t\ A(f, i) \leftrightarrow \forall i < t\ A(g, i)$. Argue analogously for the bounded existential quantifier.

2. Each occurrence of f in $A(f)$ must be in the form $t \in f$. Let $B(m)$ be the formula obtained from $A(f)$ by replacing each such occurrence $t \in f$ by the Σ^0_0 formula

$$C(m, t, n) :\equiv m \in \mathrm{Seq} \wedge \exists i, j \le t \\ \Big[t = (i, j) \wedge \Big(i < \mathrm{lh}\, m \to j = \min\big(n, m(i)\big)\Big) \wedge (i \ge \mathrm{lh}\, m \to j = 0)\Big],$$

where $[\dots]$ expresses that $j = \min\big(n, (m^\frown o)(i)\big)$. Then $B(m)$ is a Σ^0_0 formula. Let us prove $\forall f : \mathbb{N} \to [n]\ A(f) \leftrightarrow \forall m\ B(m)$.

'$\to$': Take any $m \in \mathrm{Seq}$. We define $f : \mathbb{N} \to [n]$ by $f(i) := \min\big(n, m(i)\big)$ if $i < \mathrm{lh}\, m$ and $f(i) := 0$ if $i \ge \mathrm{lh}\, m$. Then $t \in f \leftrightarrow C(m, t, n)$, so $A(f) \leftrightarrow B(m)$.

'$\leftarrow$': Take any $f : \mathbb{N} \to [n]$. By point 1, let y be such that $\forall g : \mathbb{N} \to [n]\ \big[\bar{f} y = \bar{g} y \to \big(A(f) \leftrightarrow A(g)\big)\big]$. In particular, taking $g = \bar{f} y^\frown o$ we get $A(f) \leftrightarrow A(\bar{f} y^\frown o)$. Let $m := \bar{f} y$. Then $t \in \bar{f} y^\frown o \leftrightarrow C(m, t, n)$, so $B(m) \leftrightarrow A(\bar{f} y^\frown o) \leftrightarrow A(f)$.

3. First we easily show, by induction on the structure of the term $t(i)$, that RCA_0 proves $i \le q \to t(i) \le t(q)$.

Let $A'(a)$ be the formula obtained from $A(f)$ by (adding the assumption $a \in \mathit{Seq}$ and) replacing each instance of $q \in f$ by $\exists i, j \le q\ [q = (i, j) \wedge a(i) = j]$. We show, by induction on the structure of $A(f)$, that there exists a term t such that RCA_0 proves $\forall f : \mathbb{N} \to [n]\ \big[m \ge t \to \big(A(f) \leftrightarrow A'(\bar{f} m)\big)\big]$. For an atomic formula A, if f doesn't occur, then the result is obvious; if f occurs in A, then A is of the form $q \in f$, that is equivalent to $\exists i, j \le q\ [q = (i, j) \wedge f(i) = j]$, so $t := q + 1$ works. For $\neg A$ we take the same t that works for A. For $A \wedge B$, $A \vee B$, $A \to B$ and $A \leftrightarrow B$ we take the sum of the t's working for A and B. For $\forall i < q\ A(i)$ we have by induction hypothesis a term $t'(i)$ that works for $A(i)$, so using the previous paragraph we see that the term $t(q)$ works for $\forall i < q\ A(i)$. Argue analogously for $\exists i < q\ A(i)$.

Finally, using the previous paragraph we easily see that $C(a) :\equiv m \ge t \to A'(a)$ works. $\dashv$

LEMMA 14. RCA_0 *proves the following.*

1. *For all $F \in \mathit{ASNIS}$ and for all infinite sets A, there exists an c such that for all sequences (l_m) weakly convergent to A, we have $\exists i\ \forall j \ge i\ [F(l_j) = c]$.*
2. *For all $F \in \mathit{ASNIS}$ and for all infinite sets A, there exist c and d such that $\forall l \in \mathit{Seq}\ [A_l \cap [d] = A \cap [d] \to F(l) = c]$.*

PROOF. 1. Let $F \in \mathit{ASNIS}$ and let A be a infinite set. First we define c. We define by primitive recursion a sequence (l_m) where each l_m is such that $A_{l_m} = A \cap [m]$. Clearly (l_m) weakly converges to A, so since $F \in \mathit{ASNIS}$ we have $\exists h\ \forall j \ge h\ [F(l_h) = F(l_j)]$. Let $c := F(l_h)$.

Consider an arbitrary sequence (l'_m) weakly converging to A. Let us define i. We have $\exists h'\ \forall j \ge h'\ [F(l'_{h'}) = F(l'_j)]$. By primitive recursion define the sequence (l''_m)

by

$$l''_m := \begin{cases} l_{m/2} & \text{if } m \text{ is even,} \\ l'_{(m-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

Since both (l_m) and (l'_m) weakly converge to A, then also (l''_m) weakly converges to A, and so since $F \in ASNIS$ we have $\exists h'' \ \forall j \geq h'' \ [F(l''_{h''}) = F(l''_j)]$. Let $i := \max(2h, 2h', h'')$.

It remains to prove $\forall j \geq i \ [F(l'_j) = c]$. Since $i \geq h'$, and so $\forall j \geq i \ [F(l'_{h'}) = F(l'_j)]$, it is enough to prove $F(l'_{h'}) = c$. So take an even $j \geq i$. Then $l''_j = l_{j/2}$ and $l''_{j+1} = l'_{j/2}$. Since $j \geq 2h$ and $j \geq 2h'$, we have $j/2 \geq h$ and $j/2 \geq h'$. Thus $F(l''_j) = F(l_{j/2}) = F(l_h) = c$ and $F(l''_{j+1}) = F(l'_{j/2}) = F(l'_{h'})$. But $F(l''_j) = F(l''_{h''}) = F(l''_{j+1})$ since $j \geq h''$. We conclude that $F(l'_{h'}) = c$.

2. Let $F \in ASNIS$ and, by contradiction, let us assume that A is an infinite set such that for all c and d there exists an $l \in \mathrm{Seq}$ such that $A_l \cap [d] = A \cap [d] \wedge F(l) \neq c$. Notice that the latter formula is equivalent to a Σ^0_0 formula. Let c be the number given by the previous point. We define a sequence (l_m) by $l_m := \min l \ \left[l \in \mathrm{Seq} \wedge A_l \cap [m] = A \cap [m] \wedge F(l) \neq c\right]$ so that $\forall m \ \left[A_{l_m} \cap [m] = A \cap [m] \wedge F(l_m) \neq c\right]$. Then (l_m) weakly converges to A while $\forall m \ [F(l_m) \neq c]$, contradicting point 1. $\dashv$

§4. Counterexample to $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_1$.

In this section we give a counterexample to FIPP_1. In particular, FIPP_1 is not equivalent to the true IPP.

THEOREM 15. RCA_0 *refutes* FIPP_1.

PROOF. We define $F \in AS$, n and a sequence of colorings $f_k : [k] \to [n]$.

We take $n := 1$. Let us write $\dot{m}$ to mean that the number m was given the color 0 and $\ddot{m}$ to mean that it was given the color 1.

Let $\mathbb{O} := \{1, 3, 5, \dots\}$ be the set of the odd natural numbers and $\mathbb{E} := \{0, 2, 4, \dots\}$ be the set of the even natural numbers. Let us make the (non-standard) convention $\min \emptyset := 0$. We define

$$\begin{aligned} F &: \mathbb{N} \to \mathbb{N} \\ l &\mapsto \begin{cases} \min(A_l \cap \mathbb{O}) + \min(A_l \cap \mathbb{E}) + 2 & \text{if } l \in \mathrm{Seq}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let us prove $F \in AS$. Clearly F is extensional. Consider a nested sequence with union (l_m). Then we have a nested sequence $A_{l_0} \cap \mathbb{O} \subseteq A_{l_1} \cap \mathbb{O} \subseteq A_{l_2} \cap \mathbb{O} \subseteq \cdots$. So eventually the numbers $\min(A_{l_m} \cap \mathbb{O})$ will become constant. In an analogous way, eventually the numbers $\min(A_{l_m} \cap \mathbb{E})$ will become constant. So $F(l_m)$ will eventually become constant.

We color each set $[k]$ in the following way:

1. the odd numbers are given the color 0 and the even numbers are given the color 1;
2. except for the last two numbers $k - 1$ and k, where the odd number is given the color 1 and the even number is given the color 0.

In the cases of $k = 0$ and $k = 1$, i.e., in the cases of the sets [0] and [1], we consider that 0 and 1 are the last two numbers, so we apply the second rule to them.

Let us write the colored sets $[0], [1], [2], \dots$ and, on the left of each set, the value of F over the 0- and 1-color classes:

$$\begin{array}{cc c ccccccccc}
\dot{2} & \ddot{2} & \quad & \{ & \dot{0} & & & & & & & \} \\
\dot{2} & \ddot{3} & & \{ & \dot{0}, & \ddot{1} & & & & & & \} \\
\dot{4} & \ddot{3} & & \{ & \ddot{0}, & \ddot{1}, & \dot{2} & & & & & \} \\
\dot{5} & \ddot{5} & & \{ & \ddot{0}, & \dot{1}, & \dot{2}, & \ddot{3} & & & & \} \\
\dot{7} & \ddot{5} & & \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \ddot{3}, & \dot{4} & & & \} \\
\dot{7} & \ddot{7} & & \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \dot{3}, & \dot{4}, & \ddot{5} & & \} \\
\dot{9} & \ddot{7} & & \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \dot{3}, & \ddot{4}, & \ddot{5}, & \dot{6} & \} \\
\dot{9} & \ddot{9} & & \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \dot{3}, & \ddot{4}, & \dot{5}, & \dot{6}, & \ddot{7} \quad \} \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ddots
\end{array}$$

Notice that the cardinality of any f_k-color class is less than or equal to $|[k]| = k+1$ which in turn is less than or equal to the value of F over (a code for) that color class. So we have $\forall k\ \forall l \in \mathrm{Seq}\ \forall c \in [1]\ [A_l = (f_k)^{-1}(c) \to |A_l| \leq F(l)]$, which falsifies FIPP_1. $\dashv$

§5. **Proofs of** $\mathsf{FIPP}_2 \to \mathsf{IPP}$ **and** $\mathsf{FIPP}_3 \to \mathsf{IPP}$**.** In this section we give proofs in RCA_0 of the implications $\mathsf{FIPP}_2 \to \mathsf{IPP}$ and $\mathsf{FIPP}_3 \to \mathsf{IPP}$. Latter we study the reverse implications.

THEOREM 16. 1. RCA_0 *proves* $\mathsf{FIPP}_2 \to \mathsf{IPP}$.
2. RCA_0 *proves* $\mathsf{FIPP}_3 \to \mathsf{IPP}$.

PROOF. 1. FIPP_2 implies

$$\forall n\ \forall F \in AS\ \forall f : \mathbb{N} \to [n]\ \underbrace{\exists k\ \exists l \in \mathrm{Seq}\ \left[A_l \subseteq [k] \wedge |A_l| > F(l) \wedge f|_{A_l} \text{ constant}\right]}_{\equiv: B(F,f)}. \tag{1}$$

Assume $\neg\mathsf{IPP}$. Then there exists n and $f : \mathbb{N} \to [n]$ such that

$$\forall A\ \exists m\ (|A| \leq m \vee f|_A \text{ not constant}). \tag{2}$$

If A is given by a code l, then by point 2 of Lemma 10 the formula "$|A_l| \leq m \vee f|_{A_l}$ not constant" is equivalent to some Σ^0_0 formula (using that the image of f is in $[n]$), thus by primitive recursion we can define the function

$$F : \mathbb{N} \to \mathbb{N}$$
$$l \mapsto \begin{cases} \min m\ (|A_l| \leq m \vee f|_{A_l} \text{ not constant}) & \text{if } l \in \mathrm{Seq}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove $F \in AS$. Take any nested sequence (l_m) with union $A := \bigcup_m A_{l_m}$. If A is finite, then $\exists i\ \forall j \geq i\ (A_{l_i} = A_{l_j})$ (using $B\Sigma^0_1$ which is derivable in RCA_0). Hence F eventually stabilizes over (l_m); if A is infinite, then by (2) we have $\exists x, y \in A\ [x \neq y \wedge f(x) \neq f(y)]$ and $\exists i\ (x, y \in A_{l_i})$ which yields $\forall j \geq i\ (x, y \in A_{l_j})$. Thus $\forall j \geq i\ [F(l_i) = 0 = F(l_j)]$. This concludes the proof of $F \in AS$.

So we have found n, $F \in AS$ and $f : \mathbb{N} \to [n]$ such that $\neg B(F, f)$, contradicting (1).

2. The proof is analogous to the proof of point 1, except for the argument that $F \in ASNIS$. Let us prove $F \in ASNIS$. Take any sequence (l_m) weakly convergent to an infinite set A. By (2) we get $\exists x, y \in A\ [x \neq y \wedge f(x) \neq f(y)]$. Let $z := \max(x, y)$. Since (l_m) is weakly convergent to A we have $\exists i\ \forall j \geq i\ (A_{l_j} \cap [z] = A \cap [z])$. But $x, y \in A \cap [z]$ and so $\forall j \geq i\ (x, y \in A_{l_j})$, Thus $\forall j \geq i\ [F(l_i) = 0 = F(l_j)]$. ⊣

Together with Proposition 9 we get

Corollary 17. $\mathsf{WKL_0}$ *does not prove* $\mathsf{FIPP_2}$. *Also* $\mathsf{WKL_0}$ *does not prove* $\mathsf{FIPP_3}$.

§6. Continuous uniform boundedness.

In Definition 18 we will define a predicate $\mathrm{cont}(A)$ that, in particular, expresses the continuity of the functional

$$\phi : [n]^{\mathbb{N}} \to \mathbb{N}, \quad f \mapsto \min x\ [A(f, x)].$$

Then we define a compactness principle CUB that roughly speaking expresses that if ϕ is continuous and total, then it is bounded on the compact $[n]^{\mathbb{N}}$. There is also a variant CUB' that emphasizes that the conclusion only talks about an initial segment of f. However, it turns out that for the instances of CUB and CUB' in which we are interested, the two principles are equivalent, as we show in Proposition 21. In Proposition 19 we show that Σ^0_0-CUB and Σ^0_0-CUB' can be upgraded to Σ^0_1-CUB and Σ^0_1-CUB'. In Theorem 22 we calibrate the strength of Φ-CUB in terms of the "big five" subsystems of second order arithmetic.

Definition 18. The following definition is made within $\mathsf{RCA_0}$.

1. Let $A(f, \underline{x})$ be a formula with (among others) a distinguished set variable f and a distinguished tuple of number variables $\underline{x}$. We say that A is *continuous* (w.r.t. $f, \underline{x}$), and write $\mathrm{cont}(A)$ (more precisely $\mathrm{cont}(A(f, \underline{x}))$), if and only if

$$\forall f : \mathbb{N} \to [n]\ \forall z\ \exists y\ \forall g : \mathbb{N} \to [n]\ \big[\bar{f} y = \bar{g} y \to \forall \underline{x} \leq z \big(A(f, \underline{x}) \leftrightarrow A(g, \underline{x})\big)\big],$$

where the variable n doesn't occur free in A.

2. The *continuous uniform boundedness principle* CUB is the schema

$$\forall n\ [\ \big(\mathrm{cont}(A) \wedge \forall f : \mathbb{N} \to [n]\ \exists \underline{x}\ A(f, \underline{x})\big) \to \exists z\ \forall f : \mathbb{N} \to [n]\ \exists \underline{x} \leq z\ A(f, \underline{x})].$$

We denote by Γ-CUB the restriction of CUB to formulas $A(f, \underline{x})$ in Γ.

3. The *variant continuous uniform boundedness principle* CUB' is the schema

$$\forall n\ \big[\ \big(\mathrm{cont}(A) \wedge \forall f : \mathbb{N} \to [n]\ \exists \underline{x}\ A(f, \underline{x})\big) \to \\ \exists z\ \forall f : \mathbb{N} \to [n]\ \exists \underline{x} \leq z\ \big(A(f, \underline{x}) \wedge \forall g : \mathbb{N} \to [n]\ \big(\bar{f} z = \bar{g} z \to A(g, \underline{x})\big)\big)\big].$$

We denote by Γ-CUB' the restriction of CUB' to formulas $A(f, \underline{x})$ in Γ.

Proposition 19. $\mathsf{RCA_0}$ *proves the following.*

1. Σ^0_0-$\mathsf{CUB} \leftrightarrow \Sigma^0_1$-$\mathsf{CUB}$.
2. Σ^0_0-$\mathsf{CUB}' \leftrightarrow \Sigma^0_1$-$\mathsf{CUB}'$.

Proof. 1. The right-to-left implication is trivial. Let us consider the left-to-right implication. Consider any Σ^0_1 formula $\exists \underline{w}\ A$ where A is a Σ^0_0 formula. We assume the part $\forall f : \mathbb{N} \to [n]\ \exists \underline{x}\ [\exists \underline{w}\ A(f, \underline{x}, \underline{w})]$ of the assumption of Σ^0_1-CUB. By point 1 of Lemma 13 we have $\mathrm{cont}(A)$ (w.r.t. $f, \underline{x}, \underline{w}$). By Σ^0_0-CUB applied to A we get $\exists z\ \forall f : \mathbb{N} \to [n]\ \exists \underline{x}, \underline{w} \leq z\ A(f, \underline{x}, \underline{w})$. From here we get the conclusion $\exists z\ \forall f : \mathbb{N} \to [n]\ \exists \underline{x} \leq z\ [\exists \underline{w}\ A(f, \underline{x}, \underline{w})]$ of Σ^0_1-CUB.

2. The proof is analogous to the proof of the previous point. ⊣

Remark 20. By point 1 of Lemma 13, $\mathrm{cont}(A)$ is always satisfied for Σ^0_0-formulas A and so can be dropped in Σ^0_0-CUB. The proof of point 1 in Proposition 19 above shows that dropping $\mathrm{cont}(A)$ also in Σ^0_1-CUB results in an equivalent principle. For Π^0_1-CUB this is no longer the case (see the comments at the end of this paper).

PROPOSITION 21. RCA_0 *proves the following.*
1. Σ^0_0-$\mathsf{CUB} \leftrightarrow \Sigma^0_0$-$\mathsf{CUB}'$.
2. Π^0_1-$\mathsf{CUB} \leftrightarrow \Pi^0_1$-$\mathsf{CUB}'$.
3. $\mathsf{CUB} \leftrightarrow \mathsf{CUB}'$.

PROOF. 1. The right-to-left implication is trivial. Let us prove the left-to-right implication. We assume the premise $\mathrm{cont}(A) \wedge \forall f : \mathbb{N} \to [n]\, \exists \underline{x}\, A(f, \underline{x})$ of Σ^0_0-CUB', where $A(f, \underline{x})$ is a Σ^0_0 formula. Then by Σ^0_0-CUB we have

$$\exists z'\, \forall f : \mathbb{N} \to [n]\, \exists \underline{x} \leq z'\, A(f, \underline{x}, y). \tag{3}$$

By point 1 of Lemma 13 we have

$$\exists y\, \forall f, g : \mathbb{N} \to [n]\, \forall \underline{x} \leq z' \left[\bar{f} y = \bar{g} y \to \big(A(f, \underline{x}) \leftrightarrow A(g, \underline{x})\big)\right]. \tag{4}$$

Let $z := \max(y, z')$. Then from (3) and (4) we get the conclusion of Σ^0_0-CUB'.

2. The the right-to-left implication is trivial. Let us prove the left-to-right implication. We assume the premise $\mathrm{cont}(\forall \underline{w}\, A(f, \underline{x}, \underline{w})) \wedge \forall f : \mathbb{N} \to [n]\, \exists \underline{x}\, [\forall \underline{w}\, A(f, \underline{x}, \underline{w})]$ of Π^0_1-CUB', where $\forall \underline{w}\, A(f, \underline{x}, \underline{w})$ is a Π^0_1 formula and $A(f, \underline{x}, \underline{w})$ is a Σ^0_0 formula. Then

$$\forall f : \mathbb{N} \to [n]\, \exists \underline{x}\, \exists y\, \underbrace{\left[\forall \underline{w}\, A(f, \underline{x}, \underline{w}) \wedge \forall g : \mathbb{N} \to [n]\, \big(\bar{f} y = \bar{g} y \to \forall \underline{w}\, A(g, \underline{x}, \underline{w})\big)\right]}_{\equiv: B}.$$

Note that $\bar{f} y = \bar{g} y$ is equivalent to

$$\forall i < y\, \forall z \leq n\, [(i, z) \in f \leftrightarrow (i, z) \in g] \in \Sigma^0_0.$$

Moving the quantifiers $\forall \underline{w}$ and $\forall g : \mathbb{N} \to [n]$ in B to the front of B we get an equivalent formula of the form $\forall \underline{w}\, \forall g : \mathbb{N} \to [n]\, C$ were C is a Σ^0_0 formula. By point 2 of Lemma 13, $\forall g : \mathbb{N} \to [n]\, C$ is equivalent to a Π^0_1 formula, so B is equivalent to a Π^0_1 formula. Therefore we can apply Π^0_1-CUB to B (note that $\mathrm{cont}(B)$ w.r.t. $f, \underline{x}, y$ since $\mathrm{cont}(\forall \underline{w}\, A(f, \underline{x}, \underline{w}))$) getting

$$\exists z\, \forall f : \mathbb{N} \to [n]\, \exists \underline{x}, y \leq z\, \big[\forall \underline{w}\, A(f, \underline{x}, \underline{w}) \wedge \forall g : \mathbb{N} \to [n]\, \big(\bar{f} y = \bar{g} y \to \forall \underline{w}\, A(g, \underline{x}, \underline{w})\big)\big].$$

Now replacing y by z in $\bar{f} y = \bar{g} y$ we get the conclusion of Π^0_1-CUB'.

3. The proof is analogous to the proof of point 2, disregarding the considerations about the complexity of B. ⊣

THEOREM 22. 1. RCA_0 *proves* Σ^0_0-$\mathsf{CUB} \leftrightarrow \mathsf{WKL}_0$.
2. RCA_0 *proves* Π^0_1-$\mathsf{CUB} \leftrightarrow \mathsf{ACA}_0$.
3. RCA (*not* RCA_0) *proves* $\mathsf{CUB} \leftrightarrow \mathsf{Z}_2$.

PROOF. 1. '$\to$': We assume Σ^0_0-CUB and, by contradiction, $\neg\mathsf{WKL}_0$. Then we have an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ with no infinite path, i.e., $\forall f : \mathbb{N} \to [1]$ $\exists x\, (\bar{f} x \notin T)$ where the formula $\bar{f} x \notin T$ is Δ^0_1 and hence Σ^0_1. By Σ^0_1-CUB and so (using Proposition 19) also by Σ^0_0-CUB we have $\exists z\, \forall f : \mathbb{N} \to [1]\, \exists x \leq z\, (\bar{f} x \notin T)$.

This means that every branch in T has length bounded by $z-1$, so the binary tree T is finite, contradicting the fact that it is infinite.

'$\leftarrow$': First we show that RCA_0 proves

$$\mathsf{WKL}_0 \to \big[\big(\forall k\ \exists f : \mathbb{N} \to [n]\ \forall \underline{x} \le k\ A(f, \underline{x})\big) \to \big(\exists f : \mathbb{N} \to [n]\ \forall \underline{x}\ A(f, \underline{x})\big)\big],$$

where A is Σ^0_0. We assume WKL_0 and $\forall k\ \exists f : \mathbb{N} \to [n]\ \forall \underline{x} \le k\ A(f, \underline{x})$. By point 3 of Lemma 13 we can write $A(f, \underline{x})$ as $\forall m\ B(\bar{f}m, \underline{x})$ where B is Σ^0_0. By Σ^0_0 comprehension we define the bounded tree $T := \{\tau \in [n]^{<\mathbb{N}} : \forall \underline{x}, m \le \mathrm{lh}(\tau)\ B(\bar{\tau}m, \underline{x})\}$. We have $\forall p\ \exists \tau \in T\ [\mathrm{lh}(\tau) = p]$: taking $k = p$ in our assumption we get an $f : \mathbb{N} \to [n]$ such that $\forall \underline{x}, m \le p\ B(\bar{f}m, \underline{x})$ where $\bar{f}m = \bar{\tau}m$ for $\tau := \bar{f}p \in T$ with length $\mathrm{lh}(\tau) = p$. So T is infinite, thus by WKL_0 (actually by bounded König's lemma that is equivalent to WKL_0 over RCA_0 as proved in Lemma IV.1.4 in [9]) there is a infinite path $f : \mathbb{N} \to [n]$ through T. Then $\forall \underline{x}\ A(f, \underline{x})$, i.e., $\forall \underline{x}, m\ B(\bar{f}m, \underline{x})$: for $p := \max(\underline{x}, m)$ we have $\tau := \bar{f}p \in T$, i.e., $\forall \underline{x'}, m' \le p\ B(\bar{\tau}m', \underline{x'})$ where $\bar{\tau}m' = \bar{f}m'$, and so taking $\underline{x'} = \underline{x}$ and $m' = m$ we get $B(\bar{f}m, \underline{x})$.

Finally, we show that RCA_0 proves $\mathsf{WKL}_0 \to \Sigma^0_0$-$\mathsf{CUB}$ taking the contraposition of the inner implication proved in the previous paragraph.

2. It is enough to show that RCA_0 proves that Π^0_1-CUB is equivalent to Π^0_1 comprehension, since Π^0_1 comprehension is equivalent to Σ^0_1 comprehension and in turn, as proved in Lemma III.1.3 in [9], Σ^0_1 comprehension is equivalent over RCA_0 to ACA_0.

'$\rightarrow$': Let us prove that Π^0_1-CUB implies Π^0_1 comprehension. Consider any Π^0_1 formula $\forall \underline{m}\ A(x, \underline{m})$ where $A(x, \underline{m})$ is a Σ^0_0 formula. By contradiction, we assume $\neg\exists X\, \forall x\, [x \in X \leftrightarrow \forall \underline{m}\, A(x, \underline{m})]$, i.e., $\neg\exists f : \mathbb{N} \to [1]\, \forall x\, [f(x) = 0 \leftrightarrow \forall \underline{m}\, A(x, \underline{m})]$. Then

$$\forall f : \mathbb{N} \to [1]\, \exists x, \underline{m}\ \underbrace{\forall \underline{m'}\ \neg\big[\big(f(x) = 0 \to A(x, \underline{m})\big) \wedge \big(A(x, \underline{m'}) \to f(x) = 0\big)\big]}_{\equiv: B(f, x, \underline{m})}.$$

By Π^0_1-CUB applied to the (continuous) Π^0_1 formula B we get a z such that

$$\forall f : \mathbb{N} \to [1]\, \exists x \le z\ \neg\big[\big(f(x) = 0 \to \forall \underline{m} \le z\ A(x, \underline{m})\big) \wedge \\ \big(\forall \underline{m'}\ A(x, \underline{m'}) \to f(x) = 0\big)\big].$$

But that is contradicted by the function

$$f : \mathbb{N} \to [1]$$
$$x \mapsto \begin{cases} 0 & \text{if } \forall \underline{m} \le z\ A(x, \underline{m}), \\ 1 & \text{otherwise} \end{cases}$$

which is definable by Σ^0_0 comprehension.

'$\leftarrow$:' Now let us see that Π^0_1 comprehension implies Π^0_1-CUB. We assume Π^0_1 comprehension, and, therefore, we have ACA_0. We assume the premise $\mathrm{cont}(A) \wedge \forall f : \mathbb{N} \to [n]\ \exists \underline{x}\ A(f, \underline{x})$ of Π^0_1-CUB, where $A(f, \underline{x})$ is a Π^0_1 formula, and we want to prove the conclusion $\exists z\ \forall f : \mathbb{N} \to [n]\ \exists \underline{x} \le z\ A(f, \underline{x})$ of Π^0_1-CUB. Take any $f : \mathbb{N} \to [n]$. In ACA_0 there exists $\min z\, [\exists \underline{x} \le z\ A(f, \underline{x})]$. Consider the functional

$$\phi : [n]^{\mathbb{N}} \to \mathbb{N}$$
$$f \mapsto \min z\, [\exists \underline{x} \le z\ A(f, \underline{x})].$$

This functional cannot directly be formed in ACA_0 as it is a 3rd order object. However, we will show now that it has a (2nd order) code as a continuous function in the sense of [9]. Let

$$\begin{aligned} l \in \mathrm{Seq}_{\leq n} &:\equiv l \in \mathrm{Seq} \wedge l^\frown o : \mathbb{N} \to [n], \\ B(l) &:\equiv l \in \mathrm{Seq}_{\leq n} \wedge \forall l' \in \mathrm{Seq}_{\leq n} \\ &\quad \big[l \subseteq l' \to \min z\, [\exists \underline{x} \leq z\, A(l^\frown o, \underline{x})] = \min z\, [\exists \underline{x} \leq z\, A(l'^\frown o, \underline{x})]\big]. \end{aligned}$$

In ACA_0 there exists the following function α which – as we will argue now – is an associate for ϕ:

$$\alpha(l) := \begin{cases} \min z\, [\exists \underline{x} \leq z\, A(l^\frown o, \underline{x})] + 1 & \text{if } B(l), \\ 0 & \text{otherwise.} \end{cases}$$

Take any $\beta : \mathbb{N} \to [n]$. By $\mathrm{cont}(A)$ there exists a y such that

$$\forall g : \mathbb{N} \to [n]\ \big[\bar{\beta} y = \bar{g} y \to \forall \underline{x} \leq \min z\, [\exists \underline{\tilde{x}} \leq z\, A(\beta, \underline{\tilde{x}})]\ \big(A(\beta, \underline{x}) \leftrightarrow A(g, \underline{x})\big)\big],$$

thus

$$\forall g : \mathbb{N} \to [n]\ \big[\bar{\beta} y = \bar{g} y \to \min z\, [\exists \underline{x} \leq z\, A(\beta, \underline{x})] = \min z\, [\exists \underline{x} \leq z\, A(g, \underline{x})]\big]. \quad (5)$$

(a) First we prove that there exists an m such that $\alpha(\bar{\beta} m) > 0$. Let $m := y$. We have $B(\bar{\beta} m)$: for all $l' \in \mathrm{Seq}_{\leq n}$ such that $\bar{\beta} m \subseteq l'$, taking $g = \bar{\beta} m^\frown o$ and $g = l'^\frown o$ in (5) we get, respectively,

$$\begin{aligned} \min z\, [\exists \underline{x} \leq z\, A(\beta, \underline{x})] &= \min z\, [\exists \underline{x} \leq z\, A(\bar{\beta} m^\frown o, \underline{x})], \\ \min z\, [\exists \underline{x} \leq z\, A(\beta, \underline{x})] &= \min z\, [\exists \underline{x} \leq z\, A(l'^\frown o, \underline{x})]. \end{aligned}$$

Thus $\min z\, [\exists \underline{x} \leq z\, A(\bar{\beta} m^\frown o, \underline{x})] = \min z\, [\exists \underline{x} \leq z\, A(l'^\frown o, \underline{x})]$. Since we have $B(\bar{\beta} m)$, then by definition of α we have $\alpha(\bar{\beta} m) > 0$.

(b) Now we take the least m such that $\alpha(\bar{\beta} m) > 0$ and we prove $\alpha(\bar{\beta} m) = \phi(\beta) + 1$. Since $\alpha(\bar{\beta} m) > 0$ we have $B(\bar{\beta} m)$. Let $w := \max(m, y)$. By $B(\bar{\beta} m)$ and taking $g = \bar{\beta} w^\frown o$ in (5) we get, respectively,

$$\begin{aligned} \overbrace{\min z\, [\exists \underline{x} \leq z\, A(\bar{\beta} m^\frown o, \underline{x})]}^{=\alpha(\bar{\beta} m) - 1} &= \min z\, [\exists \underline{x} \leq z\, A(\bar{\beta} w^\frown o, \underline{x})], \\ \underbrace{\min z\, [\exists \underline{x} \leq z\, A(\beta, \underline{x})]}_{=\phi(\beta)} &= \min z\, [\exists \underline{x} \leq z\, A(\bar{\beta} w^\frown o, \underline{x})]. \end{aligned}$$

Thus $\alpha(\bar{\beta} m) = \phi(\beta) + 1$.

This concludes the proof that α is an associate for ϕ. Thus by Lemma 11, ϕ has a code as continuous function. Since $[n]^{\mathbb{N}}$ is (provably already in RCA_0) a compact metric space (see [9] (examples III.2.6)) it follows from [1] (Theorem 4.1) that (provably in WKL_0 and so a-fortiori in ACA_0) ϕ has an upper bound z. Then $\forall f : \mathbb{N} \to [n]\, \exists \underline{x} \leq \phi(f) \leq z\, A(f, \underline{x})$.

3. We prove that Z_2 implies CUB essentially in the same way that we proved in the previous point that Π^0_1 comprehension implies Π^0_1-CUB. To see that, conversely, CUB implies (relative to RCA) Z_2 it is enough to show that CUB implies the comprehension axiom for arbitrary formulas A. By induction on z we prove

$\forall z\ \exists m \in \mathrm{Seq}_{\leq 1}\ \left[\mathrm{lh}\, m = z+1 \wedge \forall x \leq z\ \left(m(x) = 0 \leftrightarrow A(x)\right)\right]$. By contradiction assume $\neg\exists X\ \forall x\ [x \in X \leftrightarrow A(x)]$, that is, $\forall f : \mathbb{N} \to [1]\ \exists x\ B(f,x)$ where $B(f,x) :\equiv \neg[f(x) = 0 \leftrightarrow A(x)]$. Clearly $\mathrm{cont}(B)$, so applying CUB to B we get $\exists z\ \forall f : \mathbb{N} \to [1]\ \exists x \leq z\ B(f,x)$. But this is contradicted by $f = m^\frown o$. ⊣

Remark 23. As the proof above shows, the strength of the various CUB-principles considered does not depend on whether they are formulated with general n or just with $n = 1$ (this can also be seen directly using the construction on page 220 in [12]). Note, however, that IPP restricted to $n = 1$ or any fixed n is much weaker (and essentially provable in pure logic) than IPP.

§7. Proofs of $\mathsf{IPP} \to \mathsf{FIPP}_2$ and $\mathsf{IPP} \to \mathsf{FIPP}_3$ using continuous uniform boundedness.

In the previous section we calibrated the strength of Σ^0_0-CUB and Π^0_1-CUB in terms the "big five". In this section we give upper bounds on the strength of the implications $\mathsf{IPP} \to \mathsf{FIPP}_2$ and $\mathsf{IPP} \to \mathsf{FIPP}_3$.

Theorem 24. 1. $\mathsf{RCA}_0 + \Sigma^0_0$-$\mathsf{CUB}$ *proves* $\mathsf{IPP} \to \mathsf{FIPP}_2$.
2. $\mathsf{RCA}_0 + \Pi^0_1$-CUB *proves* $\mathsf{IPP} \to \mathsf{FIPP}_3$.

Proof. 1. Take any n and $F \in AS$. Let

$$B(f,k) :\equiv \exists l \in \mathrm{Seq}\ \left[A_l \subseteq [k] \wedge |A_l| > F(l) \wedge f|_{A_l} \text{ constant}\right].$$

Let us prove $\forall f : \mathbb{N} \to [n]\ \exists k\ B(f,k)$. Take any $f : \mathbb{N} \to [n]$. By IPP there exists an infinite color class $f^{-1}(c)$ with $c \in [n]$. By primitive recursion in f define a sequence (l_m) where each $l_m \in \mathrm{Seq}$ is such that $A_{l_m} = f^{-1}(c) \cap [m]$. Then (l_m) is a nested sequence with union $f^{-1}(c)$, so there exists a k' such that $\forall m \geq k'\ [F(l_m) = F(l_{k'})]$. Since $f^{-1}(c)$ is infinite, by points 3 and 4 of Lemma 10 there exists a k'' such that $\forall m \geq k''\ [|A_{l_m}| > F(l_{k'})]$. Let $k := \max(k', k'')$. Then we have $B(f,k)$. This finishes the proof of $\forall f : \mathbb{N} \to [n]\ \exists k\ B(f,k)$.

Notice that $A_l \subseteq [k]$ and "$f|_{A_l}$ constant" are equivalent to some bounded formulas and $|A_l| > F(l)$ is equivalent to the Σ^0_0 formula

$$\exists j \leq l\ \exists i < j\ [|A_l| = j \wedge (l,i) \in F].$$

Thus $B(f,k)$ is equivalent to a Σ^0_1 formula. Also notice that we have $\mathrm{cont}(B)$.

By Σ^0_1-CUB we get $\exists k\ \forall f : \mathbb{N} \to [n]\ \exists k''' \leq k\ B(f,k''')$. Since $B(f,k)$ is monotone in k, i.e., $k''' \leq k \wedge B(f,k''') \to B(f,k)$ we get $\exists k\ \forall f : \mathbb{N} \to [n]\ B(f,k)$. Now since f is only applied in $B(f,k)$ to arguments in $[k]$, we can consider only functions f with domain $[k]$: $\exists k\ \forall f : [k] \to [n]\ B(f,k)$.

2. Take any n and $F \in ASNIS$. Let

$$B(f,k) :\equiv \exists c \in [n]\ \forall l \in \mathrm{Seq}\ \left[A_l \cap [k] = f^{-1}(c) \cap [k] \to |A_l| > F(l)\right].$$

Let us prove $\forall f : \mathbb{N} \to [n]\ \exists k\ B(f,k)$. Take any $f : \mathbb{N} \to [n]$. As in the proof of the previous point we have an infinite color class $f^{-1}(c)$ and a sequence (l_m) where each $l_m \in \mathrm{Seq}$ is such that $A_{l_m} = f^{-1}(c) \cap [m]$. Then (l_m) weakly converges to $f^{-1}(c)$, so there exists a k' such that

$$\forall m \geq k'\ [F(l_m) = F(l_{k'})]. \tag{6}$$

Since $f^{-1}(c)$ is infinite, then by points 3 and 4 of Lemma 10 there exists a k'' such that

$$\forall m \geq k'' \, [|f^{-1}(c) \cap [m]| > F(l_{k'})]. \tag{7}$$

By Lemma 14.2 there exist c' and k''' such that

$$\forall l \in \text{Seq} \left[A_l \cap [k'''] = f^{-1}(c) \cap [k'''] \to F(l) = c'\right]. \tag{8}$$

Let $k := \max(k', k'', k''')$. Taking $m = k$ in (6) and $l = l_k$ in (8) we get $c' = F(l_{k'})$. For all $l \in \text{Seq}$, if $A_l \cap [k] = f^{-1}(c) \cap [k]$, then $A_l \cap [k'''] = f^{-1}(c) \cap [k''']$ and by point 4 of Lemma 10 we have $|A_l| \geq |f^{-1}(c) \cap [k]|$, thus by (7) and (8) we get $|A_l| > F(l)$. This finishes the proof of $\forall f : \mathbb{N} \to [n] \, \exists k \, B(f, k)$.

Notice that $A_l \cap [k] = f^{-1}(c) \cap [k]$ is equivalent to a Σ^0_0 formula and by point 2 of Lemma 10 $|A_l| > F(l)$ is (as shown above) equivalent to a Σ^0_0 formula, so (using $B\Sigma^0_1$) B is equivalent to a Π^0_1 formula. Also notice that we have $\text{cont}(B)$ since the only occurrence of f in B is $f^{-1}(c) \cap [k]$, i.e., $(f|_{[k]})^{-1}(c)$.

Now, analogously to the proof of the previous point, apply Π^0_1-CUB, use the monotonicity of $B(f, k)$ on k, and notice that we can restrict the functions f to $[k]$. Finally, taking $l = l_k$ (so $A_{l_k} = f^{-1}(c)$ where $f : [k] \to [n]$) we get the FIPP_3. ⊣

COROLLARY 25. 1. WKL_0 *proves* IPP $\leftrightarrow$ FIPP_2.
2. ACA_0 *proves* IPP $\leftrightarrow$ FIPP_3.

While the first equivalence shows that FIPP_2 is a nontrivial finitization of IPP as neither principle is derivable in WKL_0, the second equivalence does not establish this for FIPP_3 since ACA_0 not only proves IPP (and hence FIPP_3) but even much stronger principles (e.g., Ramsey's Theorem RT(k) for every fixed k, see [9], or, on the arithmetical side, $B\Sigma^0_\infty$). So while the fact that RCA_0 suffices to prove $\mathsf{FIPP}_3 \to$ IPP shows that FIPP_3 is strong enough to count as a "finitization" of IPP, the fact that for the other direction we only have proofs using ACA_0 leaves open the possibility that FIPP_3 may be too strong to be a faithful finitization of IPP.

§8. Historical comments on CUB. Without the continuity assumption $\text{cont}(A)$, principles of the form CUB feature prominently in intuitionistic mathematics under the label of "fan principles". In fact, in intuitionistic analysis it is common to assume (classically inconsistent) continuity principles that, in particular, imply $\text{cont}(A)$ (see [12]). In our language context of 2nd order arithmetic, Σ^0_0-formulas A automatically satisfy $\text{cont}(A)$ and so in Σ^0_0-CUB and (by its reduction to Σ^0_0-CUB) even in Σ^0_1-CUB one can drop the assumption $\text{cont}(A)$. However, in contexts formulated in the language in all finite types over $\mathbb{N}$, the corresponding version without $\text{cont}(A)$, called Σ^0_1-UB, is not valid in the full type structure over $\mathbb{N}$ but satisfies very useful conservation results. Σ^0_1-UB was first introduced in [4] and is studied in detail in [8] (for a systematic proof-theoretic treatment of even more general forms of uniform boundedness by a specially designed so-called bounded functional interpretation see [2]). Recently in [7, 8], Σ^0_1-UB was generalized to a principle $\exists$-UB^X dealing with uniformities in the absence of compactness for abstract bounded metric and hyperbolic spaces. Again, while not valid in the intended model, the principle satisfies strong conservation theorems and so can be used safely for proofs of large classes of statements.

With classical logic alone (essentially), however, even Π^0_1-CUB becomes inconsistent (and so in particular over RCA_0 and much weaker systems) if the assumption $\mathrm{cont}(A)$ is dropped. E.g. just consider the logically valid statement

$$\forall f : \mathbb{N} \to [1]\, \exists x \in \mathbb{N}\, \underbrace{\forall y \in \mathbb{N}\, [f(y) = 0 \to f(x) = 0]}_{\equiv: A(f,x) \in \Pi^0_1}.$$

Then CUB without the continuity assumption $\mathrm{cont}(A)$ (which does not hold here) would imply that

$$\exists z\, \forall f : \mathbb{N} \to [1]\, \big[\exists y \in \mathbb{N}\, \big(f(y) = 0\big) \to \exists x \leq z\, \big(f(x) = 0\big)\big]$$

which obviously is wrong. A syntactic condition that guarantees $\mathrm{cont}(A)$ to hold is that $A(f,x)$ can be written as $\tilde{A}\big(\bar{f}(t(x)), x\big)$ for some number term t (possibly with further number parameters of A), where $\tilde{A}(z,x) \in \Pi^0_1$ does not contain f. This is the case in the use of Π^0_1-CUB in the proof of Theorem 24.2 (with $t(k,c) := k+1$). In fact, (a version of) such a form (denoted by Π^0_1-UB$\restriction$) of CUB is considered in [5], where it is shown to imply the Bolzano-Weierstraß property of $[0,1]^d$ (over an extremely weak base system). Moreover, the proof of Theorem 22.2 immediately shows that Π^0_1-UB$\restriction$ still implies Π^0_1 comprehension.

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