Proof interpretations: theoretical and practical aspects

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Abstract/Zusammenfassung

Abstract

We study theoretical and practical aspects of proof theoretic tools called proof interpretations.

Theoretical contributions

Completeness and ω-rule Using a proof interpretation, we prove that Peano arithmetic with the ω-rule is a complete theory.

Proof interpretations with truth Proof interpretations without truth give information about the interpreted formula, not the original formula. We give three heuristics on hardwiring truth and apply them to several proof interpretations.

Copies of classical logic in intuitionistic logic The usual proof interpretations embedding classical logic in intuitionistic logic give the same copy of classical logic, suggesting uniqueness. We present three different copies.

Practical contributions

"Finitary" infinite pigeonhole principles Terence Tao studied finitisations of statements in analysis. We take a logic view at Tao's finitisations through the lenses of proof interpretations and reverse mathematics.

Proof mining Hillam's theorem Hillam's theorem characterises the convergence of fixed point iterations. We proof mine it, getting a "finitary rate of convergence" of the fixed point iteration.

Zusammenfassung

Wir untersuchen theoretische und praktische Aspekte von Beweisinterpretationen.

Theoretische Ergebnisse

Vollständigkeit und ω-Regel Mit Hilfe einer Beweisinterpretation zeigen wir, dass die Peano-Arithmetik mit der ω-Regel eine vollständige Theorie ist.

Beweisinterpretationen mit Wahrheitsprädikat Beweisinterpretationen ohne Wahrheitsprädikat geben Informationen über die interpretierte Formel und nicht mehr über die ursprüngliche Formel. Wir präsentieren drei Heuristiken, um Wahrheitsprädikate zu Beweisinterpretationen hinzuzufügen, und geben einige Beispiele.

Kopien von klassischer Logik in intuitionistischer Logik Die üblichen Einbettungen von klassischer Logik in intuitionistische Logik mit Hilfe von Beweisinterpretationen erzeugen alle die gleiche Kopie der klassischen Logik. Dies deutet darauf hin, dass diese Kopie eindeutig sein könnte. Wir zeigen, dass dies nicht der Fall ist und präsentieren drei verschiedene Kopien.

Angewandte Ergebnisse

"Finitisierungen" des unendlichen Schubfachprinzips Terence Tao untersucht Finitisierungen von Sätzen der Analysis. Wir betrachten Taos Ergebnisse aus dem Blickwinkel der Beweisinterpretationen und *reverse mathematics*.

Proof mining des Satzes von Hillam Der Satz von Hillam charakterisiert die Konvergenz von Fixpunktiterationen. Wir extrahieren mit Hilfe von *proof mining* eine Rate der Konvergenz für die Fixpunktiteration.

Introduction

What are proof interpretations

A proof interpretation I is a mapping of formulas, mapping a formula A of a theory S to a formula $A^{\mathrm{I}} \equiv \exists x\, A_{\mathrm{I}}(x)$ of a theory T

$$\begin{aligned} \mathrm{I}\colon\ \mathsf{S} &\to \mathsf{T} \\ A &\mapsto A^{\mathrm{I}} \equiv \exists x\, A_{\mathrm{I}}(x) \end{aligned}$$

such that I maps a theorem A of S to a theorem A^{I} of T:

$$\mathsf{S} \vdash A \quad \Rightarrow \quad \mathsf{T} \vdash A^{\mathrm{I}}. \tag{1}$$

Even better, I gives us a term t witnessing the quantification $\exists x$ in A^{I}:

$$\mathsf{S} \vdash A \quad \Rightarrow \quad \mathsf{T} \vdash A_{\mathrm{I}}(t). \tag{2}$$

Proof interpretations have many applications. We summarise the main ones.

Relative consistency The proof interpretation I shows that S is consistent relatively to T. Indeed, $\perp^{\mathrm{I}} \equiv \perp$, so (1) becomes $\mathsf{S} \vdash \perp \;\Rightarrow\; \mathsf{T} \vdash \perp$.

Conservation The proof interpretation I shows that S is conservative over T with respect to formulas in a certain set Γ. Indeed, $A^{\mathrm{I}} \equiv A$ for $A \in \Gamma$, so (1) becomes $\mathsf{S} \vdash A \;\Rightarrow\; \mathsf{T} \vdash A$.

Closure under rules If $\mathsf{S} = \mathsf{T}$, then I gives us the closure of S for some rules. For example, $(A \vee B)_{\mathrm{I}}(t)$ is equivalent to A or to B, so (2) gives us $\mathsf{S} \vdash A \vee B \;\Rightarrow\; (\mathsf{S} \vdash A$ or $\mathsf{S} \vdash B)$.

Unprovability The proof interpretation I gives us the unprovability in S of some formulas A. Indeed, if $\mathsf{T} \nvdash A_{\mathrm{I}}(t)$, then (2) gives us $\mathsf{S} \nvdash A$.

Computational content The proof interpretation I gives us a term t encapsulating computational content about a theorem A. For example, for $A \equiv \forall x\, \exists y\, B(x,y)$ we have $A_{\mathrm{I}}(t) \equiv \forall x\, B(x, t(x))$, so if $\mathsf{S} \vdash A$, then (2) gives us a t such that $\mathsf{T} \vdash \forall x\, B(x, t(x))$, that is t gives us $y = t(x)$ as a function of x.

What is done in this thesis

Framework In the first part of the thesis we construct the theories T that we will consider: versions of Peano arithmetic that talk not only about $\mathbb{N}$, but also about $\mathbb{N}^{\mathbb{N}}$, $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, and so on.

Proof interpretations In the second part of the thesis we present the proof interpretations I that we will consider. They all have different features: for example, if $\exists x\, A(x)$, then some proof interpretations give an exact witness (that is a term t such that $A(t)$), while others give a bound (that is a term t such that $\exists x \leq t\, A(x)$). For each proof interpretation we give applications: relative consistency results, extraction of computational content, and so on.

The first two parts of the thesis read like an introduction to proof interpretations.

Theoretical contributions In the third part of the thesis we give three theoretical contributions by means of proof interpretations.

Completeness and ω-rule The ω-rule (essentially) states that from $A(0), A(1), A(2), \ldots$ we infer $\forall n\, A(n)$. Using a proof interpretation, we prove that Peano arithmetic with the ω-rule is a complete theory.

Proof interpretations with truth A proof interpretation I gives information about A^{I}, but usually we want information about A. One way of transferring the information from A^{I} to A is to hardwire truth in I: to change I so that A^{I} implies A. We give three heuristics on how to hardwire truth and apply them to several proof interpretations.

Copies of classical logic in intuitionistic logic Some proof interpretations copy (that is embed) classical logic (that is the usual logic in mathematics) into intuitionistic logic (that is the logic of constructive mathematics). The usual proof interpretations all give the same copy, suggesting that the copy is unique. We refute this and present three different copies.

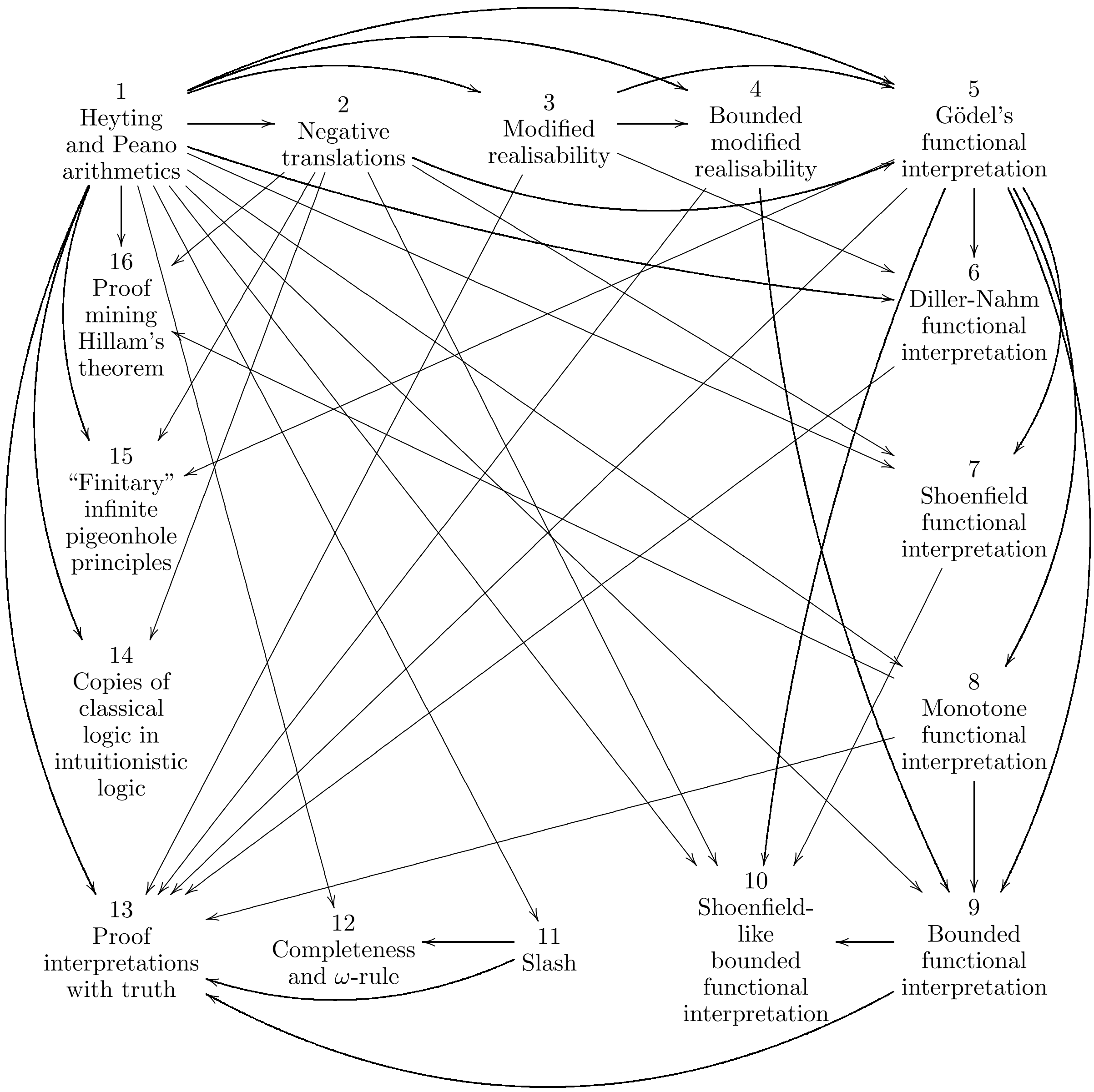
Practical contributions In the fourth part of the thesis we give two practical contributions by means of proof interpretations.

"Finitary" infinite pigeonhole principles Terence Tao studied finitisations of statements in analysis: assigning to qualitative-infinitary statements equivalent quantitative-finitary statements. One of his prime examples is a finitisation of the infinite pigeonhole principle (that is "if we colour the natural numbers with finitely many colours, then some colour occurs infinitely often"). We take a logic view at Tao's finitisations: we give a counterexample to a mistaken finitisation, we obtain a correction by a proof interpretation, and we compare in the context of reverse mathematics our correction with Tao's correction.

Proof mining Hillam's theorem Hillam's theorem characterises the convergence of a fixed point iteration (x_n) of a continuous function $f\colon [0,1] \to [0,1]$: the fixed point iteration (x_n) converges if and only if $x_{n+1} - x_n \to 0$.

We proof mine Hillam's theorem, that is using a proof interpretation we extract computational content from Hillam's proof: a "finitary rate of convergence" of (x_n) in terms of a "finitary rate of convergence" of $(x_{n+1} - x_n)$ and a rate of uniform continuity of f.

The following diagram gives an overall impression of the many connections between the chapters of the thesis:



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Contents

Part I

Framework

Chapter 1

Heyting and Peano arithmetics

1.1 Introduction

1.1. In this chapter we lay out our framework: a version HA^ω of Peano arithmetic that

1. does not have the law of excluded middle $A \vee \neg A$;

2. talks not only about $\mathbb{N}$, but also about $\mathbb{N}^\mathbb{N}$, $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$, and so on.

To set up our framework, we mainly have to do three big tasks.

Define HA^ω For HA^ω to talk about $\mathbb{N}, \mathbb{N}^\mathbb{N}, (\mathbb{N}^\mathbb{N})^\mathbb{N}, \mathbb{N}^{(\mathbb{N}^\mathbb{N})}, \ldots$, we have to introduce the so-called types. The idea is simple: we assign to each term of HA^ω an object, called type, that tells us in which of the sets $\mathbb{N}, \mathbb{N}^\mathbb{N}, (\mathbb{N}^\mathbb{N})^\mathbb{N}, \mathbb{N}^{(\mathbb{N}^\mathbb{N})}, \ldots$ the term takes values.

To bar the law of excluded middle from HA^ω, we introduce intuitionistic logic. Intuitionistic logic is, roughly speaking, the usual logic in mathematics without the law of excluded middle.

Functions Once we have defined HA^ω, it is time to set up all the machinery for constructing functions in $\mathbb{N}, \mathbb{N}^\mathbb{N}, (\mathbb{N}^\mathbb{N})^\mathbb{N}, \mathbb{N}^{(\mathbb{N}^\mathbb{N})}, \ldots$ in the language of HA^ω.

The first thing to do is set up λ-abstraction. Informally, given a term $t(x)$, λ-abstraction allows us to define the function $x \mapsto t(x)$.

The second thing to do is to show that every primitive recursive function can be represented in HA^ω by a term. This allows HA^ω to talk about addition, multiplication, and so on.

The third thing to do is to show that quantifier-free formulas $A_{\mathrm{qf}}(x)$ have characteristic terms that are (roughly speaking) characteristic functions of sets $\{x : A_{\mathrm{qf}}(x)\}$. Characteristic terms allow us to make in HA^ω definitions by cases like

$$f(x) := \begin{cases} x^2 & \text{if } \quad A_{\mathrm{qf}}(x) \\ x^3 & \text{if } \neg A_{\mathrm{qf}}(x) \end{cases}.$$

These definitions by cases play an important role in a delicate point of Gödel's functional interpretation: the interpretation of the seemly innocuous axiom $A \to A \wedge A$.

Majorisability Some of the proof interpretations that we will consider later, in the face of a theorem $\exists x\, A(x)$, seek to find not an exact witness t for x such that $A(t)$, but a bound t on x such that $\exists x \leq t\, A(x)$. The majorisability $\leq$ in question is such that $f \leq g$ (roughly speaking) means "f is pointwise smaller than or equal to g, and g is non-decreasing". To work fluently with this majorisability, we need to prove its basic properties.

1.2. Our (admittedly modest) main contribution to this topic is checking [19, *capítulos* 1 and 7] that the standard material on λ-abstraction, terms for primitive recursive functions, characteristic terms, term definition by cases, and so on, goes through in the neutral setting with an intensional majorisability $\mathsf{HA}^{\omega}_{\mathrm{i}}$ (introduced later on). This led to three tiny patches to the literature and filling in a common small omission in the literature (in the proofs of point 1 of theorem 1.30, point 1 of theorem 1.61, point 1 of proposition 1.66, and theorem 1.34).

1.2 Notation

1.3. We collect here the non-standard notation and conventions that we will be using later on.

1.4 Notation. Let A be a formula, $\underline{x} \equiv x_1, \ldots, x_n$ be a tuple of variables and $\underline{t} \equiv t_1, \ldots, t_n$ be a tuple of terms.

1. We use $\equiv$ to denote syntactic/literal equality.
2. We use $\Rightarrow$ and $\Leftrightarrow$ for implication and equivalence in meta-level.
3. We denote (possibly empty) tuples $t_1, \ldots, t_n$ of terms by an underlined letter $\underline{t}$.
4. When we write $A[\underline{t}/\underline{x}]$ or $A(\underline{t})$ to denote the simultaneous substitution of $\underline{x}$ by $\underline{t}$ in $A(\underline{x})$, we implicitly assume that each t_i is free for x_i in $A(\underline{x})$.
5. We denote the sets of free and bounded variables of A by $\mathrm{FV}(A)$ and $\mathrm{BV}(A)$, respectively. We denote by $\mathrm{FV}(\underline{t})$ the set of variables of $\underline{t}$.
6. We abbreviate $\{x_1, \ldots, x_n\}$ by $\{\underline{x}\}$.
7. We denote by p^n_k the projection $\mathrm{p}^n_k \colon \mathbb{N}^n \to \mathbb{N}$ defined by $\mathrm{p}^n_k(x_1, \ldots, x_n) := x_k$.
8. We denote by $\mu n \,.\, A(n)$ the least n such that the condition $A(n)$ holds true.
9. We denote the set $\{0, 1, 2, \ldots, n-1\}$ by n.

1.3 Intuitionistic and classical logics

1.5. Intuitionistic logic is a formalisation of intuitionism. Historically, intuitionism was introduced by Brouwer [6] and formalised by Heyting [32]. Informally, classical logic CL is the usual logic in mathematics, and intuitionistic logic IL is CL without:

1. proof by contradiction $\dfrac{\begin{array}{c}\neg A\\ \vdots\\ \bot\end{array}}{A}$;
2. law of double negation $\neg\neg A \to A$;
3. law of excluded middle $A \vee \neg A$;

(as a curiosity, these three principles are equivalent in IL). Formally, here CL and IL are axiomatised in definition 1.8 by a Hilbert-style deductive system (essentially) due to Gödel [28] [30, page 280].

1.6. There are two antagonistic ways of comparing IL and CL.

<u>IL is poorer than CL</u> Intuitionistic logic IL is weaker than classical logic CL, that is IL proves fewer theorems than CL: $\mathsf{IL} \vdash A \overset{\Rightarrow}{\not\Leftarrow} \mathsf{CL} \vdash A$.

<u>IL is richer than CL</u> Classical logic CL does not see the difference between $\neg(\neg A \wedge \neg B)$ and $A \vee B$, and between $\neg\forall x\, \neg A(x)$ and $\exists x\, A(x)$. Intuitionistic logic IL refines this situation by making a difference:

1. $\neg(\neg A \wedge \neg B)$ has the usual meaning "A or B", while $A \vee B$ has the stronger meaning "A or B, and we can point to one that holds true";
2. $\neg\forall x\, \neg A(x)$ has the usual meaning "there exists an x such that $A(x)$", while $\exists x\, A(x)$ has the stronger meaning "there exists an x such that $A(x)$, and we know such an x".

1.7. Another comparison between IL and CL is in terms of constructivity. The key criteria to determine if a logic is constructive (arguably) is if it satisfies the following properties:

<u>Disjunction property</u> if $\vdash A \vee B$, then $\vdash A$ or $\vdash B$ (where $A \vee B$ is a sentence);

<u>Existence property</u> if $\vdash \exists x\, A(x)$, then there exists a closed term t such that $\vdash A(t)$ (where $\exists x\, A(x)$ is a sentence).

In this sense, IL is constructive but CL is not.

1.8 Definition.

1. Let us define *intuitionistic logic* IL [75, section 1.1.4] [50, section 3.1].
   (a) The language of IL is the following.
      i. The language of IL has the following symbols.
         A. The logical constants $\bot$, $\wedge$, $\vee$, $\to$, $\forall$ and $\exists$.

B. Countable many variables $x_1, x_2, x_3, \ldots$.

C. For each arity $n \geq 0$, at most countable many (possibly none) n-ary function symbols $f_1, f_2, f_3, \ldots$.

D. For each arity $n \geq 0$, at most countable many (possibly none) n-ary predicate symbols $P_1, P_2, P_3, \ldots$.

ii. Terms are defined as follows.

A. Variables and (non-logical) constants (that is 0-ary function symbols) are terms.

B. If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is also a term.

iii. Formulas are defined as follows.

A. The logical constant $\bot$ is an atomic formula.

B. If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is an atomic formula.

C. Formulas are built from atomic formulas by means of $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$.

(b) We define the following in IL.

i. $\neg A :\equiv A \rightarrow \bot$.

ii. $A \leftrightarrow B :\equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

(c) We adopt the following convention to save on parentheses: $\neg$, $\forall$ and $\exists$ bind stronger than $\wedge$ and $\vee$, which in turn bind stronger than $\rightarrow$ and $\leftrightarrow$.

(d) The axioms and rules of IL are given in table 1.1 [50, section 3.1].

contraction axioms	$A \rightarrow A \wedge A \quad A \vee A \rightarrow A$
weakening axioms	$A \wedge B \rightarrow A \quad A \rightarrow A \vee B$
permutation axioms	$A \wedge B \rightarrow B \wedge A \quad A \vee B \rightarrow B \vee A$
ex falso quodlibet	$\bot \rightarrow A$
quantifier axioms	$\forall x\, A \rightarrow A[t/x] \quad A[t/x] \rightarrow \exists x\, A$
modus ponens rule	$\dfrac{A \quad A \rightarrow B}{B}$
syllogism rule	$\dfrac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}$
exportation rule	$\dfrac{A \wedge B \rightarrow C}{A \rightarrow (B \rightarrow C)}$
importation rule	$\dfrac{A \rightarrow (B \rightarrow C)}{A \wedge B \rightarrow C}$
expansion rule	$\dfrac{A \rightarrow B}{C \vee A \rightarrow C \vee B}$
quantifier rules	$\dfrac{A \rightarrow B}{A \rightarrow \forall x\, B} \quad \dfrac{B \rightarrow A}{\exists x\, B \rightarrow A}$ $(x \notin \mathrm{FV}(A))$

Table 1.1: axioms and rules of IL.

2. *Classical logic* CL is IL plus the *law of excluded middle* $A \vee \neg A$ [50, section 3.1].

1.9. The deductive systems given for IL and CL are suitable to prove properties of IL and CL by induction on the length of derivations, but unsuitable to actually find derivations in IL and CL. For this purpose, a much more practical system is the (equivalent) natural deduction [75, section 1.1.5 and theorem 1.1.11].

1.4 Types

1.10. We are going to work with a version of Peano arithmetic that talks not only about $\mathbb{N}$, but also about $\mathbb{N}^{\mathbb{N}}$, $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, and so on. The elements of these sets can only mix in a proper way: for example, given $n \in \mathbb{N}$, $f \in \mathbb{N}^{\mathbb{N}}$ and $F \in \mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, it makes sense to write $f(n)$ and $F(f)$, but not $f(F)$ and $F(n)$. So the syntax of our Peano arithmetic has to somehow keep track of the sets in which the terms take values. This is achieved by the types: to each term we assign a type, and the type identifies the set according to the "dictionary" given in table 1.2. (We could directly assign to each term a set, but traditionally we assign a type.)

Set	$\mathbb{N}$	$\mathbb{N}^{\mathbb{N}}$	$(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$	$\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$	$\cdots$
Type	0	00	(00)0	0(00)	$\cdots$

Table 1.2: sets and their types.

1.11 Definition. Consider an alphabet $\{0, \to, (,)\}$. *Finite types*, or simply *types* [75, section 1.6.2] [50, section 3.3], are words on this alphabet generated recursively by:

1. 0 is a type;
2. if σ and ρ are types, then $(\sigma \to \rho)$ is also a type.

We adopt the following notation, where σ and ρ are types and $\underline{\rho} \equiv \rho_1, \ldots, \rho_m$ and $\underline{\sigma} \equiv \sigma_1, \ldots, \sigma_n$ are tuples of types.

1. We denote $(\sigma \to \rho)$ by $\rho\sigma$ (note the inversion of the position of the letters).
2. In $\rho_1 \cdots \rho_m$ we associate to the left, that is we read $\big(((\rho_1\rho_2)\rho_3)\rho_4\big) \cdots \rho_m$.
3. We define $\underline{\rho}\underline{\sigma} :\equiv \rho_1\sigma_1 \cdots \sigma_n, \ldots, \rho_m\sigma_1 \cdots \sigma_n$
4. We define $\underline{\rho}^{\mathrm{t}} :\equiv \rho_m, \ldots, \rho_1$.

1.12 Remark. All types ρ can be decomposed as $\rho = 0\rho_1 \cdots \rho_n$ (with possible no ρ_is) [75, section 1.6.2] [50, section 3.1].

1.13. We can interpret types in the following way:

1. the type 0 is interpreted as the set $\mathbb{N}$;
2. if the types ρ and σ are interpreted as sets A and B respectively, then the type $\rho\sigma$ is interpreted as the set A^B [75, section 1.6.2].

This interpretation produces table 1.2. Then we can interpret the statement "x has type ρ" as meaning "x is in the set interpreting ρ".

1.5 Heyting and Peano arithmetics

1.14. Now we introduce a version PA^ω of Peano arithmetic that, informally, talks not only about $\mathbb{N}$ but also about $\mathbb{N}^\mathbb{N}$, $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$ and so on. More formally, the syntax of PA^ω has the following two devices that mimic the sets $\mathbb{N}, \mathbb{N}^\mathbb{N}, (\mathbb{N}^\mathbb{N})^\mathbb{N}, \mathbb{N}^{(\mathbb{N}^\mathbb{N})}, \ldots$.

<u>Assigning types to terms</u> Each term has a type associated that, informally, says to which of the sets $\mathbb{N}, \mathbb{N}^\mathbb{N}, (\mathbb{N}^\mathbb{N})^\mathbb{N}, \mathbb{N}^{(\mathbb{N}^\mathbb{N})}, \ldots$ the term belongs.

<u>Applying terms</u> In the same way that given $F \in \mathbb{N}^{(\mathbb{N}^\mathbb{N})}$ and $f \in \mathbb{N}^\mathbb{N}$ we can apply them getting $F(f) \in \mathbb{N}$, given two terms s of type $\rho\sigma$ and t of type σ we can apply them getting a term st of type ρ. We can think of applying s and t as applying $s \in \rho^\sigma$ and $t \in \sigma$ getting $st \in \rho$.

1.15. Before we proceed to the definition of PA^ω, we need to compile some notation about terms and their types. In the definition of PA^ω some axioms are restricted to certain classes of formulas, so we also need to compile the classes of formulas that we will need.

1.16 Definition. Let $\leq_\rho$ be some majorisability infixed between terms of type ρ, $=_\rho$ be some equality infixed between terms of type ρ, and $\max_\rho$ be some maximum of two terms of type ρ (they will be defined later on).

1. If t is a term of type ρ, then we write t^ρ to express this fact. More generally, if $\underline{t} \equiv t_1, \ldots, t_n$ is a tuple of terms, $\underline{\rho} \equiv \rho_1, \ldots, \rho_n$ is a tuple of types, and each t_i has type ρ_i, then we write $\underline{t}^{\underline{\rho}}$ to express this fact. If it is not important to make the types explicit, then we write simply t, $\leq$, $=$ and $\max$ instead of t^ρ, $\leq_\rho$, $=_\rho$ and $\max_\rho$.

2. Let $\underline{s} \equiv s_1, \ldots, s_m$ and $t \equiv t_1, \ldots, t_n$ be tuples of terms. In $t_1 \cdots t_n$ we associate to the left, that is we read $\big(((t_1t_2)t_3)t_4\big) \cdots t_n$. We define $\underline{s}\underline{t} :\equiv s_1t_1 \cdots t_n, \ldots, s_mt_1 \cdots t_n$.

3. Given terms s and t, when we write st we implicitly assume that the types of the terms s and t fit, that is s has type $\rho\sigma$ and t has type σ for some types ρ and σ. Given a formula $A(\underline{x})$ with a distinguished tuple of variables $\underline{x} \equiv x_1, \ldots, x_n$, and terms $\underline{t} \equiv t_1, \ldots, t_n$, when we write $A[\underline{t}/\underline{x}]$ or $A(\underline{t})$ we implicitly assume that each t_i has the same type that x_i.

4. (a) We call *bounded quantifiers* [15, section 2] to quantifiers of the form

$$\forall x \leq_\rho t\, A,$$
$$\exists x \leq_\rho t\, A$$

(where t is a term and $x \notin \mathrm{FV}(t)$).

(b) We call *monotone quantifiers* [15, section 2.1] to quantifiers of the form

$$\tilde{\forall} x^\rho\, A :\equiv \forall x\, (x \leq_\rho x \to A),$$
$$\tilde{\exists} x^\rho\, A :\equiv \exists x\, (x \leq_\rho x \wedge A).$$

(c) We call *monotone bounded quantifiers* to quantifiers of the form

$$\tilde{\forall} x \leq_\rho t\, A :\equiv \forall x \leq_\rho t\, (x \leq_\rho x \to A),$$
$$\tilde{\exists} x \leq_\rho t\, A :\equiv \exists x \leq_\rho t\, (x \leq_\rho x \wedge A).$$

5. Let $\underline{\rho} \equiv \rho_1, \ldots, \rho_n$ be a tuple of types, $\underline{x}^{\underline{\rho}} \equiv x_1^{\rho_1}, \ldots, x_n^{\rho_n}$ and $\underline{y}^{\underline{\rho}} \equiv y_1^{\rho_1}, \ldots, y_n^{\rho_n}$ be tuples of variables and $\underline{s}^{\underline{\rho}} \equiv s_1^{\rho_1}, \ldots, s_n^{\rho_n}$ and $\underline{t}^{\underline{\rho}} \equiv t_1^{\rho_1}, \ldots, t_n^{\rho_n}$ be tuples of terms. We define

$$\begin{aligned}
\underline{s} =_{\underline{\rho}} \underline{t} &:\equiv s_1 =_{\rho_1} t_1 \wedge \cdots \wedge s_n =_{\rho_n} t_n, \\
\underline{x} \leq_{\underline{\rho}} \underline{y} &:\equiv s_1 \leq_{\rho_1} t_1 \wedge \cdots \wedge s_n \leq_{\rho_n} t_n, \\
\max_{\underline{\rho}}(\underline{s}, \underline{t}) &:\equiv \max_{\rho_1}(s_1, t_1), \ldots, \max_{\rho_n}(s_n, t_n), \\
\forall \underline{x}^{\underline{\rho}}\, A &:\equiv \forall x_1^{\rho_1} \ldots \forall x_n^{\rho_n}\, A, \\
\exists \underline{x}^{\underline{\rho}}\, A &:\equiv \exists x_1^{\rho_1} \ldots \exists x_n^{\rho_n}\, A, \\
\tilde{\forall} \underline{x}^{\underline{\rho}}\, A &:\equiv \tilde{\forall} x_1^{\rho_1} \ldots \tilde{\forall} x_n^{\rho_n}\, A, \\
\tilde{\exists} \underline{x}^{\underline{\rho}}\, A &:\equiv \tilde{\exists} x_1^{\rho_1} \ldots \tilde{\exists} x_n^{\rho_n}\, A, \\
\forall \underline{x} \leq_{\underline{\rho}} \underline{t}\, A &:\equiv \forall x_1 \leq_{\rho_1} t_1 \ldots \forall x_n \leq_{\rho_n} t_n\, A, \\
\exists \underline{x} \leq_{\underline{\rho}} \underline{t}\, A &:\equiv \exists x_1 \leq_{\rho_1} t_1 \ldots \exists x_n \leq_{\rho_n} t_n\, A, \\
\tilde{\forall} \underline{x} \leq_{\underline{\rho}} \underline{t}\, A &:\equiv \tilde{\forall} x_1 \leq_{\rho_1} t_1 \ldots \tilde{\forall} x_n \leq_{\rho_n} t_n\, A, \\
\tilde{\exists} \underline{x} \leq_{\underline{\rho}} \underline{t}\, A &:\equiv \tilde{\exists} x_1 \leq_{\rho_1} t_1 \ldots \tilde{\exists} x_n \leq_{\rho_n} t_n\, A.
\end{aligned}$$

1.17 Definition.

1. We reserve the subscript "at" (as in A_{at}) for atomic formulas.

2. A *quantifier-free* formula is a formula without (bounded and unbounded) quantifiers. We reserve the subscript "qf" (as in A_{qf}) for quantifier-free formulas.

3. A *bounded* formula [15, section 2] is a formula without unbounded quantifiers. We reserve the subscript "b" (as in A_{b}) for bounded formulas.

4. An $\exists$-*free* formula [50, definition 5.2.1)] is a formula:

   (a) without disjunctions;

   (b) without (bounded and unbounded) existential quantifiers.

   We reserve the subscript "$\exists$f" (as in $A_{\exists \mathrm{f}}$) for $\exists$-free formulas.

5. An $\tilde{\exists}$-*free* formula [14, definition 3] is a formula:

   (a) without disjunctions;

   (b) without unbounded existential quantifiers;

   (c) whose universal quantifiers are all monotone.

   We reserve the subscript "$\tilde{\exists}$f" (as in $A_{\tilde{\exists} \mathrm{f}}$) for $\tilde{\exists}$-free formulas.

1.18 Definition.

1. Let us define the (typed) *Heyting arithmetic* HA^ω [76, section 3.1] [15, section 2].

   (a) The language of HA^ω is the following.

      i. The language of HA^ω has the following symbols.

         A. The logical constants $\bot$, $\wedge$, $\vee$, $\to$, $\forall$ and $\exists$.

         B. Countable many variables $x_1^\rho, x_2^\rho, x_3^\rho, \ldots$ for each type ρ.

         C. The constant *zero* 0.

         D. The constant *successor* S.

         E. A constant *projector* $\Pi_{\rho,\sigma}$ for each types ρ and σ.

         F. A constant *combinator* $\Sigma_{\rho,\sigma,\tau}$ for each types ρ, σ and τ.

         G. A tuple of constants *recursors* $\underline{\mathrm{R}}_{\underline{\rho}} \equiv (\mathrm{R}_1)_{\underline{\rho}}, \ldots, (\mathrm{R}_n)_{\underline{\rho}}$ for each tuple of types $\underline{\rho} = \rho_1, \ldots, \rho_n$.

         H. The binary relation *equality* $=_0$.

      ii. Terms are defined as follows (their types indicated in superscripts).

         A. Variables x^ρ, and the constants 0^0, S^{00}, $\Pi_{\rho,\sigma}^{\rho\sigma\rho}$, $\Sigma_{\rho,\sigma,\tau}^{\tau\rho(\sigma\rho)(\tau\sigma\rho)}$ and $(\mathrm{R}_i)_{\underline{\rho}}^{\rho_i(\underline{\rho}^{\mathrm{t}}0\underline{\rho}^{\mathrm{t}})\underline{\rho}^{\mathrm{t}}0}$ are terms.

         B. If $s^{\rho\sigma}$ and t^σ are terms, then $(st)^\rho$ is a term.

      iii. Formulas are defined as follows.

         A. The logical constant $\bot$ is an atomic formula.

         B. The expressions $s =_0 t$ are atomic formulas (where s^0 and t^0 are terms).

         C. Formulas are built from atomic formulas by means of $\wedge$, $\vee$, $\to$, $\forall$ and $\exists$.

   (b) We define the following in HA^ω.

      i. The formula $A \vee_t B :\equiv (t =_0 0 \to A) \wedge (t \neq_0 0 \to B)$, where t^0 is a term of HA^ω and A and B are formulas of HA^ω.

      ii. The *extensional equality* $s =_\rho t :\equiv \forall \underline{x}\, (s\underline{x} =_0 t\underline{x})$, where s and t are terms of HA^ω of type $\rho = 0\rho_n \cdots \rho_1$ and $\underline{x} \equiv x_1^{\rho_1}, \ldots, x_n^{\rho_n}$.

      iii. The *hereditary equality* $s^\rho \approx_\rho t^\rho$ [75, section 2.7.2], where s and t are terms of HA^ω, by recursion on the structure of ρ by:

         A. $s \approx_0 t :\equiv s =_0 t$;

         B. $s \approx_{\rho\sigma} t :\equiv \forall x^\sigma, y^\sigma\, (x \approx_\sigma y \to sx \approx_\rho ty)$.

      iv. A. The *type* 0 *inequality* $s \leq_0 t :\equiv s \dot{-} t =_0 0$, where s^0 and t^0 are terms of HA^ω and $\dot{-}$ (a term of HA^ω standing for the cut-off/limited/truncated subtraction) is defined in point 3 of definition 1.37.

         B. We extended $\leq_0$ to higher types by $s \leq_\rho t :\equiv \forall \underline{x}\, (s\underline{x} \leq_0 t\underline{x})$, where s and t are terms of HA^ω of type $\rho = 0\rho_n \cdots \rho_1$ and $\underline{x} \equiv x_1^{\rho_1}, \ldots, x_n^{\rho_n}$.

v. The *extensional majorisability* $s \leq^{\mathrm{e}}_{\rho} t$ [39, section 2] [5, paragraph 1.1] [50, definition 3.34], where s^{ρ} and t^{ρ} are terms of HA^{ω}, by recursion on the structure of ρ by:

A. $s \leq^{\mathrm{e}}_{0} t :\equiv s \leq_{0} t$;

B. $s \leq^{\mathrm{e}}_{\rho\sigma} t :\equiv \forall x^{\sigma}, y^{\sigma}\, (x \leq^{\mathrm{e}}_{\sigma} y \to sx \leq^{\mathrm{e}}_{\rho} ty \wedge tx \leq^{\mathrm{e}}_{\rho} ty)$.

(c) The axioms and rules of HA^{ω} are the ones of IL plus the ones given in table 1.3.

axioms of $=_0$	$x =_0 x$ $x =_0 y \wedge A_{\mathrm{at}}[x/z] \to A_{\mathrm{at}}[y/z]$
axioms of S	$\mathrm{S}x \neq_0 0$ $\mathrm{S}x =_0 \mathrm{S}y \to x =_0 y$
axioms of Π, Σ and $\underline{\mathrm{R}}$	$A_{\mathrm{at}}[\Pi_{\rho,\sigma}xy/w] \leftrightarrow A_{\mathrm{at}}[x/w]$ $A_{\mathrm{at}}[\Sigma_{\rho,\sigma,\tau}xyz/w] \leftrightarrow A_{\mathrm{at}}[xz(yz)/w]$ $A_{\mathrm{at}}[\underline{\mathrm{R}}_{\underline{\rho}}0\underline{y}\underline{z}/\underline{w}] \leftrightarrow A_{\mathrm{at}}[\underline{y}/\underline{w}]$ $A_{\mathrm{at}}[\underline{\mathrm{R}}_{\underline{\rho}}(\mathrm{S}x)\underline{y}\underline{z}/\underline{w}] \leftrightarrow A_{\mathrm{at}}[\underline{z}(\underline{\mathrm{R}}_{\underline{\rho}}x\underline{y}\underline{z})x/\underline{w}]$
induction rule	$\dfrac{A(0) \quad A(x) \to A(\mathrm{S}x)}{A(x)}$

Table 1.3: axioms and rules of HA^{ω} (in addition to the ones of IL).

2. The (typed) *Heyting arithmetic with weak extensionality* [75, section 1.6.12] [50, section 3.3] $\mathsf{WE\text{-}HA}^{\omega}$ is HA^{ω} but with the *extensionality rule* [66, page 12]

$$\frac{A_{\mathrm{at}} \to s =_{\rho} t}{A_{\mathrm{at}} \to r[s/x] =_0 r[t/x]}.$$

where r^0, s^{ρ} and t^{ρ} are terms of $\mathsf{WE\text{-}HA}^{\omega}$.

3. The (typed) *Heyting arithmetic with extensionality* $\mathsf{E\text{-}HA}^{\omega}$ [75, section 1.6.12] [50, section 3.3] is HA^{ω} but with the *extensionality axioms*

$$\forall \underline{x}^{\underline{\rho}}, \underline{y}^{\underline{\rho}}, z^{0\underline{\rho}^{\mathrm{t}}}\, (\underline{x} =_{\underline{\rho}} \underline{y} \to z\underline{x} =_0 z\underline{y}).$$

4. The (typed) *Heyting arithmetic with extensional majorisability* $\mathsf{HA}^{\omega}_{\mathrm{e}}$ [14, section 4.1] is HA^{ω} with primitive bounded quantifications $\forall x \leq^{\mathrm{e}}_{\rho} t\, A$ and $\exists x \leq^{\mathrm{e}}_{\rho} t\, A$ (for each type ρ and with the restriction $x \notin \mathrm{FV}(t)$) and their axioms

$$\forall x \leq^{\mathrm{e}}_{\rho} tA \leftrightarrow \forall x\, (x \leq^{\mathrm{e}}_{\rho} t \to A),$$
$$\exists x \leq^{\mathrm{e}}_{\rho} tA \leftrightarrow \exists x\, (x \leq^{\mathrm{e}}_{\rho} t \wedge A).$$

In $\mathsf{HA}^{\omega}_{\mathrm{e}}$ we redefine $\leq^{\mathrm{e}}$ by (the equivalent) [14, page 333]

(a) $s \leq^{\mathrm{e}}_{0} t :\equiv s \leq_0 t$;

(b) $s \leq^{\mathrm{e}}_{\rho\sigma} t :\equiv \tilde{\forall} y^{\sigma}\, \forall x \leq^{\mathrm{e}}_{\sigma} y\, (sx \leq^{\mathrm{e}}_{\rho} ty \wedge tx \leq^{\mathrm{e}}_{\rho} ty)$.

5. The (typed) *Heyting arithmetic with intensional majorisability* $\mathsf{HA}^{\omega}_{\mathrm{i}}$ [15, definition 5] is HA^{ω} with the following additions.

   (a) A primitive binary relation $\leq^{\mathrm{i}}_{\rho}$ (for each type ρ) infixed between terms of $\mathsf{HA}^{\omega}_{\mathrm{i}}$ of type ρ, called *intensional majorisability*, and its axioms and rule

$$x \leq^{\mathrm{i}}_{0} y \leftrightarrow x \leq_{0} y, \qquad x \leq^{\mathrm{i}}_{\rho\sigma} y \to \forall u \leq^{\mathrm{i}}_{\sigma} v\, (xu \leq^{\mathrm{i}}_{\rho} yv \wedge yu \leq^{\mathrm{i}}_{\rho} yv),$$

$$\frac{A_{\mathrm{b}} \wedge x \leq^{\mathrm{i}}_{\sigma} y \to sx \leq^{\mathrm{i}}_{\rho} ty \wedge tx \leq^{\mathrm{i}}_{\rho} ty}{A_{\mathrm{b}} \to s \leq^{\mathrm{i}}_{\rho\sigma} t},$$

   where s and t are terms of $\mathsf{HA}^{\omega}_{\mathrm{i}}$ and in the rule we have the restriction $x, y \notin \mathrm{FV}(A_{\mathrm{b}}) \cup \mathrm{FV}(s) \cup \mathrm{FV}(t)$. We declare the formulas $s \leq^{\mathrm{i}}_{\rho} t$ atomic, where s and t are terms of $\mathsf{HA}^{\omega}_{\mathrm{i}}$.

   (b) Primitive bounded quantifications $\forall x \leq^{\mathrm{i}}_{\rho} t\, A$ and $\exists x \leq^{\mathrm{i}}_{\rho} t\, A$ (for each type ρ and with the restriction $x \notin \mathrm{FV}(t)$) and their axioms

$$\forall x \leq^{\mathrm{i}}_{\rho} t\, A \leftrightarrow \forall x\, (x \leq^{\mathrm{i}}_{\rho} t \to A),$$
$$\exists x \leq^{\mathrm{i}}_{\rho} t\, A \leftrightarrow \exists x\, (x \leq^{\mathrm{i}}_{\rho} t \wedge A).$$

6. The (typed) *Peano arithmetics* PA^{ω}, $\mathsf{WE\text{-}PA}^{\omega}$, $\mathsf{E\text{-}PA}^{\omega}$, $\mathsf{PA}^{\omega}_{\mathrm{e}}$ and $\mathsf{PA}^{\omega}_{\mathrm{i}}$ are, respectively, HA^{ω}, $\mathsf{WE\text{-}HA}^{\omega}$, $\mathsf{E\text{-}HA}^{\omega}$, $\mathsf{HA}^{\omega}_{\mathrm{e}}$ and $\mathsf{HA}^{\omega}_{\mathrm{i}}$ with the addition of the law of excluded middle.

1.19. The role of Π, Σ, $\underline{\mathrm{R}}$, $=_{\rho}$, $\approx_{\rho}$, $\leq^{\mathrm{e}}_{\rho}$, $\leq^{\mathrm{i}}_{\rho}$ and extensionality may be a bit obscure, so let us explain it.

Π and Σ The role of Π and Σ is to, given a term $t(x)$, construct a term doing the job of the function $x \mapsto t(x)$. This will be treat in detail in section 1.7.

$\underline{\mathrm{R}}$ The tuple of recursors $\underline{\mathrm{R}}$ is used to define terms by recursion. For example, if the tuple has only one recursor R, then $\mathrm{R}xyz$ stands for the sequence $(r_x)_{x \in \mathbb{N}}$ defined by recursion on x by $r_0 := y$ and $r_{x+1} := z(r_x, x)$. So, in $\mathrm{R}xyz$, x is the recursion variable, y is the initial value and z is the function that performs the recursion step. The use of tuples of recursors $\underline{\mathrm{R}} \equiv \mathrm{R}_1, \ldots, \mathrm{R}_n$ allows us to define multiple sequences $(r^1_x)_{x \in \mathbb{N}}, \ldots, (r^n_x)_{x \in \mathbb{N}}$ by simultaneous recursion.

We should note that $r_{x+1} := z(r_x, x)$ is (in general) not a numeric equality but a function equality; this feature takes our recursors beyond the scope of primitive recursive functions (for example, we can define the Ackermann function [34, pages 185-186]).

$=_{\rho}$ and $\approx_{\rho}$ The equality $=_{\rho}$ just mimics the usual equality between, for example, functions $f, g \colon \mathbb{N}^n \to \mathbb{N}$: $f = g$ if and only if $\forall \underline{x} \in \mathbb{N}^n\, (f(\underline{x}) = g(\underline{x}))$.

The equality $\approx_{\rho}$ is used for technicalities in points 3 and 6 of proposition 1.26: to give an alternative formulation of the extensionality axioms in the form $\forall z\, (z \approx_{\rho} z)$, and then to prove that every closed term t is extensional in the sense of $t \approx_{\rho} t$.

$\underline{\leq^{\mathrm{e}}_\rho \text{ and } \leq^{\mathrm{i}}_\rho}$ For $\rho = 00$, the majorisability $f \leq^{\mathrm{e}}_\rho g$ means "$(*_1)$ f is pointwise smaller than or equal to g, and $(*_2)$ g is non-decreasing". By adding $(*_2)$ we gain the property $m \leq n \to f(m) \leq g(n)$ which plays an important role for some proof interpretations. For higher types ρ it is difficult to nicely describe $f \leq^{\mathrm{e}}_\rho g$.

The majorisability $\leq^{\mathrm{e}}_\rho$ trivially satisfies $(*)$ $s \leq^{\mathrm{e}}_{\rho\sigma} t \leftrightarrow \forall x^\sigma, y^\sigma\, (x \leq^{\mathrm{e}}_\sigma y \to sx \leq^{\mathrm{e}}_\rho ty \wedge tx \leq^{\mathrm{e}}_\rho ty)$. The majorisability $\leq^{\mathrm{i}}_\rho$ is (essentially) $\leq^{\mathrm{e}}_\rho$ but with the right-to-left implication of $(*)$ weakened to a rule because some proof interpretations do not seem to interpret that implication.

Extensionality To better explain extensionality, let us advance that in point 4 of proposition 1.26 we will show that the extensionality rule of $\mathsf{WE\text{-}HA}^\omega$ implies $s =_\rho t \,/\, A(s) \to A(t)$, and the extensionality axioms of $\mathsf{E\text{-}HA}^\omega$ imply $s =_\rho t \to (A(s) \to A(t))$. So we see that extensionality is just an equality axiom for $=_\rho$, and that the extensionality rule is (essentially) the weakening of the extensionality axioms to a rule because some proof interpretations do not interpret the axioms.

1.20. Sometimes the axioms of Π, Σ and $\underline{\mathrm{R}}$ are given as term equalities like $t[\Pi xy/w] = t[x/w]$ [75, section 1.6.15]. In the case of $\mathsf{HA}^\omega_{\mathrm{i}}$ this would leave some atomic formulas out of reach of the axioms because not all atomic formulas are term equalities (we also have the atomic formulas $s \leq^{\mathrm{i}} t$). So we formulated the axioms of Π, Σ and $\underline{\mathrm{R}}$ as equivalences like $A_{\mathrm{at}}[\Pi xy/w] \leftrightarrow A_{\mathrm{at}}[x/w]$ covering all atomic formulas.

This situation is somewhat typical: much of what is said below is well-known for $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$, but we should be careful with HA^ω, $\mathsf{HA}^\omega_{\mathrm{e}}$, and especially $\mathsf{HA}^\omega_{\mathrm{i}}$ (because of $\leq^{\mathrm{i}}$), as sometimes some tweak are necessary. So we carefully check the details below.

1.21. Due to the multiplicity of theories defined above, it is useful to draw a picture clarifying the relation between the languages and theorems of the theories. Let us denote by term(HA^ω) the set of all terms of HA^ω, and by form(HA^ω) the set of all formulas of HA^ω, and analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$. We have:

1. term(HA^ω) = term($\mathsf{WE\text{-}HA}^\omega$) = term($\mathsf{E\text{-}HA}^\omega$) = term($\mathsf{HA}^\omega_{\mathrm{e}}$) = term($\mathsf{HA}^\omega_{\mathrm{i}}$);

2. form(HA^ω) = form($\mathsf{WE\text{-}HA}^\omega$) = form($\mathsf{E\text{-}HA}^\omega$) $\subsetneq$ form($\mathsf{HA}^\omega_{\mathrm{e}}$) $\subsetneq$ form($\mathsf{HA}^\omega_{\mathrm{i}}$) (modulo considering $\forall x \leq^{\mathrm{e}}_\rho t\, A \equiv \forall x \leq^{\mathrm{i}}_\rho t\, A$ and $\exists x \leq^{\mathrm{e}}_\rho t\, A \equiv \exists x \leq^{\mathrm{i}}_\rho t\, A$);

3. for all formulas A of HA^ω we have $\mathsf{HA}^\omega_{\mathrm{i}} \vdash A \Leftrightarrow \mathsf{HA}^\omega_{\mathrm{e}} \vdash A \Leftrightarrow \mathsf{HA}^\omega \vdash A \underset{\not\Leftarrow}{\Rightarrow} \mathsf{WE\text{-}HA}^\omega \vdash A \underset{\not\Leftarrow}{\Rightarrow} \mathsf{E\text{-}HA}^\omega \vdash A$ [15, proposition 11] [39, theorem 3.2].

In figure 1.1 we picture the inclusions and main differences between between HA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

1.22. The next lemma is used to generalise axioms like $x =_0 y \wedge A_{\mathrm{at}}[x/z] \to A_{\mathrm{at}}[y/z]$ from atomic formulas A_{at} to arbitrary formulas A. Roughly speaking, the lemma says that if an axiom holds for atomic formulas, then it holds for all formulas.

1.23 Lemma. Let $\underline{s} \equiv s_1, \ldots, s_n$ and $\underline{t} \equiv t_1, \ldots, t_n$ be tuples of terms of HA^ω and A a formula of HA^ω. If $\mathsf{HA}^\omega \vdash A \to (B_{\mathrm{at}}[\underline{s}/\underline{x}] \leftrightarrow B_{\mathrm{at}}[\underline{t}/\underline{x}])$ for all atomic formulas B_{at} of HA^ω and for all tuples $\underline{x} \equiv x_1, \ldots, x_n$ of variables of HA^ω, then:

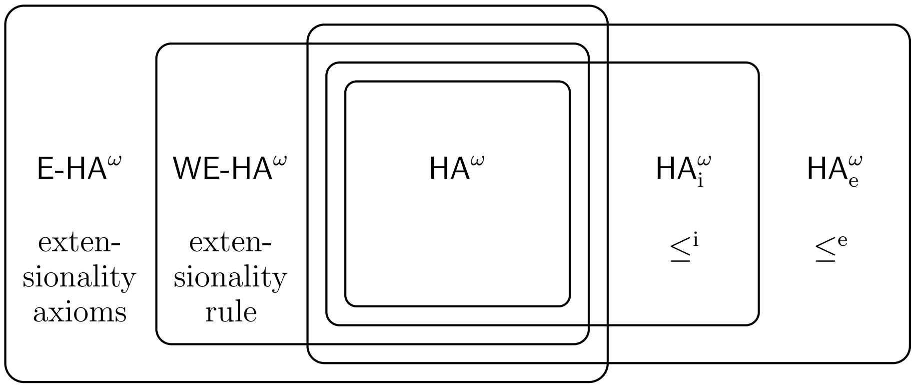

Figure 1.1: inclusions and main differences between HA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_\mathrm{e}$ and $\mathsf{HA}^\omega_\mathrm{i}$.

1. $\mathsf{HA}^\omega \vdash A \to (B[\underline{s}/\underline{x}] \leftrightarrow B[\underline{t}/\underline{x}])$ for all formulas B of HA^ω such that $\mathrm{FV}(A) \cap \mathrm{BV}(B) = \emptyset$ and for all tuples $\underline{x} \equiv x_1, \dots, x_n$ of variables of HA^ω;
2. $\mathsf{HA}^\omega \vdash A \to \underline{s} = \underline{t}$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_\mathrm{e}$ and $\mathsf{HA}^\omega_\mathrm{i}$ [19, *lemas* 28 and 144].

1.24 Proof. Let us do the proof only for $\mathsf{HA}^\omega_\mathrm{i}$; the cases of the other theories are analogous.

1. The proof is by induction on the structure of B. Let us only see the case of $\forall \leq^\mathrm{i}$; the other cases are analogous. Let $\underline{x}'$ be the tuple obtained from $\underline{x}$ by replacing y by a variable $z \notin \mathrm{FV}(B)$. By induction hypothesis we have $\mathsf{HA}^\omega_\mathrm{i} \vdash A \to (B[\underline{s}/\underline{x}'] \leftrightarrow B[\underline{t}/\underline{x}'])$, so $(*_1)$ $\mathsf{HA}^\omega_\mathrm{i} \vdash A \to (\forall y \leq^\mathrm{i} r[\underline{s}/\underline{x}']\, B[\underline{s}/\underline{x}'] \leftrightarrow \forall y \leq^\mathrm{i} r[\underline{s}/\underline{x}']\, B[\underline{t}/\underline{x}'])$ (since $y \notin \mathrm{FV}(A)$ because $\mathrm{FV}(A) \cap \mathrm{BV}(\forall y \leq^\mathrm{i} r\, B) = \emptyset$). By the premise of the lemma we have $(*_2)$ $A \to (y \leq^\mathrm{i} r[\underline{s}/\underline{x}'] \leftrightarrow y \leq^\mathrm{i} r[\underline{t}/\underline{x}'])$. In the following, the last formula is provable by $(*_1)$ and $(*_2)$, so the first formula is also provable:
$$\begin{aligned} A \to ((\forall y \leq^\mathrm{i} r\, B)[\underline{s}/\underline{x}] \leftrightarrow (\forall y \leq^\mathrm{i} r\, B)[\underline{t}/\underline{x}]) &\equiv \\ A \to (\forall y \leq^\mathrm{i} r[\underline{s}/\underline{x}']\, B[\underline{s}/\underline{x}'] \leftrightarrow \forall y \leq^\mathrm{i} r[\underline{t}/\underline{x}']\, B[\underline{t}/\underline{x}']) &\leftrightarrow \\ A \to (\forall y\, (y \leq^\mathrm{i} r[\underline{s}/\underline{x}'] \to B[\underline{s}/\underline{x}']) \leftrightarrow \forall y\, (y \leq^\mathrm{i} r[\underline{t}/\underline{x}'] \to B[\underline{t}/\underline{x}'])). \end{aligned}$$
2. Taking $B :\equiv \underline{s} = \underline{x}$ (with $\underline{x} \notin \mathrm{FV}(\underline{s})$) in point 1 we get $A \to (\underline{s} = \underline{s} \leftrightarrow \underline{s} = \underline{t})$ where $\underline{s} = \underline{s}$ is provable.

1.25. Some axioms and rules of HA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_\mathrm{e}$ and $\mathsf{HA}^\omega_\mathrm{i}$ where formulated with restrictions on the classes of formulas and on the types. We always choose the minimal formulation, that is the formulation in which the classes and the types are as low as possible, because this makes easier to prove the so-called soundness theorems of the proof interpretations. For example, we could have formulated the extensionality rule $(*_1)$ $A_\mathrm{at} \to s =_\rho t \,/\, A_\mathrm{at} \to r[s/x] =_0 r[t/x]$ as $(*_2)$ $A_\mathrm{qf} \to s =_\rho t \,/\, A_\mathrm{qf} \to r[s/x] =_\sigma r[t/x]$ (note that in the latter we have A_qf instead of A_at, and $=_\sigma$ instead of $=_0$), but we chose the minimal $(*_1)$. In the next proposition we show that some minimal axioms and rules imply more liberal formulations.

In the next proposition we also collect some properties about the theories HA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$. Some of these properties are just expected properties, like that $=_\rho$ is an equivalence relation, but anyway we should make sure that they are really provable. Other properties, like that the extensionality axioms can be equivalently replaced by $\forall z\,(z \approx z)$, give us alternative axiomatisations sometimes more convenient to prove the soundness theorems.

1.26 Proposition.

1. The theory HA^ω proves

$$\begin{aligned} A[\Pi_{\rho,\sigma}xy/w] &\leftrightarrow A[x/w], & \Pi_{\rho,\sigma}xy &= x,\\ A[\Sigma_{\rho,\sigma,\tau}xyz/w] &\leftrightarrow A[xz(yz)/w], & \Sigma_{\rho,\sigma,\tau}xyz &= xz(yz),\\ A[\underline{\mathrm{R}}_{\underline{\rho}}0\underline{y}\underline{z}/\underline{w}] &\leftrightarrow A[\underline{y}/\underline{w}], & \underline{\mathrm{R}}_{\underline{\rho}}0\underline{y}\underline{z} &= \underline{y},\\ A[\underline{\mathrm{R}}_{\underline{\rho}}(\mathrm{S}x)\underline{y}\underline{z}/\underline{w}] &\leftrightarrow A[\underline{z}(\underline{\mathrm{R}}_{\underline{\rho}}x\underline{y}\underline{z})x/\underline{w}], & \underline{\mathrm{R}}_{\underline{\rho}}(\mathrm{S}x)\underline{y}\underline{z} &= \underline{z}(\underline{\mathrm{R}}_{\underline{\rho}}x\underline{y}\underline{z})x, \end{aligned}$$

   for all formulas A of HA^ω. Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [15, proposition 2].

2. The theory HA^ω proves:
   (a) $x =_\rho y$;
   (b) $x =_\rho y \to y =_\rho x$;
   (c) $x =_\rho y \wedge y =_\rho z \to x =_\rho z$.

   Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$ [50, remark 3.11.2)], $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

3. The following three theories, with three different formulations of extensionality, are equal [50, remark 3.11.3)] [75, section 2.7.2]:

$$\begin{aligned} \mathsf{E\text{-}HA}^\omega &:= \mathsf{HA}^\omega + \forall \underline{x}^{\underline{\sigma}}, \underline{y}^{\underline{\rho}}, z^{0\underline{\rho}^{\mathrm{t}}}\,(\underline{x} =_{\underline{\rho}} \underline{y} \to z\underline{x} =_0 z\underline{y}),\\ \mathsf{E\text{-}HA}^{\omega\prime} &:= \mathsf{HA}^\omega + \forall x^\sigma, y^\sigma, z^{\rho\sigma}\,(x =_\rho y \to zx =_\sigma zy),\\ \mathsf{E\text{-}HA}^{\omega\prime\prime} &:= \mathsf{HA}^\omega + \forall z^\rho\,(z \approx_\rho z). \end{aligned}$$

4. (a) The theory HA^ω proves

$$x =_0 y \to t[x/z] =_0 t[y/z], \qquad x =_0 y \wedge A[x/z] \to A[y/z]$$

   for all terms t^0 and formulas A of HA^ω. Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [15, proposition 1].

   (b) The rules

$$\frac{A_{\mathrm{qf}} \to s =_\rho t}{A_{\mathrm{qf}} \to r[s/x] =_\sigma r[t/x]}, \qquad \frac{A_{\mathrm{qf}} \to s =_\rho t}{A_{\mathrm{qf}} \wedge A[s/x] \to A[t/x]}$$

   hold in $\mathsf{WE\text{-}HA}^\omega$ [50, remark 3.13]. Analogously for $\mathsf{E\text{-}HA}^\omega$.

(c) The theory $\mathsf{E\text{-}HA}^\omega$ proves

$$x =_\rho y \to t[x/z] =_\sigma t[y/z], \qquad x =_\rho y \wedge A[x/z] \to A[y/z]$$

for all terms t^σ and formulas A of $\mathsf{E\text{-}HA}^\omega$ [50, remark 3.11.2)].

5. (a) The theory HA^ω proves the induction axiom (schema) [50, remark 3.3.2)]

$$A(0) \wedge \forall x\, (A(x) \to A(\mathrm{S}x)) \to \forall x\, A(x).$$

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_\mathrm{e}$ and $\mathsf{HA}^\omega_\mathrm{i}$.

(b) The following double induction rule holds in HA^ω [78, proposition 2.6 in chapter 3]:

$$\frac{A(0,y) \quad A(x,0) \quad A(x,y) \to A(\mathrm{S}x, \mathrm{S}y)}{A(x,y)}.$$

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_\mathrm{e}$ and $\mathsf{HA}^\omega_\mathrm{i}$.

6. For all closed terms t of HA^ω we have $\mathsf{HA}^\omega \vdash t \approx t$ [75, theorem 2.7.3]. Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_\mathrm{e}$ and $\mathsf{HA}^\omega_\mathrm{i}$.

1.27 Proof.

1. Follow from the axioms of Π, Σ and $\underline{\mathrm{R}}$, and lemma 1.23.

2. Let us only prove point 2c; points 2a and 2b are analogous. First we prove the claim for $\rho = 0$: from $x =_0 y \to y =_0 x$ and $y =_0 x \wedge A_\mathrm{at}(y) \to A_\mathrm{at}(x)$ we get $x =_0 y \wedge A_\mathrm{at}(y) \to A_\mathrm{at}(x)$; taking $A_\mathrm{at}(w) :\equiv w =_0 z$ we get $x =_0 y \wedge y =_0 z \to x =_0 z$. The claim for an arbitrary ρ follows from the claim for $\rho = 0$.

3. $\underline{\mathsf{E\text{-}HA}^\omega = \mathsf{E\text{-}HA}^{\omega\prime}}$

   $\underline{\mathsf{E\text{-}HA}^\omega \vdash \forall x^\sigma, y^\sigma, z^{\rho\sigma}\, (x =_\sigma y \to zx =_\rho zy)}$ Taking $\underline{x} \equiv x, \underline{w}$ and $\underline{y} \equiv y, \underline{w}$ in $\forall \underline{x}, \underline{y}, z\, (\underline{x} = \underline{y} \to z\underline{x} =_0 z\underline{y})$ we get $\forall x, y, z, \underline{w}\, (x = y \to zx\underline{w} =_0 zy\underline{w})$, that is $\forall x, y, z\, (x = y \to zx =_\sigma zy)$.

   $\underline{\mathsf{E\text{-}HA}^{\omega\prime} \vdash \forall \underline{x}^{\underline{\rho}}, \underline{y}^{\underline{\rho}}, z^{0\underline{\rho}^\mathrm{t}}\, (\underline{x} =_{\underline{\rho}} \underline{y} \to z\underline{x} =_0 z\underline{y})}$ The proof is by induction on the length n of the tuples $\underline{x}$ and $\underline{y}$. The base case $n = 1$ is trivial, so let us see the induction step. We take arbitrary $\underline{x} \equiv x_1, \ldots, x_{n+1}$, $\underline{y} \equiv y_1, \ldots, y_{n+1}$ and z, assume $\underline{x} = \underline{y}$ and prove $z\underline{x} =_0 z\underline{y}$. From $x_1 = y_1$ we get $zx_1 = zy_1$, so $(*_1)$ $zx_1x_2 \cdots x_{n+1} = zy_1x_2 \cdots x_{n+1}$. We have $x_2 = y_2 \wedge \cdots \wedge x_n = y_n \to zy_1x_2 \cdots x_n =_0 zy_1y_2 \cdots y_n$ by induction hypothesis, so $(*_2)$ $zy_1x_2 \cdots x_n =_0 zy_1y_2 \cdots y_n$ by $\underline{x} = \underline{y}$. From $(*_1)$ and $(*_2)$ we get $z\underline{x} =_0 z\underline{y}$.

$\underline{\mathsf{E\text{-}HA}^{\omega\prime} = \mathsf{E\text{-}HA}^{\omega\prime\prime}}$ It suffices to show that both $\mathsf{E\text{-}HA}^{\omega\prime}$ and $\mathsf{E\text{-}HA}^{\omega\prime\prime}$ prove $x =_\rho y \leftrightarrow x \approx_\rho y$. We do only the proof for $\mathsf{E\text{-}HA}^{\omega\prime\prime}$; the case of $\mathsf{E\text{-}HA}^{\omega\prime}$ is analogous. The proof is by induction on the structure of ρ. The base case is trivial, so let us see the induction step. Using the induction hypothesis in the equivalences and $\mathsf{E\text{-}HA}^{\omega\prime\prime} \vdash y \approx_{\rho\sigma} y$ in the implication, we get $(*)$ $u =_\sigma v \leftrightarrow u \approx_\sigma v \to yu \approx_\rho yv \leftrightarrow yu =_\rho yv$. Using the induction hypothesis in the first equivalence, taking $v := u$ in the left-to-right implication of the second equivalence, and using $(*)$ in the right-to-left implication of the second equivalence, we get

$$\begin{aligned} x \approx_{\rho\sigma} y \equiv \\ \forall u^\sigma, v^\sigma\, (u \approx_\sigma v \to xu \approx_\rho yv) \leftrightarrow \\ \forall u^\sigma, v^\sigma\, (u =_\sigma v \to xu =_\rho yv) \leftrightarrow \\ \forall u^\sigma\, (xu =_\rho yu) \equiv \\ x =_{\rho\sigma} y. \end{aligned}$$

4. (a) Let us prove the claim for formulas: we have $x =_0 y \to (A_{\rm at}[x/z] \leftrightarrow A_{\rm at}[y/z])$ for all $A_{\rm at}$, so by point 1 of lemma 1.23 we get $x =_0 y \to (A[x/z] \leftrightarrow A[y/z])$ (with x and y are free for z in A, which implies $\mathrm{FV}(x =_0 y) \cap \mathrm{BV}(A) = \emptyset$). To prove the claim for terms, we apply the claim for formulas to $A :\equiv t[x/z] =_0 t$.

   (b) Let us prove the claim for terms. We will prove in theorem 1.44 that $A_{\rm qf}$ is equivalent in HA^ω to an atomic formula $A_{\rm at}$, so we replace $A_{\rm qf}$ by $A_{\rm at}$. Let $\underline{y}$ be a tuple of variables such that $r\underline{y}$ has type 0, x does not occur in $\underline{y}$ and $\underline{y} \notin \mathrm{FV}(A_{\rm at})$. Using the extensionality rule in the first implication we get

$$\begin{aligned} \mathsf{WE\text{-}HA}^\omega \vdash A_{\rm at} \to s =_\rho t \ &\Rightarrow \\ \mathsf{WE\text{-}HA}^\omega \vdash A_{\rm at} \to (r\underline{y})[s/x] =_0 (r\underline{y})[t/x] \ &\Rightarrow \\ \mathsf{WE\text{-}HA}^\omega \vdash A_{\rm qf} \to r[s/x]\underline{y} =_0 r[t/x]\underline{y} \ &\Rightarrow \\ \mathsf{WE\text{-}HA}^\omega \vdash A_{\rm qf} \to \forall \underline{y}\, (r[s/x]\underline{y} =_0 r[t/x]\underline{y}) \ &\Rightarrow \\ \mathsf{WE\text{-}HA}^\omega \vdash A_{\rm qf} \to r[s/x] =_\sigma r[t/x]. \end{aligned}$$

   Let us prove the claim for formulas. It suffices to prove it for atomic formulas $A \equiv r_1 =_0 r_2$ and $A \equiv \perp$ (the latter being trivial) by point 1 of lemma 1.23. If $\mathsf{WE\text{-}HA}^\omega \vdash A_{\rm at} \to s =_\rho t$, then $\mathsf{WE\text{-}HA}^\omega \vdash A_{\rm at} \to r_i[s/x] =_0 r_i[t/x]$ for $i = 1, 2$, so $\mathsf{WE\text{-}HA}^\omega \vdash A_{\rm at} \to ((r_1 =_0 r_2)[s/x] \leftrightarrow (r_1 =_0 r_2)[t/x])$.

   (c) Let us prove the claim for terms by induction on the structure of t.

   Base case If t is a variable w, then $x = y \to t[x/z] = t[y/z]$ is provable because its conclusion is $x = y$ if $w \equiv z$, and $w = w$ if $w \not\equiv z$. Analogously for 0, S, Π, Σ and $\underline{\mathrm{R}}$.

   Induction step Let us assume $x = y$ and prove $(*_1)$ $(st)[x/z] = (st)[y/z]$. By induction hypothesis we have $s[x/z] = s[y/z]$ and $t[x/z] = t[y/z]$. So $(*_2)$ $s[x/z]t[x/z] = s[x/z]t[y/z]$ (by extensionality formulated as $x = y \to zx = zy$) and $(*_2)$ $s[x/z]t[y/z] = s[y/z]t[y/z]$. From $(*_2)$ and $(*_3)$ we get $(*_1)$.

   The claim for formulas follows from the claim for terms analogously to point 4b.

5. (a) Let $B(x) :\equiv A(0) \wedge \forall x\, (A(x) \to A(\mathrm{S}x)) \to A(x)$. We can prove $B(0)$ and $B(x) \to B(\mathrm{S}x)$, so by the induction rule we prove $B(x)$, thus $\forall x\, B(x)$, that is the induction axiom.

   (b) We assume the premises of the rule and prove its conclusion by induction on x.

   $\underline{A(0, y)}$ It is the first premise.

   $\underline{A(x, y) \to A(\mathrm{S}x, y)}$ We assume $A(x, y)$ and prove $A(\mathrm{S}x, y)$ by induction on y.

   $\underline{A(\mathrm{S}x, 0)}$ It is an instance of the second premise.

   $\underline{A(\mathrm{S}x, y) \to A(\mathrm{S}x, \mathrm{S}y)}$ Follows from $A(x, y)$ and the third premise.

6. The proof is by induction on the structure of t. The induction step is easy, so let us see the base case.

   $\underline{\mathrm{S}}$ We have $\forall u, v\, (u =_0 v \to \mathrm{S}u =_0 \mathrm{S}v)$, that is $\mathrm{S} \approx \mathrm{S}$. Analogously for 0.

   $\underline{\underline{\mathrm{R}}}$ Note $x \approx_\rho y \leftrightarrow \forall \underline{u}, \underline{v}\, (\underline{u} \approx_{\underline{\tau}} \underline{v} \to x\underline{u} \approx_\sigma y\underline{v})$ where $\rho = \sigma\underline{\tau}^{\mathrm{t}}$. Say $\underline{\mathrm{R}} \equiv \mathrm{R}_1, \ldots, \mathrm{R}_n$. Let us prove $\mathrm{R}_i \approx \mathrm{R}_i$ by proving $A(x) :\equiv \forall x', \underline{y}, \underline{y}', \underline{z}, \underline{z}'\, (x =_0 x' \wedge \underline{y} \approx \underline{y}' \wedge \underline{z} \approx \underline{z}' \to \bigwedge_{i=1}^n \mathrm{R}_i x\underline{y}\underline{z} \approx \mathrm{R}_i x'\underline{y}'\underline{z}')$ by induction on x.

   Base case: The premise of $A(0)$ implies $x' =_0 0$, so the conclusion of $A(0)$ is equivalent to $\bigwedge_{i=1}^n y_i \approx y_i'$, which is implied by the premise.

   Induction step: The premise of $A(\mathrm{S}x)$ implies $x' =_0 \mathrm{S}x$, so the conclusion of $A(\mathrm{S}x)$ is equivalent to $\bigwedge_{i=1}^n z_i(\underline{\mathrm{R}}x\underline{y}\underline{z})x \approx z_i'(\underline{\mathrm{R}}x\underline{y}'\underline{z}')x$, which is implied by the premise together with the induction hypothesis $A(x)$.

   Analogous for Π and Σ.

1.6 Term reduction

1.28. Every natural number n can be represented in HA^ω by a closed term of type 0, namely the numeral $\bar{n} :\equiv \mathrm{S} \cdots \mathrm{S}0$. But is the reciprocal true: every closed term of type 0 represents a natural number? If it is not, then HA^ω is not faithfully capturing the natural numbers; it is also talking about some foreign numbers.

We are going to prove that the reciprocal is indeed true. Our strategy to prove this has two main ideas.

1. Informally speaking, the axioms of Σ say $\Sigma xyz = xz(yz)$, so they put Σxyz and $xz(yz)$ at the same level. However, we think of $\Sigma xyz = xz(yz)$ as meaning "Σxyz reduces to $xz(yz)$", not "$xz(yz)$ reduces to Σxyz", so the axioms suggest a direction. Analogously for Π and $\underline{\mathrm{R}}$.

   Given a term t, we can reduce in t all occurrences of Πxy, Σxyz and $\underline{\mathrm{R}}x\underline{y}\underline{z}$, getting a term t^{n} that says the same that t and cannot be reduced any further. We think of t^{n} as a normal form of t.

2. We show that if t is closed and has type 0, then t^{n} is a numeral $\bar{n}$.

Combining the two points above, as schematically in

$$t \rightsquigarrow t^{\mathrm{n}} \rightsquigarrow \bar{n},$$

we conclude that every closed term t of type 0 represents a numeral $\bar{n}$.

1.29 Definition. Let p and q be terms of HA^ω.

1. We write $p \succ_1 q$ if and only if q is obtained from p by replacing exactly one subterm of p of the form

$$\Pi rs, \qquad \Sigma rst, \qquad \mathrm{R}_i 0 \underline{s}\underline{t}, \qquad \mathrm{R}_i(\mathrm{S}r)\underline{s}\underline{t}$$

(where $\underline{\mathrm{R}} \equiv \mathrm{R}_1, \ldots, \mathrm{R}_n$, $\underline{s} \equiv s_1, \ldots, s_n$ and $\underline{t} \equiv t_1, \ldots, t_n$) by, respectively,

$$r, \qquad rt(st), \qquad s_i, \qquad t_i(\underline{\mathrm{R}}r\underline{s}\underline{t})r.$$

2. We say that p *reduces* to q, and write $p \succeq q$, if and only if there exists a sequence $p \succ_1 \cdots \succ_1 q$ (possibly $p \equiv q$).

3. We say that p is *normal* if and only if there is no term q such that $p \succ_1 q$.

4. We call *normal form* of p to a normal term p^{n} such that $p \succeq p^{\mathrm{n}}$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [75, section 2.2.2].

1.30 Theorem.

1. Every term of HA^ω reduces to a unique normal form [78, proposition 2.10 and section 2.22 in chapter 9] [75, theorem 2.2.23].

2. Every closed normal term of HA^ω of type 0 is a numeral [78, proposition 2.5(i) in chapter 9] [75, lemma 2.2.8].

3. For all closed terms t^0 of HA^ω, there exists a unique numeral $\bar{n}$ such that $\mathsf{HA}^\omega \vdash t =_0 \bar{n}$ [78, corollary 2.12 in chapter 9] [75, theorem 2.2.9].

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

1.31 Proof. We only do the proof for HA^ω; the cases of the other theories are analogous. We only prove the existence of the normal form; the references given contain proofs of the rest. Let N be the set of terms of HA^ω that reduce to normal form. Let C be the union, for all types ρ, of Tait's computability predicates C_ρ [70, pages 198–199] defined by recursion on the structure of ρ by

1. C_0 is the set of all $t^0 \in \mathrm{N}$;

2. $\mathrm{C}_{\rho\sigma}$ is the set of all $t^{\rho\sigma} \in \mathrm{N}$ such that for all $s \in \mathrm{C}_\sigma$ we have $ts \in \mathrm{C}_\rho$.

Let us make two remarks.

1. Let $\rho = \sigma\sigma_n \cdots \sigma_1$. We have $t \in \mathrm{C}_\rho$ if and only if for all $t_i \in \mathrm{C}_{\sigma_i}$ $(i = 1, \ldots, n)$ we have $t, tt_1, \ldots, tt_1 \cdots t_{n-1} \in \mathrm{N}$ and $tt_1 \cdots t_n \in \mathrm{C}_\sigma$ (this contains a tiny patch to the literature).

2. If $s \succeq t$ and $t \in \mathrm{C}$, then $s \in \mathrm{C}$.

We want to prove that every term is in N; it suffices to prove that every term is in C by induction on the structure of the term. The induction step is easy, so let us see the base case.

$\underline{x}$ Accordingly to remark 1 with $\sigma = 0$, we prove that for all $t_1, \ldots, t_n \in \mathrm{C}$ we have $x, xt_1, \ldots, xt_1 \cdots t_n \in \mathrm{N}$. Since $t_i \in \mathrm{C}$, then $t_i \succeq t_i^{\mathrm{n}}$, so $x \in \mathrm{N}$, $xt_1 \succeq xt_1^{\mathrm{n}} \in \mathrm{N}$, and so on. Analogously for 0 and S.

$\underline{\Sigma}$ We prove that for all $r, s, t \in \mathrm{C}$ we have $(*_1)$ $\Sigma, \Sigma r, \Sigma rs \in \mathrm{N}$ and $(*_2)$ $\Sigma rst \in \mathrm{C}$. To prove $(*_1)$ we note $\Sigma \in \mathrm{N}$, $\Sigma r \succeq \Sigma r^{\mathrm{n}} \in \mathrm{N}$ and $\Sigma rs \succeq \Sigma r^{\mathrm{n}} s^{\mathrm{n}} \in \mathrm{N}$. To prove $(*_2)$ we note $\Sigma rst \succeq rt(st) \in \mathrm{C}$.

$\underline{\mathrm{R}_i}$ Say $\underline{\mathrm{R}} \equiv \mathrm{R}_1, \ldots, \mathrm{R}_n$, $\underline{s} \equiv s_1, \ldots, s_n$ and $\underline{t} \equiv t_1, \ldots, t_n$. We prove, simultaneously for $i = 1, \ldots, n$, that for all $r, \underline{s}, \underline{t} \in \mathrm{C}$ we have $(*_1)$ $\mathrm{R}_i, \mathrm{R}_i r, \mathrm{R}_i rs_1, \ldots, \mathrm{R}_i rs_1 \cdots s_n$, $\mathrm{R}_i rs_1 \cdots s_n t_1, \ldots, \mathrm{R}_i rs_1 \cdots s_n t_1 \cdots t_{n-1} \in \mathrm{N}$ and $(*_2)$ $\mathrm{R}_i r\underline{s}\underline{t} \in \mathrm{C}$. We have $r^{\mathrm{n}} \equiv S^k r'$ for some $k \in \mathbb{N}$ and $r' \in \mathrm{N}$ not of the form $r' \equiv \mathrm{S}r''$. To prove $(*_1)$ we note $\mathrm{R}_i \in \mathrm{N}$, $\mathrm{R}_i r \succeq \mathrm{R}_i r^{\mathrm{n}} \in \mathrm{N}$, and so on. The proof of $(*_2)$ is by induction on k.

<u>Base case</u> If $r' \equiv 0$, then $\mathrm{R}_i r\underline{s}\underline{t} \succeq \mathrm{R}_i 0\underline{s}\underline{t} \succeq s_i \in \mathrm{C}$. If $r' \not\equiv 0$, then $\mathrm{R}_i r\underline{s}\underline{t} \succeq \mathrm{R}_i r'\underline{s}^{\mathrm{n}}\underline{t}^{\mathrm{n}} \in \mathrm{C}$ since, accordingly to remark 1 with $\sigma = 0$, for all $q_1, \ldots, q_m \in \mathrm{C}$ we have $\mathrm{R}_i r'\underline{s}^{\mathrm{n}}\underline{t}^{\mathrm{n}} \in \mathrm{N}$, $\mathrm{R}_i r'\underline{s}^{\mathrm{n}}\underline{t}^{\mathrm{n}} q_1 \succeq \mathrm{R}_i r'\underline{s}^{\mathrm{n}}\underline{t}^{\mathrm{n}} q_1^{\mathrm{n}} \in \mathrm{N}$, and so on.

<u>Induction step</u> We have $\mathrm{R}_i r\underline{s}\underline{t} \succeq \mathrm{R}_i(\mathrm{S}^{k+1}r')\underline{s}\underline{t} \succeq t_i(\underline{\mathrm{R}}(\mathrm{S}^k r')\underline{s}\underline{t})(\mathrm{S}^k r') \in \mathrm{C}$.

1.7 λ-abstraction

1.32. Now we are going to see that given a term $t(x)$, we can construct a term $\lambda x \,.\, t(x)$ that behaves like the function $x \mapsto t(x)$. This is important so that HA^ω can talk not only about terms like $2x$, but also about functions like $x \mapsto 2x$. The sole role of the constants Π and Σ is to construct the term $\lambda x \,.\, t(x)$:

1. the role of Π is to construct the term $\lambda x \,.\, c :\equiv \Pi c$ for a constant c;

2. the role of Σ is to combine two terms $\lambda x \,.\, s$ for s and $\lambda x \,.\, t$ for t into a new term $\lambda x \,.\, st :\equiv \Sigma(\lambda x \,.\, s)(\lambda x \,.\, t)$ for st.

1.33 Definition.

1. Let t^ρ be a term and x^σ a variable of HA^ω. We define the term $(\lambda x \,.\, t)^{\rho\sigma}$ [75, theorem 1.6.8] [50, lemma 3.15] of HA^ω (essentially) by recursion on the structure of t by

   (a) $\lambda x^\sigma \,.\, t^\rho :\equiv \Pi_{\rho,\sigma} t$ if $x \notin \mathrm{FV}(t)$;

(b) $\lambda x^\sigma \,.\, x^\sigma :\equiv \Sigma_{\sigma,\sigma 0,\sigma}\Pi_{\sigma,\sigma 0}\Pi_{\sigma,0}$;

(c) $\lambda x^\sigma \,.\, (s^{\rho\tau}t^\tau) :\equiv \Sigma_{\sigma,\tau,\rho}(\lambda x \,.\, s)(\lambda x \,.\, t)$ if $x \in \mathrm{FV}(st)$.

2. We extent the definition to tuples of

   (a) variables $\underline{x} \equiv x_1, \ldots, x_m$ by $\lambda \underline{x} \,.\, t :\equiv \lambda x_1 \,.\, \cdots \lambda x_m \,.\, t$ [50, page 50];

   (b) terms $\underline{t} \equiv t_1, \ldots, t_n$ by $\lambda \underline{x} \,.\, \underline{t} :\equiv \lambda \underline{x} \,.\, t_1, \ldots, \lambda \underline{x} \,.\, t_n$.

3. We adopt the following convention to save on parentheses:

   (a) $\lambda \underline{x} \,.\, rs$ means $\lambda \underline{x} \,.\, (rs)$, not $(\lambda \underline{x} \,.\, r)s$;

   (b) $\lambda \underline{x} \,.\, t[\underline{s}/\underline{y}]$ means $\lambda \underline{x} \,.\, (t[\underline{s}/\underline{y}])$, not $(\lambda \underline{x} \,.\, t)[\underline{s}/\underline{y}]$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

1.34 Theorem. For all tuples of terms $\underline{q} = q_1, \ldots, q_m$ and $\underline{t} = t_1, \ldots, t_n$, tuples of variables $\underline{x} = x_1, \ldots, x_m$ and $\underline{y} = y_1, \ldots, y_n$, and formulas A of HA^ω, we have:

1. $\mathsf{HA}^\omega \vdash A[(\lambda \underline{x} \,.\, \underline{t})\underline{q}/\underline{y}] \leftrightarrow A[\underline{t}[\underline{q}/\underline{x}]/\underline{y}]$ [75, section 1.6.15] [19, *teorema* 43];

2. $\mathsf{HA}^\omega \vdash (\lambda \underline{x} \,.\, \underline{t})\underline{q} = \underline{t}[\underline{q}/\underline{x}]$ [75, theorem 1.6.8] [50, lemma 3.15].

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

1.35 Proof. We sketch the proof for HA^ω; the cases of the other theories are analogous. By lemma 1.23 it suffices to prove the claim for atomic formulas.

1. First we prove the claim for $m = n = 1$ by induction on the structure of t. If $x \in \mathrm{FV}(rs)$, then using induction hypothesis in the second equivalence we get

$$\begin{aligned} A_{\mathrm{at}}[(\lambda x \,.\, st)q/y] &\equiv \\ A_{\mathrm{at}}[\Sigma(\lambda x \,.\, s)(\lambda x \,.\, t)q/y] &\leftrightarrow \\ A_{\mathrm{at}}[((\lambda x \,.\, s)q)((\lambda x \,.\, t)q)/y] &\leftrightarrow \\ A_{\mathrm{at}}[s[q/x]t[q/x]/y] &\equiv \\ A_{\mathrm{at}}[(st)[q/x]/y]&. \end{aligned}$$

   Analogously for $x \notin \mathrm{FV}(t)$ and for $t \equiv x$.

2. Now we consider a tuple $\underline{x}' \equiv x'_1, \ldots, x'_m$ of variables such that $\underline{x}, \underline{x}'$ are distinct and $\underline{x}' \notin \mathrm{FV}(t, \underline{q}) \cup \mathrm{FV}(A_{\mathrm{at}})$, and remark:

   (a) $(\lambda x'_i \,.\, t)[q_j/x'_j] \equiv \lambda x'_i \,.\, t[q_j/x'_j]$ for $i \neq j$ [19, *lema* 37];

   (b) $t[\underline{q}/\underline{x}] \equiv t[\underline{x}'/\underline{x}][q_1/x'_1] \cdots [q_m/x'_m]$ [19, *lema* 39.1];

   (c) $A_{\mathrm{at}}[\underline{q}/\underline{x}] \equiv A_{\mathrm{at}}[\underline{x}'/\underline{x}][q_1/x'_1] \cdots [q_m/x'_m]$ [19, *lema* 39.2];

   (d) $\lambda \underline{x} \,.\, t \equiv \lambda \underline{x}' \,.\, t[\underline{x}'/\underline{x}]$ [19, *lema* 41].

3. Now we generalise to an arbitrary m [19, *teorema* 43] (a common small omission in the literature). Actually, we argue for $m = 2$ since the argument for $m > 2$ is just an iteration of the argument for $m = 2$:

$$\begin{aligned}
A_{\mathrm{at}}[(\lambda x_1, x_2 \,.\, t)q_1 q_2/y] &\equiv && \text{(by 2d)}\\
A_{\mathrm{at}}[(\lambda x_1', x_2' \,.\, t[x_1', x_2'/x_1, x_2])q_1 q_2/y] &\equiv &&\\
A_{\mathrm{at}}[(\lambda x_1' \,.\, (\lambda x_2' \,.\, t[x_1', x_2'/x_1, x_2]))q_1 q_2/y] &\leftrightarrow && \text{(by 1)}\\
A_{\mathrm{at}}[(\lambda x_2' \,.\, t[x_1', x_2'/x_1, x_2])[q_1/x_1']q_2/y] &\equiv && \text{(by 2a)}\\
A_{\mathrm{at}}[(\lambda x_2' \,.\, t[x_1', x_2'/x_1, x_2][q_1/x_1'])q_2/y] &\leftrightarrow && \text{(by 1)}\\
A_{\mathrm{at}}[t[x_1', x_2'/x_1, x_2][q_1/x_1'][q_2/x_2']/y] &\equiv && \text{(by 2b)}\\
A_{\mathrm{at}}[t[q_1, q_2/x_1, x_2]/y]. &&&
\end{aligned}$$

4. Now we generalise to an arbitrary n [19, *teorema* 43] (a common small omission in the literature). Actually, we argue for $n = 2$, since the argument for $n > 2$ is just an iteration of the argument for $n = 2$:

$$\begin{aligned}
A_{\mathrm{at}}[(\lambda \underline{x} \,.\, t_1)\underline{q}, (\lambda \underline{x} \,.\, t_2)\underline{q}/y_1, y_2] &\equiv && \text{(by 2c)}\\
A_{\mathrm{at}}[y_1', y_2'/y_1, y_2][(\lambda \underline{x} \,.\, t_1)\underline{q}/y_1'][(\lambda \underline{x} \,.\, t_2)\underline{q}/y_2'] &\leftrightarrow && \text{(by 3)}\\
A_{\mathrm{at}}[y_1', y_2'/y_1, y_2][t_1[\underline{q}/\underline{x}]/y_1'][t_2[\underline{q}/\underline{x}]/y_1'] &\equiv && \text{(by 2c)}\\
A_{\mathrm{at}}[t_1[\underline{q}/\underline{x}], t_2[\underline{q}/\underline{x}]/y_1, y_2]. &&&
\end{aligned}$$

1.8 Terms for primitive recursive functions

1.36. Now we show that every primitive recursive function can be represented in HA^ω by a term of HA^ω. In particular, we can introduce in HA^ω the operations $+$ and $\cdot$, and the order relation $\leq$, which are conspicuously missing in our arithmetic. The idea to represent primitive recursive functions by terms is fairly simple.

1. Primitive recursive function are constructed
   (a) from the basic functions 0, S and p^n_k;
   (b) by means of (generalised) composition;
   (c) and primitive recursion.
2. We can represent
   (a) the basic functions 0, S and p^n_k the terms 0, S and $\lambda x_1, \ldots, x_n \,.\, x_k$;
   (b) composition by term application st of two terms s and t;
   (c) and primitive recursion using the recursor R_0.

1.37 Definition.

1. In the following, let all functions be primitive recursive, and f be introduced by the equalities stated. To each derivation of a primitive recursive function $f\colon \mathbb{N}^n \to \mathbb{N}$ we assign a closed term $\mathrm{T}f$ of HA^ω of type $0 \cdots 0$ ($n+1$ times) by recursion on the derivation of f by [75, paragraph 1.6.9] [19, *teorema* 47.1]:

(a) $\mathrm{T}0 :\equiv 0$;

(b) $\mathrm{TS} :\equiv \mathrm{S}$;

(c) $\mathrm{Tp}^n_k :\equiv \lambda x_1, \ldots, x_n \,.\, x_k$;

(d) if $f(\underline{x}) = g(h_1(\underline{x}), \ldots, h_n(\underline{x}))$, then $\mathrm{T}f :\equiv \lambda \underline{x} \,.\, \mathrm{T}g(\mathrm{T}h_1\,\underline{x}) \cdots (\mathrm{T}h_n\,\underline{x})$;

(e) if $\begin{cases} f(0,\underline{x}) = g(\underline{x}) \\ f(y+1,\underline{x}) = h(y, f(y,\underline{x}), \underline{x}) \end{cases}$, then $\mathrm{T}f :\equiv \lambda y, \underline{x} \,.\, \mathrm{R}_0 y(\mathrm{T}g\,\underline{x})(\lambda z, y \,.\, \mathrm{T}h\, yz\underline{x})$.

2. Let $f \in \{+, \mathrm{z}, \cdot, \overline{\mathrm{sg}}, \mathrm{pd}, \dot{-}, |\cdot - \cdot|, \max_0\}$. We denote by just f the term $\mathrm{T}f$ of HA^ω assigned to derivation of the function f given in table 1.4 [50, page 45] [75, section 1.3.9].

$+$	$x + 0 = \mathrm{p}^1_1(x)$ $x + (y+1) = \mathrm{S}(\mathrm{p}^3_2(y, x+y, x))$
z	$\mathrm{z}(0) = 0$ $\mathrm{z}(x+1) = \mathrm{p}^2_1(x, \mathrm{z}(x))$
$\cdot$	$x \cdot 0 = \mathrm{z}(x)$ $x \cdot (y+1) = \mathrm{p}^3_2(x, x\cdot y, x) + \mathrm{p}^3_3(y, x \cdot y, x)$
$\overline{\mathrm{sg}}$	$\overline{\mathrm{sg}}(0) = \mathrm{S}(0)$ $\overline{\mathrm{sg}}(x+1) = \mathrm{z}(\mathrm{p}^2_1(x, \overline{\mathrm{sg}}\,x))$
pd	$\mathrm{pd}(0) = 0$ $\mathrm{pd}(x+1) = \mathrm{p}^2_1(x, \mathrm{pd}\,x)$
$\dot{-}$	$x \dot{-} 0 = \mathrm{p}^1_1(x)$ $x \dot{-} (y+1) = \mathrm{pd}(\mathrm{p}^3_2(y, x \dot{-} y, x))$
$\lvert\cdot - \cdot\rvert$	$\lvert x - y\rvert = (x \dot{-} y) + (\mathrm{p}^2_2(x,y) \dot{-} \mathrm{p}^2_1(x,y))$
$\max_0$	$\max_0(x,y) = \mathrm{p}^2_1(x,y) + (\mathrm{p}^2_2(x,y) \dot{-} \mathrm{p}^2_1(x,y))$

Table 1.4: derivations of the primitive recursive functions mentioned in definition 1.37.

3. We define $s \leq_0 t :\equiv s \dot{-} t =_0 0$, where s and t are terms of HA^ω [78, definition 2.7 in chapter 3] [19, *definição* 128].

4. We define the term $\max_\rho$ of HA^ω by recursion on the structure of ρ by [15, section 2.1]:

(a) $\max_0$ is already defined;

(b) $\max_{\rho\sigma} :\equiv \lambda x^{\rho\sigma}, y^{\rho\sigma}, z^\sigma \,.\, \max_\rho(xz)(yz)$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ or $\mathsf{HA}^\omega_{\mathrm{i}}$.

1.38. Just to be sure that the term $\mathrm{T}f$ behaves (inside HA^ω) like the function f does (on $\mathbb{N}$), we state the following proposition.

1.39 Proposition. In the following, let all functions be primitive recursive, and f be introduced by the equalities stated. We have

1. $\mathsf{HA}^\omega \vdash \mathrm{T}0 =_0 0$;
2. $\mathsf{HA}^\omega \vdash \mathrm{TS}\, x =_0 \mathrm{S}x$;
3. $\mathsf{HA}^\omega \vdash \mathrm{Tp}^n_k\, x_1 \cdots x_n =_0 x_k$;
4. if $f(\underline{x}) = g(h_1(\underline{x}), \dots, h_n(\underline{x}))$, then $\mathsf{HA}^\omega \vdash \mathrm{T}f\, \underline{x} =_0 \mathrm{T}g(\mathrm{T}h_1\, \underline{x}) \cdots (\mathrm{T}h_n\, \underline{x})$;
5. if $\begin{cases} f(0,\underline{x}) = g(\underline{x}) \\ f(y+1,\underline{x}) = h(y, f(y,\underline{x}), \underline{x}) \end{cases}$, then $\begin{cases} \mathsf{HA}^\omega \vdash \mathrm{T}f\, 0\underline{x} =_0 \mathrm{T}g\, \underline{x} \\ \mathsf{HA}^\omega \vdash \mathrm{T}f\, (\mathrm{S}y)\underline{x} =_0 \mathrm{T}h\, y(\mathrm{T}f\, y\underline{x})\underline{x} \end{cases}$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [19, *teorema* 47.2].

1.40 Proof. Let us only prove $\mathsf{HA}^\omega \vdash \mathrm{T}f\, (\mathrm{S}y)\underline{x} =_0 \mathrm{T}h\, y(\mathrm{T}f\, y\underline{x})\underline{x}$; the remaining claims are analogous. We have

$$\begin{aligned} &\mathrm{T}f\,(\mathrm{S}y)\underline{x} \equiv \\ (\lambda y, \underline{x}\,.\,\mathrm{R}_0 y(\mathrm{T}g\,\underline{x})(\lambda z, y\,.\,\mathrm{T}h\,yz\underline{x}))(\mathrm{S}y)\underline{x} &=_0 \\ \mathrm{R}_0(Sy)(\mathrm{T}g\,\underline{x})(\lambda z, y\,.\,\mathrm{T}h\,yz\underline{x}) &=_0 \\ (\lambda z, y\,.\,\mathrm{T}h\,yz\underline{x})(\underbrace{\mathrm{R}_0 y(\mathrm{T}g\,\underline{x})(\lambda z, y\,.\,\mathrm{T}h\,yz\underline{x})}_{=_0 \mathrm{T}f\,y\underline{x}})y &=_0 \\ \mathrm{T}h\,y(\mathrm{T}f\,y\underline{x})\underline{x}. \end{aligned}$$

1.9 Characteristic terms for quantifier-free formulas

1.41. Now we are going to see that each quantifier-free formula A_{qf} has a characteristic term $\chi_{A_{\mathrm{qf}}}$ such that $A_{\mathrm{qf}} \leftrightarrow \chi_{A_{\mathrm{qf}}} =_0 0$. These terms are important for two reasons:

1. they play a main role in interpreting the axiom $A \to A \wedge A$ with Gödel's functional interpretation;
2. they are used to show that HA^ω (despite being an intuitionistic theory) proves the law of excluded middle for quantifier-free formulas.

1.42. The idea to construct the terms is quite simple: we replace the logical operations $\wedge$, $\vee$ and $\to$ on formulas by the arithmetic operations $+$, $\cdot$ and $\overline{\mathrm{sg}}$ on terms. For example, if we already have characteristic terms χ_A for A and χ_B for B, then we can construct the characteristic term $\chi_{A\wedge B} :\equiv \chi_A + \chi_B$ for $A \wedge B$:

$$\underbrace{A \wedge B}_{\text{logical operation on formulas}} \quad\leftrightarrow\quad \chi_A =_0 0 \wedge \chi_B =_0 0 \quad\leftrightarrow\quad \underbrace{\chi_A + \chi_B}_{\text{arithmetic operation on terms}} =_0 0.$$

1.43 Definition. Let A_{qf} be a quantifier-free formula of HA^ω. We define a term $\chi_{A_{\mathrm{qf}}}$ of HA^ω with $\mathrm{FV}(\chi_{A_{\mathrm{qf}}}) = \mathrm{FV}(A)$, called *characteristic term* of A_{qf}, by recursion on the structure of A_{qf} by:

1. $\chi_\bot :\equiv \mathrm{S}0$;
2. $\chi_{s=_0 t} :\equiv |s - t|$;
3. $\chi_{A_{\mathrm{qf}} \wedge B_{\mathrm{qf}}} :\equiv \chi_{A_{\mathrm{qf}}} + \chi_{B_{\mathrm{qf}}}$;
4. $\chi_{A_{\mathrm{qf}} \vee B_{\mathrm{qf}}} :\equiv \chi_{A_{\mathrm{qf}}} \cdot \chi_{B_{\mathrm{qf}}}$;
5. $\chi_{A_{\mathrm{qf}} \to B_{\mathrm{qf}}} :\equiv (\overline{\mathrm{sg}}\, \chi_{A_{\mathrm{qf}}}) \cdot \chi_{B_{\mathrm{qf}}}$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [50, proposition 3.8].

1.44 Theorem. For all quantifier-free formulas A_{qf} of HA^ω, we have $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}} \leftrightarrow \chi_{A_{\mathrm{qf}}} =_0 0$. Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [50, proposition 3.8].

1.45 Proof.

First, we prove four properties of $+$, $\cdot$, $\overline{\mathrm{sg}}$, and $|\cdot - \cdot|$ by double induction on x and y [50, lemma 3.7].

$\underline{A(x,y) :\equiv x + y =_0 0 \leftrightarrow x =_0 0 \wedge y =_0 0}$

$\underline{A(0,y)}$ It is provable because we can prove $0 + y =_0 y$ by induction on y.

$\underline{A(x,0)}$ It is provable because $x + 0 =_0 x$.

$\underline{A(x,y) \to A(\mathrm{S}x, \mathrm{S}y)}$ It is provable because $\mathrm{S}x + \mathrm{S}y =_0 \mathrm{S}(\mathrm{S}x + y) \neq_0 0$, $\mathrm{S}x \neq_0 0$ and $\mathrm{S}y \neq_0 0$.

$\underline{B(x,y) :\equiv x \cdot y =_0 0 \leftrightarrow x =_0 0 \vee y =_0 0}$ Analogously to $A(x,y)$.

$\underline{C(x,y) :\equiv (\overline{\mathrm{sg}}\, x) \cdot y =_0 0 \leftrightarrow (x =_0 0 \to y =_0 0)}$

$\underline{C(0,y)}$ It is provable because $\overline{\mathrm{sg}}\, 0 \cdot y =_0 (\mathrm{S}0) \cdot y$ and we can prove $(\mathrm{S}0) \cdot y =_0 y$ by induction on y.

$\underline{C(x,0)}$ It is provable because $\overline{\mathrm{sg}}\, x \cdot 0 =_0 0$.

$\underline{C(x,y) \to C(\mathrm{S}x, \mathrm{S}y)}$ It is provable because $\overline{\mathrm{sg}}(\mathrm{S}x) \cdot \mathrm{S}y =_0 0 \cdot \mathrm{S}y =_0 0$.

$\underline{D(x,y) :\equiv |x - y| =_0 0 \leftrightarrow x =_0 y}$

$\underline{D(0,y)}$ It is provable because $|0 - y| =_0 (0 \dot{-} y) + (y \dot{-} 0) =_0 0 + y =_0 y$ since we can prove $0 \dot{-} y =_0 0$ by induction on y.

$\underline{D(x,0)}$ Analogous to $D(0,y)$.

$\underline{D(x,y) \to D(\mathrm{S}x, \mathrm{S}y)}$ It is provable because $|\mathrm{S}x - \mathrm{S}y| =_0 (\mathrm{S}x \dot{-} \mathrm{S}y) + (\mathrm{S}y \dot{-} \mathrm{S}x) =_0 (x \dot{-} y) + (y \dot{-} x) =_0 |x - y|$ since we can prove $\mathrm{pd}(\mathrm{S}x \dot{-} 0) =_0 x$ by induction on x and then $\mathrm{S}x \dot{-} \mathrm{S}y =_0 x \dot{-} y$ by induction on y.

Finally, using $A(x,y)$, $B(x,y)$, $C(x,y)$ and $D(x,y)$, it is easy to prove the claim of the theorem by induction on the structure of A_{qf}.

1.10 Definition by quantifier-free cases

1.46. Now we are going to show that given terms r^ρ and s^ρ and a quantifier-free formula A_{qf}, we can define a term t^ρ by cases by

$$t =_\rho \begin{cases} r & \text{if} \quad A_{\mathrm{qf}} \\ s & \text{if} \ \neg A_{\mathrm{qf}} \end{cases}.$$

This definition by cases is important because it plays a major role in interpreting the axiom $A \to A \wedge A$ with Gödel's functional interpretation.

The idea for the definition of t is simply: since A_{qf} and $\neg A_{\mathrm{qf}}$ reduce to $\chi_{A_{\mathrm{qf}}} =_0 0$ and $\chi_{A_{\mathrm{qf}}} \neq_0 0$, then we only need a term that distinguishes between a number being zero and non-zero; the recursor R is such a term since Rx distinguishes between $x =_0 0$ and $x \neq_0 0$.

1.47 Definition.

1. For each terms $\underline{r}^{\underline{\rho}} \equiv r_1^{\rho_1}, \ldots, r_n^{\rho_n}$, s^0 and $\underline{t}^{\underline{\rho}} \equiv t_1^{\rho_1}, \ldots, t_n^{\rho_n}$ of HA^ω we define the terms $\underline{r} \vee_s \underline{t} :\equiv \underline{\mathrm{R}} s \underline{r} (\lambda \underline{x}, y \,.\, \underline{t})$ where $\underline{x}, y \notin \mathrm{FV}(\underline{t})$ [50, proposition 3.19].

2. For each type $\rho = 0\rho_n \cdots \rho_1$ (possibly with no ρ_is) we define the term $\mathcal{O}^\rho :\equiv \lambda x_1^{\rho_1}, \ldots, x_n^{\rho_n} \,.\, 0$ [50, page 98].

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

1.48 Proposition. For all terms $\underline{r}^{\underline{\rho}} \equiv r_1^{\rho_1}, \ldots, r_n^{\rho_n}$, s^0 and $\underline{t}^{\underline{\rho}} \equiv t_1^{\rho_1}, \ldots, t_n^{\rho_n}$ of HA^ω and formulas $A(\underline{z}^{\underline{\rho}})$ of HA^ω, we have:

1. $\mathsf{HA}^\omega \vdash \big(s =_0 0 \to (A(\underline{r} \vee_s \underline{t}) \leftrightarrow A(\underline{r}))\big) \wedge \big(s \neq_0 0 \to (A(\underline{r} \vee_s \underline{t}) \leftrightarrow A(\underline{t}))\big)$;

2. $\mathsf{HA}^\omega \vdash (s =_0 0 \to \underline{r} \vee_s \underline{t} = \underline{r}) \wedge (s \neq_0 0 \to \underline{r} \vee_s \underline{t} = \underline{t})$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ [50, proposition 3.19].

1.49 Proof.

1. We can prove $\forall x \, (x =_0 0 \vee x \neq_0 0)$ by induction on x, so $s =_0 0 \vee s \neq_0 0$. If $s \neq_0 0$, then $s =_0 \mathrm{S}(\mathrm{pd}\, s)$, therefore

$$\begin{aligned} A(\underline{r} \vee_s \underline{t}) &\leftrightarrow \\ A\big((\lambda \underline{x}, y \,.\, \underline{t})(\underline{\mathrm{R}} s \underline{r} (\lambda \underline{x}, y \,.\, \underline{t}))s\big) &\leftrightarrow \\ A(\underline{t}). \end{aligned}$$

   Analogously if $s =_0 0$.

2. Just take $A(\underline{z}) :\equiv \underline{r} \vee_s \underline{t} = \underline{z}$ (with $\underline{z} \notin \mathrm{FV}(\underline{r} \vee_s \underline{t})$) in the previous point.

1.50. The term $r \vee_s t$ provably reduces to r or t according to $s =_0 0$ or $s \neq_0 0$, or in a more pictorial form,

$$r \vee_s t = \begin{cases} r & \text{if } s =_0 0 \\ t & \text{if } s \neq_0 0 \end{cases}.$$

A particularly important use of $r \vee_s t$ is when $s \equiv \chi_{A_{\mathrm{qf}}}$:

$$r \vee_{\chi_{A_{\mathrm{qf}}}} t = \begin{cases} r & \text{if } \quad A_{\mathrm{qf}} \\ t & \text{if } \neg A_{\mathrm{qf}} \end{cases}.$$

1.51. The term $\mathcal{O}$ is a dummy term that we use when we need to present a closed term but does not matter which term. For example, if we are asked to witness by a closed term the existential quantifier in $\exists x\, (\perp \to A(x))$, we can take $x = \mathcal{O}$.

1.11 Law of excluded middle for quantifier-free formulas

1.52. The Heyting arithmetic HA^ω is an intuitionistic theory, and intuitionistic logic does not prove the law of excluded middle, so naturally HA^ω does not prove the law of excluded middle. However, HA^ω does prove the law of excluded middle for quantifier-free formulas. This is a contribution not of the logical part of HA^ω, but of the arithmetical part of HA^ω, namely characteristic terms and induction. The idea of the proof is very simple:

1. we reduce quantifier-free formulas A_{qf} to $\chi_{A_{\mathrm{qf}}} =_0 0$;
2. we prove $\forall x\, (x =_0 0 \vee x \neq_0 0)$ by induction on x;
3. taking $x = \chi_{A_{\mathrm{qf}}}$ we get $A_{\mathrm{qf}} \vee \neg A_{\mathrm{qf}}$.

1.53 Theorem.

1. For all quantifier-free formulas A_{qf} of HA^ω, we have $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}} \vee \neg A_{\mathrm{qf}}$ [78, proposition 2.9 in chapter 3] [50, corollary 3.18].
2. For all quantifier-free *sentences* A_{qf} of HA^ω, we have $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}}$ or $\mathsf{HA}^\omega \vdash \neg A_{\mathrm{qf}}$ [50, proposition 3.8] [75, theorem 2.2.23].

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathsf{e}}$ and $\mathsf{HA}^\omega_{\mathsf{i}}$.

1.54 Proof. We do the proof only for HA^ω; the cases of the other theories are analogous.

1. By theorem 1.44 we have $(*_1)$ $A_{\mathrm{qf}} \leftrightarrow \chi_{A_{\mathrm{qf}}} =_0 0$. We can prove $\forall x\, (x =_0 0 \vee x \neq_0 0)$ by induction on x, so $(*_2)$ $\chi_{A_{\mathrm{qf}}} =_0 0 \vee \chi_{A_{\mathrm{qf}}} \neq_0 0$. From $(*_1)$ and $(*_2)$ we get $(*_3)$ $A_{\mathrm{qf}} \vee \neg A_{\mathrm{qf}}$.
2. By theorem 1.44 we have $(*_1)$ $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}} \leftrightarrow \chi_{A_{\mathrm{qf}}} =_0 0$ where $\chi_{A_{\mathrm{qf}}}$ is a closed term of type 0. So by point 3 of theorem 1.30 there exists $n \in \mathbb{N}$ such that $(*_2)$ $\mathsf{HA}^\omega \vdash \chi_{A_{\mathrm{qf}}} =_0 \bar{n}$. From $(*_1)$ and $(*_2)$ we get $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}} \leftrightarrow \bar{n} =_0 0$. So we have $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}}$ or $\mathsf{HA}^\omega \vdash \neg A_{\mathrm{qf}}$ accordingly to $n = 0$ or $n \neq 0$, respectively.

1.12 Majorisability and majorants

1.55. Now we turn to the basic properties of majorisability and to the existence of majorants of closed terms. These properties are necessary for some proof interpretations that, in the face of a theorem $\exists x\, A(x)$, seek to find not an exact witness t for x (such that $A(t)$) but a bound t on x (such that $\exists x \leq^{\mathrm{e}} t\, A(x)$).

Properties of $\leq^{\mathrm{e}}$, $\leq^{\mathrm{i}}$ and max We prove the basic properties of $\leq^{\mathrm{e}}$, $\leq^{\mathrm{i}}$ and max. Some properties are expected: for example, $\leq^{\mathrm{e}}$ is transitive. Other properties are a little bit more surprising: for example, $x \leq^{\mathrm{e}} \max xy$ but provided that $x \leq^{\mathrm{e}} x$ and $y \leq^{\mathrm{e}} y$. Admittedly, the proofs are tedious, so the reader may want to skip them.

Majorants We show that every closed term t has a majorant t^{m} such that $t \leq^{\mathrm{e}} t^{\mathrm{m}}$. The construction of t^{m} uses a small cute idea: we cannot simply take $t^{\mathrm{m}} \equiv t$ because $t \leq^{\mathrm{e}} t^{\mathrm{m}}$ requires t^{m} to be non-decreasing, so we (essentially) take t^{m} to be the non-decreasing version $t^{\mathrm{m}}n :\equiv \max\{t0, t1, t2, \ldots, tn\}$ of t.

1.56 Proposition. The theory HA^ω proves:

1. (a) $0 \leq_0 x$ [19, *lema* 30.1].
   (b) $x \leq_0 0 \leftrightarrow x =_0 0$ [19, *lema* 30.3];
2. (a) $x \leq_0 y \leftrightarrow \mathrm{S}x \leq_0 \mathrm{S}y$ [19, *lema* 30.2];
   (b) $x \leq_0 \mathrm{S}y \leftrightarrow x \leq_0 y \vee x =_0 \mathrm{S}y$ [19, *lema* 30.5];
3. (a) $x \leq_0 x$ [19, *teorema* 131.1];
   (b) $x \leq_0 y \wedge y \leq_0 x \to x =_0 y$ [78, proposition 2.8(ii) in chapter 3] [19, *teorema* 131.2];
   (c) $x \leq_0 y \wedge y \leq_0 z \to x \leq_0 z$ [78, proposition 2.11(iii) in chapter 3] [19, *teorema* 131.3];
4. (a) $x \leq_0 \max xy \wedge y \leq_0 \max xy$ [19, *lema* 134.1];
   (b) $x \leq_0 x' \wedge y \leq_0 y' \to \max xy \leq_0 \max x'y'$ [19, *lema* 134.3];
5. (a) $x \leq^{\mathrm{e}}_\rho y \to y \leq^{\mathrm{e}}_\rho y$ [15, lemmas 1(i) and 2(i)];
   (b) $x \leq^{\mathrm{e}}_\rho y \wedge y \leq^{\mathrm{e}}_\rho z \to x \leq^{\mathrm{e}}_\rho z$ [15, lemmas 1(i) and 2(i)];
6. (a) $\max_\rho \leq^{\mathrm{e}}_{\rho\rho\rho} \max_\rho$ [15, lemma 4(ii)];
   (b) $x \leq^{\mathrm{e}}_\rho x \wedge y \leq^{\mathrm{e}}_\rho y \to x \leq^{\mathrm{e}} \max_\rho xy \wedge y \leq^{\mathrm{e}} \max_\rho xy$ [15, lemma 4(i)];
7. The following rule holds in HA^ω [19, *proposição* 143]:

$$\frac{A_{\mathrm{b}} \wedge \underline{x} \leq^{\mathrm{e}} \underline{y} \to s\underline{x} \leq^{\mathrm{e}} t\underline{y} \wedge t\underline{x} \leq^{\mathrm{e}} t\underline{y}}{A_{\mathrm{b}} \to s \leq^{\mathrm{e}} t}.$$

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ (for $\mathsf{HA}^\omega_{\mathrm{i}}$ replacing $\leq^{\mathrm{e}}$ by $\leq^{\mathrm{i}}$).

1.57 Proof. We do the proofs only for $\mathsf{HA}_{\mathrm{i}}^{\omega}$; the cases of the other theories are analogous.

1. (a) We can prove $0 \leq_0 x \equiv 0 \dot{-} x =_0 0$ by induction on x.

   (b) Follows from $x \leq_0 0 \equiv x \dot{-} 0 =_0 0$ and $x \dot{-} 0 =_0 x$.

2. (a) Follows from $\mathrm{S}x \dot{-} \mathrm{S}y =_0 x \dot{-} y$, which we can prove by induction on y.

   (b) We prove $A(x, y) :\equiv x \leq_0 \mathrm{S}y \leftrightarrow x \leq_0 y \vee x =_0 \mathrm{S}y$ by double induction on x and y.

   $\underline{A(0, y)}$ Follows from point 1a.

   $\underline{A(x, 0)}$ The proof is by induction on x. The base case is an instance of $A(0, y)$, so let us see the induction step: by point 2a, $A(\mathrm{S}x, 0)$ is equivalent to $x \leq_0 0 \leftrightarrow \mathrm{S}x \leq_0 0 \vee x =_0 0$, which is provable by point 1b.

   $\underline{A(x, y) \to A(\mathrm{S}x, \mathrm{S}y)}$ By point 2a, $A(\mathrm{S}x, \mathrm{S}y)$ is equivalent to $A(x, y)$.

3. (a) The proof is by induction on x, using point 2a in the induction step.

   (b) We prove $A(x, y) :\equiv x \leq_0 y \wedge y \leq_0 x \to x =_0 y$ by double induction on x and y.

   $\underline{A(0, y)}$ Follows from point 1b.

   $\underline{A(x, 0)}$ Analogously to $A(0, y)$.

   $\underline{A(x, y) \to A(\mathrm{S}x, \mathrm{S}y)}$ By point 2a, $A(\mathrm{S}x, \mathrm{S}y)$ is equivalent to $A(x, y)$.

   (c) We prove $A(z) :\equiv x \leq_0 y \wedge y \leq_0 z \to x \leq_0 z$ is by induction on z. The base case follows from point 1b, so let us see the induction step: from $A(z)$ we get $x \leq_0 y \wedge (y \leq_0 z \vee y =_0 \mathrm{S}z) \to x \leq_0 z \vee x \leq_0 \mathrm{S}z$, where by point 2b the premise is equivalent to the premise of $A(\mathrm{S}z)$, and the conclusion implies the conclusion of $A(\mathrm{S}z)$.

4. (a) Let us only prove $A(x, y) :\equiv x \leq_0 \max xy$ by double induction on x and y; the case of $y \leq_0 \max xy$ is analogous. The base cases are easy, so let us see the induction step: from $A(x, y) \equiv x \dot{-} (x + (y \dot{-} x)) =_0 0$ we get $A(\mathrm{S}x, \mathrm{S}y) \equiv \mathrm{S}x \dot{-} (\mathrm{S}x + (\mathrm{S}y \dot{-} \mathrm{S}x)) =_0 0$ since $x \dot{-} (x + (y \dot{-} x)) =_0 \mathrm{S}x \dot{-} (\mathrm{S}x + (\mathrm{S}y \dot{-} \mathrm{S}x))$.

   (b) We only sketch the proof. First we prove $(*)$ $x \leq_0 z \wedge y \leq_0 z \to \max xy \leq_0 z$ by triple induction on x, y and z. Now, if $x \leq_0 x'$ and $y \leq_0 y'$, then $x \leq_0 \max x'y'$ and $y \leq_0 \max x'y'$ by points 3c and 4a, therefore $\max xy \leq_0 \max x'y'$ taking $z := \max x'y'$ in $(*)$.

5. (a) If $\rho = 0$, then the claim follows from point 3a. If ρ is a composite type, then $x \leq^{\mathrm{i}} y \wedge u \leq^{\mathrm{i}} v \to yu \leq^{\mathrm{i}} yv \wedge yu \leq^{\mathrm{i}} yv$, so $x \leq^{\mathrm{i}} y \to y \leq^{\mathrm{i}} y$ by the rule of $\leq^{\mathrm{i}}$.

   (b) The proof is by induction on the structure of ρ. The base case is point 3c, so let us see the induction step. We have $x \leq^{\mathrm{i}} y \wedge y \leq^{\mathrm{i}} z \wedge u \leq^{\mathrm{i}} v \to xu \leq^{\mathrm{i}} zv \wedge zu \leq^{\mathrm{i}} zv$: from the premise (which implies $v \leq^{\mathrm{i}} v$ and $z \leq^{\mathrm{i}} z$ by point 5a) we get $xu \leq^{\mathrm{i}} yv$, $yv \leq^{\mathrm{i}} zv$ and $zu \leq^{\mathrm{i}} zv$, so $xu \leq^{\mathrm{i}} zv$ by induction hypothesis. By the rule of $\leq^{\mathrm{i}}$ we conclude $x \leq^{\mathrm{i}} y \wedge y \leq^{\mathrm{i}} z \rightarrow x \leq^{\mathrm{i}} z$.

6. (a) Anticipating point 7, it suffices to prove $x \leq^{\mathrm{i}} x' \wedge y \leq^{\mathrm{i}} y' \rightarrow \max_\rho xy \leq^{\mathrm{i}} \max_\rho x'y'$ by induction on the structure of ρ. The base case is point 4b, so let us see the induction step. We have $x \leq^{\mathrm{i}} x' \wedge y \leq^{\mathrm{i}} y' \wedge u \leq^{\mathrm{i}} v \rightarrow \max(xu)(yu) \leq^{\mathrm{i}} \max(x'v)(y'v) \wedge \max(x'u)(y'u) \leq^{\mathrm{i}} \max(x'v')(y'v)$: from the premise we get $xu \leq^{\mathrm{i}} x'v$, $yu \leq^{\mathrm{i}} y'v$, $x'u \leq^{\mathrm{i}} x'v$ and $y'u \leq^{\mathrm{i}} y'v$, so we get the conclusion by induction hypothesis.

   (b) The proof is by induction on the structure of ρ. The base case follows from point 4a, so let us see the induction step. We prove $x \leq^{\mathrm{i}} x \wedge y \leq^{\mathrm{i}} y \rightarrow x \leq^{\mathrm{i}} \max xy$ (the part $x \leq^{\mathrm{i}} x \wedge y \leq^{\mathrm{i}} y \rightarrow y \leq^{\mathrm{i}} \max xy$ is analogous) by proving $x \leq^{\mathrm{i}} x \wedge y \leq^{\mathrm{i}} y \wedge u \leq^{\mathrm{i}} v \rightarrow xu \leq^{\mathrm{i}} \max(xv)(yv) \wedge \max(xu)(yu) \leq^{\mathrm{i}} \max(xv)(yv)$: from the premise we get $xu \leq^{\mathrm{i}} xv$, $xv \leq^{\mathrm{i}} xv$, $yu \leq^{\mathrm{i}} yv$ and $yv \leq^{\mathrm{i}} yv$ so we get the conclusion by induction hypothesis and points 5b and 6a.

7. The proof is by induction on the length of the tuple $\underline{x}$. The base case is trivial, so let us see the induction step. We assume $(*_1)$ $\mathsf{HA}^\omega_{\mathrm{i}} \vdash A_{\mathrm{b}} \wedge \underline{x}, x' \leq^{\mathrm{i}} \underline{y}, y' \rightarrow s\underline{x}x' \leq^{\mathrm{i}} t\underline{y}y' \wedge t\underline{x}x' \leq^{\mathrm{i}} t\underline{y}y'$ and prove $(*_2)$ $\mathsf{HA}^\omega_{\mathrm{i}} \vdash A_{\mathrm{b}} \rightarrow s \leq^{\mathrm{i}} t$. Taking $\underline{x} = \underline{y}$ in $(*_1)$ we get $(A_{\mathrm{b}} \wedge \underline{y} \leq^{\mathrm{i}} \underline{y}) \wedge x' \leq^{\mathrm{i}} y' \rightarrow t\underline{y}x' \leq^{\mathrm{i}} t\underline{y}y'$. So we have the premises of the following instances of the rule of $\leq^{\mathrm{i}}$, therefore we have the conclusions:

$$\frac{(A_{\mathrm{b}} \wedge \underline{x} \leq^{\mathrm{i}} \underline{y}) \wedge x' \leq^{\mathrm{i}} y' \rightarrow s\underline{x}x' \leq^{\mathrm{i}} t\underline{y}y' \wedge t\underline{y}x' \leq^{\mathrm{i}} t\underline{y}y'}{A_{\mathrm{b}} \wedge \underline{x} \leq^{\mathrm{i}} \underline{y} \rightarrow s\underline{x} \leq^{\mathrm{i}} t\underline{y}},$$
$$\frac{(A_{\mathrm{b}} \wedge \underline{x} \leq^{\mathrm{i}} \underline{y}) \wedge x' \leq^{\mathrm{i}} y' \rightarrow t\underline{x}x' \leq^{\mathrm{i}} t\underline{y}y' \wedge t\underline{y}x' \leq^{\mathrm{i}} t\underline{y}y'}{A_{\mathrm{b}} \wedge \underline{x} \leq^{\mathrm{i}} \underline{y} \rightarrow t\underline{x} \leq^{\mathrm{i}} t\underline{y}},$$

   From the conclusions we get $(*_2)$ by induction hypothesis.

1.58 Definition. Let t be a term of HA^ω and $\mathrm{FV}(t) = \{\underline{x}\}$. We say that t is *monotone* if and only if $\mathsf{HA}^\omega \vdash \lambda\underline{x} \,.\, t \leq^{\mathrm{e}} \lambda\underline{x} \,.\, t$. Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ (for $\mathsf{HA}^\omega_{\mathrm{i}}$ replacing $\leq^{\mathrm{e}}$ by $\leq^{\mathrm{i}}$) [15, definition 3].

1.59. In point 1b of the next definition we will define a term $t^{\mathrm{m}'}$ in a formal manner that turns out to be quite cryptic. So it is convenient to say that, informally, $t^{\mathrm{m}'}$ is just the following non-decreasing version of t: $t^{\mathrm{m}'}n :\equiv \max\{t0, t1, t2, \ldots, tn\}$.

1.60 Definition.

1. For each closed term t of HA^ω we define the term t^{m} of HA^ω by recursion on the structure of t by:

   (a) $t^{\mathrm{m}} :\equiv t$ for $t \in \{0, \mathrm{S}, \Pi, \Sigma\}$;

   (b) $(\mathrm{R}_i)^{\mathrm{m}} :\equiv (\mathrm{R}_i)^{\mathrm{m}'}$ where $t^{\mathrm{m}'} :\equiv \lambda x \,.\, \mathrm{R}_\rho x(t0)\big(\lambda y, x \,.\, \max_\rho y(t(\mathrm{S}x))\big)$ [50, definition 3.65].

   (c) $(st)^{\mathrm{m}} :\equiv s^{\mathrm{m}}t^{\mathrm{m}}$.

2. For each term t of HA^ω we define the term $t^{\mathrm{m}}(\underline{x}) :\equiv (\lambda \underline{x} \, . \, t)^{\mathrm{m}} \underline{x}$ where $\mathrm{FV}(t) = \{\underline{x}\}$ and $(\lambda \underline{x} \, . \, t)^{\mathrm{m}}$ was defined in the previous point.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ (for $\mathsf{HA}^\omega_{\mathrm{i}}$ replacing $\leq^{\mathrm{e}}$ by $\leq^{\mathrm{i}}$).

1.61 Theorem.

1. For all closed terms t of HA^ω we have $\mathsf{HA}^\omega \vdash t \leq^{\mathrm{e}} t^{\mathrm{m}}$.

2. For all terms $t(\underline{x})$ of HA^ω we have $\mathsf{HA}^\omega \vdash \tilde{\forall} \underline{x}' \, \forall \underline{x} \leq^{\mathrm{e}} \underline{x}' \, (t(\underline{x}) \leq^{\mathrm{e}} t^{\mathrm{m}}(\underline{x}'))$ where $\mathrm{FV}(t) = \{\underline{x}\}$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$ (for $\mathsf{HA}^\omega_{\mathrm{i}}$ replacing $\leq^{\mathrm{e}}$ by $\leq^{\mathrm{i}}$) [15, lemma 5].

1.62 Proof. We do the proof only for $\mathsf{HA}^\omega_{\mathrm{i}}$; the cases of the other theories are analogous.

1. First we prove: $(*_1)$ for all terms $s^{\rho 0}$ and $t^{\rho 0}$ of $\mathsf{HA}^\omega_{\mathrm{i}}$, if $(*_2)$ $\mathsf{HA}^\omega_{\mathrm{i}} \vdash \forall z^0 \, (sz \leq^{\mathrm{i}} tz)$ then $\mathsf{HA}^\omega_{\mathrm{i}} \vdash s \leq^{\mathrm{i}} t^{\mathrm{m}'}$ (this contains a tiny patch to the literature [19, *lema* 151]). We assume the premise and prove the conclusion by proving $A(v) :\equiv \forall u \, (u \leq_0 v \to su \leq^{\mathrm{i}} t^{\mathrm{m}'} v \wedge t^{\mathrm{m}'} u \leq^{\mathrm{i}} t^{\mathrm{m}'} v)$ by induction on v.

   Base case By point 1b of proposition 1.56, $A(0)$ is equivalent to $s0 \leq^{\mathrm{i}} t0 \wedge t0 \leq^{\mathrm{i}} t0$, which follows from $(*_2)$.

   Induction step We assume $A(v)$, take any $u \leq_0 \mathrm{S}v$ and prove the conclusion of $A(\mathrm{S}v)$, that is $(*_3)$ $su \leq^{\mathrm{i}} \max(t^{\mathrm{m}'} v)(t(\mathrm{S}v)) \wedge t^{\mathrm{m}'} u \leq^{\mathrm{i}} \max(t^{\mathrm{m}'} v)(t(\mathrm{S}v))$. By point 2b of proposition 1.56 we have two cases.

   $u \leq_0 v$ We have $su \leq^{\mathrm{i}} t^{\mathrm{m}'} v$ and $t^{\mathrm{m}'} u \leq^{\mathrm{i}} t^{\mathrm{m}'} v$ by $A(v)$, and $t(\mathrm{S}v) \leq^{\mathrm{i}} t(\mathrm{S}v)$ by $(*_2)$. So we have $(*_3)$ by points 5b and 6b of proposition 1.56.

   $u =_0 \mathrm{S}v$ We have $s(\mathrm{S}v) \leq^{\mathrm{i}} t(\mathrm{S}v)$ by $(*_2)$, and $t^{\mathrm{m}'} v \leq^{\mathrm{i}} t^{\mathrm{m}'} v$ by $A(v)$ with $u = v$. So $s(\mathrm{S}v) \leq^{\mathrm{i}} \max(t^{\mathrm{m}'} v)(t(\mathrm{S}v)) \wedge \max(t^{\mathrm{m}'} v)(t(\mathrm{S}v)) \leq^{\mathrm{i}} \max(t^{\mathrm{m}'} v)(t(\mathrm{S}v))$ by point 6a of proposition 1.56, that is $(*_3)$.

   Now we prove the claim of the theorem by induction on the structure of t. The induction step is easy, so let us see the base case.

   S We have $x \leq_0 y \to \mathrm{S}x \leq_0 \mathrm{S}y$ by point 2a of proposition 1.56, so $\mathrm{S} \leq^{\mathrm{i}} \mathrm{S}$ by the rule of $\leq^{\mathrm{i}}$. Analogously for 0.

   Σ We have $x \leq^{\mathrm{i}} x' \wedge y \leq^{\mathrm{i}} y' \wedge z \leq^{\mathrm{i}} z' \to \Sigma xyz \leq^{\mathrm{i}} \Sigma x'y'z'$ because the conclusions is equivalent to $xz(yz) \leq^{\mathrm{i}} x'z'(y'z')$. So $\Sigma \leq^{\mathrm{i}} \Sigma$ by point 7 of proposition 1.56. Analogously for Π.

   $\underline{\mathrm{R}}$ Say $\underline{\mathrm{R}} \equiv \mathrm{R}_1, \ldots, \mathrm{R}_n$. We can prove $\underline{\mathrm{R}} x \leq^{\mathrm{i}} \underline{\mathrm{R}} x$ by proving $\underline{y} \leq^{\mathrm{i}} \underline{y}' \wedge \underline{z} \leq^{\mathrm{i}} \underline{z}' \to \underline{\mathrm{R}} x \underline{y} \underline{z} \leq^{\mathrm{i}} \underline{\mathrm{R}} x \underline{y}' \underline{z}'$ by induction on x. Then $\mathrm{R}_i \leq^{\mathrm{i}} \mathrm{R}_i^{\mathrm{m}}$ by $(*_1)$.

2. By point 2d of proof 1.35 we have $\lambda \underline{x} \, . \, t(\underline{x}) \equiv \lambda \underline{x}' \, . \, t(\underline{x}')$. By the previous point we have $\lambda \underline{x} \, . \, t(\underline{x}) \leq^{\mathrm{i}} (\lambda \underline{x}' \, . \, t(\underline{x}'))^{\mathrm{m}}$. So, if $\underline{x} \leq^{\mathrm{i}} \underline{x}'$, then $(\lambda \underline{x} \, . \, t(\underline{x}))\underline{x} \leq^{\mathrm{i}} (\lambda \underline{x}' \, . \, t(\underline{x}'))^{\mathrm{m}} \underline{x}'$, that is $t(\underline{x}) \leq^{\mathrm{i}} t^{\mathrm{m}}(\underline{x}')$.

1.13 Principles

1.63. So far we have been working with with HA^ω (and its variants). However, proof interpretations interpret more than just HA^ω: they also interpret certain principles like the axiom (schema) of choice

$$\forall x\, \exists y\, A(x, y) \to \exists Y\, \forall x\, A(x, Yx).$$

So now we collect all the principles that we will be considering later.

1.64 Definition. In table 1.5 we define several principles [50, section 5.1, definitions 5.26 and 8.4] [15, section 4.1 and proposition 4.4] [14, section 3.1]. The principles using $\leq^{\mathrm{e}}$ can also be read with $\leq^{\mathrm{i}}$ instead of $\leq^{\mathrm{e}}$: in the context of $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{PA}^\omega_{\mathrm{e}}$ they use $\leq^{\mathrm{e}}$, while in the context of $\mathsf{HA}^\omega_{\mathrm{i}}$ and $\mathsf{PA}^\omega_{\mathrm{i}}$ they use $\leq^{\mathrm{i}}$. The variables introduced as bounds are not free in the formulas A, A_{qf}, $A_{\exists\mathrm{f}}$, $A_{\tilde{\exists}\mathrm{f}}$, A_{b}, B and B_{b}. For example, in BAC, x and $\underline{y}$ can be free in A, but u and $\underline{v}$ cannot.

1.65 Remark. In all principles, where are single variables x and y, we can generalise to tuples of variables $\underline{x}$ and $\underline{y}$ by induction on the length of the tuples. For example, AC generalises to $\forall \underline{x}\, \exists \underline{y}\, A(\underline{x}, \underline{y}) \to \exists \underline{Y}\, \forall \underline{x}\, A(\underline{x}, \underline{Y}\underline{x})$. (Some principles are already stated with tuples because they seemly do not generalise to tuples. For example, $\neg\neg\exists x\, A_{\mathrm{qf}} \to \exists x\, A_{\mathrm{qf}}$ does not seem to generalise to $\neg\neg\exists \underline{x}\, A_{\mathrm{qf}} \to \exists \underline{x}\, A_{\mathrm{qf}}$, so we stated $\mathsf{QF\text{-}MP}$ already with tuples $\underline{x}$.)

1.66 Proposition.

1. The theories $\mathsf{HA}^\omega_{\mathrm{e}} + \mathsf{B\text{-}BAC} + \tilde{\exists}\mathsf{F\text{-}BIP} + \mathsf{MAJ}$ and $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ}$ prove $\mathsf{B\text{-}BC}$ [15, page 92]. Analogously replacing both $\mathsf{B\text{-}BAC}$ by BAC and $\mathsf{B\text{-}BC}$ by BC [15, proposition 3] [14, proposition 2].

2. The theories $\mathsf{HA}^\omega_{\mathrm{e}} + \mathsf{B\text{-}BAC} + \tilde{\exists}\mathsf{F\text{-}BIP}$ and $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \forall\text{-}\mathsf{BIP}$ prove $\mathsf{B\text{-}MAC}$. Analogously replacing both $\mathsf{B\text{-}BAC}$ by BAC and $\mathsf{B\text{-}MAC}$ by MAC [15, proposition 3] [14, proposition 3].

1.67 Proof.

1. We do the proof only for $\mathsf{HA}^\omega_{\mathrm{e}} + \mathsf{B\text{-}BAC} + \tilde{\exists}\mathsf{F\text{-}BIP} + \mathsf{MAJ}$; the cases of the other theories are analogous. By MAJ, there exists an u' such that $u \leq^{\mathrm{e}} u'$ (this contains a tiny patch to the literature). Using $\tilde{\exists}\mathsf{F\text{-}BIP}$ in the second implication, $\mathsf{B\text{-}BAC}$ in the fourth implication, and taking $x' := u'$ (which satisfies $x' \leq^{\mathrm{e}} x'$) in the fifth implication, we get

$$\begin{aligned}
\forall x \leq^{\mathrm{e}} u\, \exists \underline{y}\, A_{\mathrm{b}} &\to \\
\forall x\, (x \leq^{\mathrm{e}} u \to \exists \underline{y}\, A_{\mathrm{b}}) &\to \\
\forall x\, \tilde{\exists} \underline{v}\, (x \leq^{\mathrm{e}} u \to \exists \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}}) &\to \\
\forall x\, \exists \underline{v}\, (\underline{v} \leq^{\mathrm{e}} \underline{v} \wedge (x \leq^{\mathrm{e}} u \to \exists \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}})) &\to \\
\tilde{\exists} \underline{v}'\, \tilde{\forall} x'\, \forall x \leq^{\mathrm{e}} x'\, \exists \underline{v} \leq^{\mathrm{e}} \underline{v}'x'\, (\underline{v} \leq^{\mathrm{e}} \underline{v} \wedge (x \leq^{\mathrm{e}} u \to \exists \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}})) &\to \\
\tilde{\exists} \underline{v}'\, \forall x \leq^{\mathrm{e}} u'\, \exists \underline{v} \leq^{\mathrm{e}} \underline{v}'u'\, (\underline{v} \leq^{\mathrm{e}} \underline{v} \wedge (x \leq^{\mathrm{e}} u \to \exists \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}})) &\to \\
\tilde{\exists} \underline{v}'\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}'u'\, A_{\mathrm{b}} &\to \\
\tilde{\exists} \underline{v}\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}}&.
\end{aligned}$$

axiom (schema) of choice AC	$\forall x\, \exists y\, A(x,y) \to \exists Y\, \forall x\, A(x, Yx)$
axiom (schema) of choice (for quantifier-free formulas) QF-AC	$\forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}(\underline{x},\underline{y}) \to \exists \underline{Y}\, \forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}, \underline{Y}\underline{x})$
bounded axiom (schema) of choice BAC	$\forall x\, \exists \underline{y}\, A \to \tilde{\exists} \underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v} u\, A$
bounded axiom (schema) of choice (for bounded formulas) B-BAC	$\forall x\, \exists \underline{y}\, A_{\mathrm{b}} \to \tilde{\exists} \underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v} u\, A_{\mathrm{b}}$
monotone axiom (schema) of choice MAC	$\tilde{\forall} x\, \tilde{\forall} z\, \tilde{\forall} y \leq^{\mathrm{e}} z\, (A(x,y) \to A(x,z)) \wedge$ $\tilde{\forall} x\, \tilde{\exists} y\, A(x,y) \to \tilde{\exists} Y\, \tilde{\forall} x\, A(x, Yx)$
monotone axiom (schema) of choice (for bounded formulas) B-MAC	$\tilde{\forall} x\, \tilde{\forall} \underline{z}\, \tilde{\forall} \underline{y} \leq^{\mathrm{e}} \underline{z}\, (A_{\mathrm{b}}(x,\underline{y}) \to A_{\mathrm{b}}(x,\underline{z})) \wedge$ $\tilde{\forall} x\, \tilde{\exists} \underline{y}\, A_{\mathrm{b}}(x,\underline{y}) \to \tilde{\exists} \underline{Y}\, \tilde{\forall} x\, A_{\mathrm{b}}(x, \underline{Y} x)$
independence of premises (for $\exists$-free premises) $\exists$F-IP	$(A_{\exists \mathrm{f}} \to \exists x\, B) \to \exists x\, (A_{\exists \mathrm{f}} \to B)$ $(x \notin \mathrm{FV}(A_{\exists \mathrm{f}}))$
independence of premises (for purely universal premises) $\forall$-IP	$(\forall \underline{x}\, A_{\mathrm{qf}} \to \exists y\, B) \to \exists y\, (\forall \underline{x}\, A_{\mathrm{qf}} \to B)$ $(y \notin \mathrm{FV}(\forall \underline{x}\, A_{\mathrm{qf}}))$
bounded independence of premises (for $\tilde{\exists}$-free premises) $\tilde{\exists}$F-BIP	$(A_{\tilde{\exists} \mathrm{f}} \to \exists x\, B) \to \tilde{\exists} y\, (A_{\tilde{\exists} \mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B)$ $(x \notin \mathrm{FV}(A_{\tilde{\exists} \mathrm{f}}))$
bounded independence of premises (for purely universal premises) $\forall$-BIP	$(\forall \underline{x}\, A_{\mathrm{b}} \to \exists y\, B) \to \tilde{\exists} z\, (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{e}} z\, B)$ $(y \notin \mathrm{FV}(\forall \underline{x}\, A_{\mathrm{b}}))$
Markov's principle (for quantifier-free formulas) QF-MP	$\neg\neg\exists \underline{x}\, A_{\mathrm{qf}} \to \exists \underline{x}\, A_{\mathrm{qf}}$
bounded Markov's principle (for bounded formulas) B-BMP	$(\forall \underline{x}\, A_{\mathrm{b}} \to B_{\mathrm{b}}) \to$ $\tilde{\exists} \underline{y}\, (\forall \underline{x} \leq^{\mathrm{e}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}})$
bounded collection BC	$\forall x \leq^{\mathrm{e}} u\, \exists y\, A \to \tilde{\exists} v\, \forall x \leq^{\mathrm{e}} u\, \exists y \leq^{\mathrm{e}} v\, A$
bounded collection (for bounded formulas) B-BC	$\forall x \leq^{\mathrm{e}} u\, \exists \underline{y}\, A_{\mathrm{b}} \to \tilde{\exists} \underline{v}\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}}$
bounded contra collection (for bounded formulas) B-BCC	$\tilde{\forall} \underline{v}\, \exists \underline{x} \leq^{\mathrm{e}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{e}} \underline{v}\, A_{\mathrm{b}} \to \exists \underline{x} \leq^{\mathrm{e}} \underline{u}\, \forall \underline{y}\, A_{\mathrm{b}}$
law of excluded middle LEM	$A \vee \neg A$
law of excluded middle (for bounded formulas) B-LEM	$A_{\mathrm{b}} \vee \neg A_{\mathrm{b}}$
bounded universal disjunction (for bounded formulas) B-BUD	$\tilde{\forall} \underline{u}, \underline{v}\, (\forall \underline{x} \leq^{\mathrm{e}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \leq^{\mathrm{e}} \underline{v}\, B_{\mathrm{b}}) \to$ $\forall \underline{x}\, A_{\mathrm{b}} \vee \forall \underline{y}\, B_{\mathrm{b}}$
majorisability axioms MAJ	$\forall x\, \exists y\, (x \leq^{\mathrm{e}} y)$
ω-rule ωR	$\dfrac{\text{for all closed terms } t^{\rho}\ \ A(t)}{\forall x^{\rho}\, A(x)}$

Table 1.5: principles.

2. We do the proof only for $\mathsf{HA}^\omega_{\mathrm{e}} + \mathsf{B\text{-}BAC} + \tilde{\exists}\mathsf{F\text{-}BIP}$; the cases of the other theories are analogous. Let us assume $(*)\ \tilde{\forall} x\, \tilde{\forall} \underline{z}\, \tilde{\forall} \underline{y} \leq^{\mathrm{e}} \underline{z}\, (A_{\mathrm{b}}(x, \underline{y}) \to A_{\mathrm{b}}(x, \underline{z}))$. Using $\tilde{\exists}\mathsf{F\text{-}BIP}$ in the third implication, $\mathsf{B\text{-}BAC}$ in the fourth implication, taking $x' := x$ (which satisfies $x' \leq^{\mathrm{e}} x'$) in the fifth implication, and using $(*)$ is the last implication, we get

$$\begin{aligned}
&\tilde{\forall} x\, \tilde{\exists} \underline{y}\, A_{\mathrm{b}}(x, \underline{y}) \to \\
&\forall x \left(x \leq^{\mathrm{e}} x \to \exists \underline{y}\, (\underline{y} \leq^{\mathrm{e}} \underline{y} \wedge A_{\mathrm{b}}(x, \underline{y}))\right) \to \\
&\forall x\, \tilde{\exists} \underline{y}' \left(x \leq^{\mathrm{e}} x \to \exists \underline{y} \leq^{\mathrm{e}} \underline{y}'\, (\underline{y} \leq^{\mathrm{e}} \underline{y} \wedge A_{\mathrm{b}}(x, \underline{y}))\right) \to \\
&\forall x\, \exists \underline{y}' \left(\underline{y}' \leq^{\mathrm{e}} \underline{y}' \wedge (x \leq^{\mathrm{e}} x \to \tilde{\exists} \underline{y} \leq^{\mathrm{e}} \underline{y}'\, A_{\mathrm{b}}(x, \underline{y}))\right) \to \\
&\tilde{\exists} \underline{Y}\, \tilde{\forall} x\, \forall x' \leq^{\mathrm{e}} x\, \exists \underline{y}' \leq^{\mathrm{e}} \underline{Y} x \left(\underline{y}' \leq^{\mathrm{e}} \underline{y}' \wedge (x' \leq^{\mathrm{e}} x' \to \tilde{\exists} \underline{y} \leq^{\mathrm{e}} \underline{y}'\, A_{\mathrm{b}}(x', \underline{y}))\right) \to \\
&\tilde{\exists} \underline{Y}\, \tilde{\forall} x\, \exists \underline{y}' \leq^{\mathrm{e}} \underline{Y} x\, \tilde{\exists} \underline{y} \leq^{\mathrm{e}} \underline{y}'\, A_{\mathrm{b}}(x, \underline{y}) \to \\
&\tilde{\exists} \underline{Y}\, \tilde{\forall} x\, \tilde{\exists} \underline{y} \leq^{\mathrm{e}} \underline{Y} x\, A_{\mathrm{b}}(x, \underline{y}) \to \\
&\tilde{\exists} \underline{Y}\, \tilde{\forall} x\, A_{\mathrm{b}}(\underline{Y} x).
\end{aligned}$$

1.14 Conclusion

1.68. We mainly saw the following three big points.

<u>HA^ω</u> Our framework is HA^ω: a version of Peano arithmetic that

1. does not have the law of excluded middle $A \vee \neg A$;
2. talks not only about $\mathbb{N}$, but also about $\mathbb{N}^{\mathbb{N}}$, $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, and so on.

<u>Functions</u> We saw that in HA^ω:

1. we have λ-abstraction $\lambda x\,.\, t(x)$ (that is, informally, $x \mapsto t(x)$);
2. we can represent primitive recursive functions ($+$, $\cdot$, and so on) by terms;
3. we can define terms by cases by $s \vee_{\chi_{A_{\mathrm{qf}}}} t = \begin{cases} s & \text{if} \quad A_{\mathrm{qf}} \\ t & \text{if} \ \neg A_{\mathrm{qf}} \end{cases}$.

<u>Majorisability</u> We have a majorisability $\leq^{\mathrm{e}}$ (and $\leq^{\mathrm{i}}$) such that $f \leq^{\mathrm{e}} g$ roughly speaking means "f is pointwise smaller than or equal to g, and g is non-decreasing".

Part II

Proof interpretations

Chapter 2

Negative translations

2.1 Introduction

2.1. Classical logic CL is (informally) the usual logic in mathematics, and intuitionistic logic IL is (informally) the constructive part of CL. So IL is weaker than CL, but IL is constructive while CL is not.

<u>IL is weaker than CL</u> We have $\mathsf{IL} \vdash A \underset{\not\Leftarrow}{\Rightarrow} \mathsf{CL} \vdash A$.

<u>IL is constructive</u> A proof $\mathsf{IL} \vdash \exists x\, A(x)$ can be turned into a proof $\mathsf{IL} \vdash A(t)$ (for some suitable term t of IL) that we regard in the following way: $\mathsf{IL} \vdash A(t)$ is a constructive proof of $\mathsf{IL} \vdash \exists x\, A(x)$ that witnesses $\exists x$ by t.

Given these differences between CL and IL, it is surprising that CL can be embedded in IL.

2.2. A negative translation N is a embedding of CL in IL. By embedding we mean a mapping of formulas $A \mapsto A^{\mathrm{N}}$ with the following two properties:

<u>Soundness theorem</u> if $\mathsf{CL} + \Gamma \vdash A$, then $\mathsf{IL} + \Gamma^{\mathrm{N}} \vdash A^{\mathrm{N}}$;

<u>Characterisation theorem</u> $\mathsf{CL} \vdash A \leftrightarrow A^{\mathrm{N}}$.

In this chapter we introduce four negative translations due to Gödel and Gentzen, Kolmogorov, Kuroda, and Krivine. We end with two applications: a conservation result and a relative consistency result.

2.3. Our (admittedly modest) main contribution to this topic is the extension of the Krivine, Gödel-Gentzen and Kolmogorov negative translations to $\mathsf{PA}^\omega_{\mathrm{i}} \pm \mathsf{B\text{-}BAC} \pm \mathsf{B\text{-}BCC} \pm \mathsf{MAJ}$ (definition 2.5 and theorem 2.9). (The Kuroda negative translation was extended by Ferreira and Oliva [15, section 5], and the soundness theorems of our extensions are corollaries to the soundness theorem of their extension.)

2.2 Definition

2.4. Many classical laws hold intuitionistically if they are double negated. For example:

$$
\begin{aligned}
&\mathsf{CL} \vdash A \vee \neg A, && \mathsf{IL} \vdash \neg\neg(A \vee \neg A),\\
&\mathsf{CL} \vdash \neg\neg A \to A, && \mathsf{IL} \vdash \neg\neg(\neg\neg A \to A),\\
&\mathsf{CL} \vdash \neg(A \wedge B) \to \neg A \vee \neg B, && \mathsf{IL} \vdash \neg\neg(\neg(A \wedge B) \to \neg A \vee \neg B),\\
&\mathsf{CL} \vdash (A \to B) \to \neg A \vee B, && \mathsf{IL} \vdash \neg\neg((A \to B) \to \neg A \vee B),\\
&\mathsf{CL} \vdash \neg\forall x\, \neg A \to \exists x\, A, && \mathsf{IL} \vdash \neg\neg(\neg\forall x\, \neg A \to \exists x\, A).
\end{aligned}
$$

This could lead us to conjecture that for all formulas A we have $\mathsf{CL} \vdash A \Leftrightarrow \mathsf{IL} \vdash \neg\neg A$, but this is false: $\mathsf{CL} \vdash A$ but $\mathsf{IL} \nvdash \neg\neg A$ for $A \equiv \neg\forall x\, B \to \exists x\, \neg B$. The correct proposition is $\mathsf{CL} \vdash A \Leftrightarrow \mathsf{IL} + \mathsf{DNS} \vdash \neg\neg A$, where the *double negation shift* DNS is the principle $\forall x\, \neg\neg A \to \neg\neg\forall x\, A$ [78, exercise 2.2.3 in chapter 3]. We may wonder if with a more involved change N (than a plain double negation) we have $\mathsf{CL} \vdash A \Leftrightarrow \mathsf{IL} \vdash A^{\mathrm{N}}$ by making DNS superfluous. We are going to use this question as a motivation to four such Ns called negative translations.

Kuroda negative translation The double negation shift DNS is superfluous when applied to a universal quantification with a negated matrix, that is $\forall x\, \neg\neg A \to \neg\neg\forall x\, A$ is already provable in IL if A is a negated formula. So one way to make DNS superfluous is to define a translation Ku that maps a formula A to the formula A^{Ku} obtained from A by double negating not only A but also all matrices of universal quantifications in A.

Kolmogorov negative translation A wasteful variant of Ku is the translation Ko that maps a formula A to the formula A^{Ko} obtained from A by double negating every subformula of A.

Krivine negative translation Another way to make DNS superfluous is to consider CL based on a language without $\exists$ and define a translation Kr that maps a formula A to the formula A^{Kr} obtained from A by double negating A and moving one of the negations inside up to the atomic subformulas of A. For example, in step (2.1) below we add a double negation, in step (2.2) we move one negation inside, and in step (2.3) we further move the negation inside up to the atomic level:

$$
\begin{aligned}
\forall x\,(A_{\mathrm{at}} \vee B_{\mathrm{at}}) &\rightsquigarrow && (2.1)\\
\neg\neg\forall x\,(A_{\mathrm{at}} \vee B_{\mathrm{at}}) &\rightsquigarrow && (2.2)\\
\neg\exists x\, \neg(A_{\mathrm{at}} \vee B_{\mathrm{at}}) &\rightsquigarrow && (2.3)\\
\neg\exists x\,(\neg A_{\mathrm{at}} \wedge \neg B_{\mathrm{at}}) &\equiv\\
(\forall x\,(A_{\mathrm{at}} \vee B_{\mathrm{at}}))^{\mathrm{Kr}}&.
\end{aligned}
$$

Moving one negation inside turns all $\forall$s into $\exists$s, so there will be no $\forall$s in A^{Kr}, thus making DNS superfluous. (Later we can put back $\exists$ and translate $\exists x\, A$ as if it were $\neg\forall x\, \neg A$.)

Gödel-Gentzen negative translation A fourth attempt is to identify a fragment of IL where for all formulas we have $\mathsf{IL} \vdash \neg\neg A \leftrightarrow A$, and then define a translation GG that maps a formula A to the formula A^{GG} obtained from A by rewriting A in the fragment. If we try to prove $\mathsf{IL} \vdash \neg\neg A \leftrightarrow A$ by induction on the structure of A we find no problems with $\wedge$, $\to$ and $\forall$, but we find problems with A_{at} and $\exists$ (below "IH" signals a use of induction hypothesis):

$$\begin{aligned} \neg\neg A_{\mathrm{at}} &\not\leftrightarrow A_{\mathrm{at}}, \\ \neg\neg(A \wedge B) &\leftrightarrow \neg\neg A \wedge \neg\neg B \overset{\mathrm{IH}}{\leftrightarrow} A \wedge B, \\ \neg\neg(A \vee B) \overset{\mathrm{IH}}{\leftrightarrow} \neg\neg(\neg\neg A \vee \neg\neg B) &\not\leftrightarrow \neg\neg A \vee \neg\neg B \overset{\mathrm{IH}}{\leftrightarrow} A \vee B, \\ \neg\neg(A \to B) &\leftrightarrow (\neg\neg A \to \neg\neg B) \overset{\mathrm{IH}}{\leftrightarrow} (A \to B), \\ \neg\neg\forall x\, A \overset{\mathrm{IH}}{\leftrightarrow} \neg\neg\forall x\, \neg\neg A &\leftrightarrow \forall x\, \neg\neg A \overset{\mathrm{IH}}{\leftrightarrow} \forall x\, A, \\ \neg\neg\exists x\, A \overset{\mathrm{IH}}{\leftrightarrow} \neg\neg\exists x\, \neg\neg A &\not\leftrightarrow \exists x\, \neg\neg A \overset{\mathrm{IH}}{\leftrightarrow} \exists x\, A. \end{aligned}$$

We solve the problem with atomic formulas by assuming that in the fragment they always occur negated, and we solve the provable with $\vee$ and $\exists$ by leaving them out of the fragment. So the fragment in question is the *negative fragment* whose formulas are build from negated atomic formulas by means of $\wedge$, $\to$ and $\forall$. Then the translation GG rewrites a formula into the negative fragment by:

1. rewriting A_{at} as $\neg\neg A_{\mathrm{at}}$;
2. rewriting $A \vee B$ as $\neg(\neg A \wedge \neg B)$;
3. rewriting $\exists x\, A$ as $\neg\forall x\, \neg A$;
4. leaving invariant $\wedge$, $\to$ and $\forall$.

Since A^{GG} is in the negative fragment, we have $\mathsf{IL} \vdash \neg\neg A^{\mathrm{GG}} \leftrightarrow A^{\mathrm{GG}}$, so hopefully DNS becomes superfluous.

2.5 Definition.

1. The *Gödel-Gentzen negative translation* [27] [29, page 287] [23] [24, theorem III] assigns to each formula A of $\mathsf{PA}^{\omega}_{\mathrm{e}}$ the formula A^{GG} of $\mathsf{HA}^{\omega}_{\mathrm{e}}$ defined by recursion on the structure of A by (where $A_{\mathrm{at}} \not\equiv \bot$)

$$\begin{aligned} \bot^{\mathrm{GG}} &:\equiv \bot, \\ A_{\mathrm{at}}^{\mathrm{GG}} &:\equiv \neg\neg A_{\mathrm{at}}, \\ (A \wedge B)^{\mathrm{GG}} &:\equiv A^{\mathrm{GG}} \wedge B^{\mathrm{GG}}, \\ (A \vee B)^{\mathrm{GG}} &:\equiv \neg(\neg A^{\mathrm{GG}} \wedge \neg B^{\mathrm{GG}}), \\ (A \to B)^{\mathrm{GG}} &:\equiv A^{\mathrm{GG}} \to B^{\mathrm{GG}}, \\ (\forall x \leq^{\mathrm{e}} t\, A)^{\mathrm{GG}} &:\equiv \forall x \leq^{\mathrm{e}} t\, A^{\mathrm{GG}}, \\ (\exists x \leq^{\mathrm{e}} t\, A)^{\mathrm{GG}} &:\equiv \neg\forall x \leq^{\mathrm{e}} t\, \neg A^{\mathrm{GG}}, \\ (\forall x\, A)^{\mathrm{GG}} &:\equiv \forall x\, A^{\mathrm{GG}}, \\ (\exists x\, A)^{\mathrm{GG}} &:\equiv \neg\forall x\, \neg A^{\mathrm{GG}}. \end{aligned}$$

2. The *Kolmogorov negative translation* [54] [53, formula (49)] assigns to each formula A of $\mathsf{PA}^\omega_\mathrm{e}$ the formula A^{Ko} of $\mathsf{HA}^\omega_\mathrm{e}$ defined by recursion on the structure of A by

$$\begin{aligned}
A_{\mathrm{at}}^{\mathrm{Ko}} &:\equiv \neg\neg A_{\mathrm{at}},\\
(A \wedge B)^{\mathrm{Ko}} &:\equiv \neg\neg(A^{\mathrm{Ko}} \wedge B^{\mathrm{Ko}}),\\
(A \vee B)^{\mathrm{Ko}} &:\equiv \neg\neg(A^{\mathrm{Ko}} \vee B^{\mathrm{Ko}}),\\
(A \to B)^{\mathrm{Ko}} &:\equiv \neg\neg(A^{\mathrm{Ko}} \to B^{\mathrm{Ko}}),\\
(\forall x \leq^{\mathrm{e}} t\, A)^{\mathrm{Ko}} &:\equiv \neg\neg\forall x \leq^{\mathrm{e}} t\, A^{\mathrm{Ko}},\\
(\exists x \leq^{\mathrm{e}} t\, A)^{\mathrm{Ko}} &:\equiv \neg\neg\exists x \leq^{\mathrm{e}} t\, A^{\mathrm{Ko}},\\
(\forall x\, A)^{\mathrm{Ko}} &:\equiv \neg\neg\forall x\, A^{\mathrm{Ko}},\\
(\exists x\, A)^{\mathrm{Ko}} &:\equiv \neg\neg\exists x\, A^{\mathrm{Ko}}.
\end{aligned}$$

3. The *Krivine negative translation* [69, sections 2 and 4] [2, page 1] [20, definition 3.1] assigns to each formula A of $\mathsf{PA}^\omega_\mathrm{e}$ the formula $A^{\mathrm{Kr}} :\equiv \neg\neg A_{\mathrm{Kr}}$ of $\mathsf{HA}^\omega_\mathrm{e}$ where A_{Kr} is defined by recursion on the structure of A by

$$\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{Kr}} &:\equiv \neg A_{\mathrm{at}},\\
(A \wedge B)_{\mathrm{Kr}} &:\equiv A_{\mathrm{Kr}} \vee B_{\mathrm{Kr}},\\
(A \vee B)_{\mathrm{Kr}} &:\equiv A_{\mathrm{Kr}} \wedge B_{\mathrm{Kr}},\\
(A \to B)_{\mathrm{Kr}} &:\equiv \neg A_{\mathrm{Kr}} \wedge B_{\mathrm{Kr}},\\
(\forall x \leq^{\mathrm{e}} t\, A)_{\mathrm{Kr}} &:\equiv \exists x \leq^{\mathrm{e}} t\, A_{\mathrm{Kr}},\\
(\exists x \leq^{\mathrm{e}} t\, A)_{\mathrm{Kr}} &:\equiv \neg\exists x \leq^{\mathrm{e}} t\, \neg A_{\mathrm{Kr}},\\
(\forall x\, A)_{\mathrm{Kr}} &:\equiv \exists x\, A_{\mathrm{Kr}},\\
(\exists x\, A)_{\mathrm{Kr}} &:\equiv \neg\exists x\, \neg A_{\mathrm{Kr}}.
\end{aligned}$$

4. The *Kuroda negative translation* [57, page 46] [15, section 5] assigns to each formula A of $\mathsf{PA}^\omega_\mathrm{e}$ the formula $A^{\mathrm{Ku}} :\equiv \neg\neg A_{\mathrm{Ku}}$ of $\mathsf{HA}^\omega_\mathrm{e}$ where A_{Ku} is defined by recursion on the structure of A by

$$\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{Ku}} &:\equiv A_{\mathrm{at}},\\
(A \wedge B)_{\mathrm{Ku}} &:\equiv A_{\mathrm{Ku}} \wedge B_{\mathrm{Ku}},\\
(A \vee B)_{\mathrm{Ku}} &:\equiv A_{\mathrm{Ku}} \vee B_{\mathrm{Ku}},\\
(A \to B)_{\mathrm{Ku}} &:\equiv A_{\mathrm{Ku}} \to B_{\mathrm{Ku}},\\
(\forall x \leq^{\mathrm{e}} t\, A)_{\mathrm{Ku}} &:\equiv \forall x \leq^{\mathrm{e}} t\, \neg\neg A_{\mathrm{Ku}},\\
(\exists x \leq^{\mathrm{e}} t\, A)_{\mathrm{Ku}} &:\equiv \exists x \leq^{\mathrm{e}} t\, A_{\mathrm{Ku}},\\
(\forall x\, A)_{\mathrm{Ku}} &:\equiv \forall x\, \neg\neg A_{\mathrm{Ku}},\\
(\exists x\, A)_{\mathrm{Ku}} &:\equiv \exists x\, A_{\mathrm{Ku}}.
\end{aligned}$$

Given a set Γ of formulas of CL, we define $\Gamma^{\mathrm{GG}} := \{A^{\mathrm{GG}} : A \in \Gamma\}$. Analogously for Ko, Kr, Ku, and for CL, PA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{HA}^\omega_\mathrm{i}$.

2.3 Soundness

2.6. Now we prove the soundness theorems of GG, Ko, Kr and Ku, essentially saying that these negative translations map theorems of CL to theorems of IL.

To avoid doing four tedious proofs by induction on derivations, we take a short-cut: in the next proposition we show that GG, Ko, Kr and Ku are equivalent in IL, so it suffices to prove the soundness theorem for one of them. This proposition is also of interest on its own since it says that four different embeddings of CL in IL turn out to be (essentially) the same; in chapter 14 we will see if this is a coincidence or not.

2.7 Proposition. For all formulas A of IL, we have $\mathsf{IL} \vdash A^{\mathrm{GG}} \leftrightarrow A^{\mathrm{Ko}} \leftrightarrow A^{\mathrm{Kr}} \leftrightarrow A^{\mathrm{Ku}}$ [58, pages 42–43] [69, proposition 2.1] [2, lemma 0.2]. Analogously for HA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$, $\mathsf{HA}^\omega_{\mathrm{e}}$ and $\mathsf{HA}^\omega_{\mathrm{i}}$.

2.8 Proof. Let us prove the proposition for $\mathsf{HA}^\omega_{\mathrm{i}}$; the cases of the other theories are analogous. First let us prove

$$\mathsf{HA}^\omega_{\mathrm{i}} \vdash \neg\neg(x \leq^{\mathrm{i}}_\rho y) \to x \leq^{\mathrm{i}}_\rho y \tag{2.4}$$

by induction on the structure of ρ [15, lemma 7]. The base case follows from $x \leq^{\mathrm{i}}_0 y \leftrightarrow x \leq_0 y$ and point 1 of theorem 1.53, so let us see the induction step. We have $x \leq^{\mathrm{i}} y \wedge u \leq^{\mathrm{i}} v \to xu \leq^{\mathrm{i}} yv \wedge yu \leq^{\mathrm{i}} yv$, so $\neg\neg(x \leq^{\mathrm{i}} y) \wedge \neg\neg(u \leq^{\mathrm{i}} v) \to \neg\neg(xu \leq^{\mathrm{i}} yv) \wedge \neg\neg(yu \leq^{\mathrm{i}} yv)$, thus $\neg\neg(x \leq^{\mathrm{i}} y) \wedge u \leq^{\mathrm{i}} v \to xu \leq^{\mathrm{i}} yv \wedge yu \leq^{\mathrm{i}} yv$ by induction hypothesis, therefore $\neg\neg(x \leq^{\mathrm{i}} y) \to x \leq^{\mathrm{i}} y$ by the rule of $\leq^{\mathrm{i}}$.

Now let us prove the proposition by induction on the structure of A.

$\underline{A^{\mathrm{Ku}} \leftrightarrow A^{\mathrm{GG}}}$ Using (2.4) in the fourth equivalence and induction hypothesis in the fifth equivalence, we get

$$\begin{aligned}
(\exists x \leq^{\mathrm{i}} t\, A)^{\mathrm{Ku}} &\equiv \\
\neg\neg\exists x \leq^{\mathrm{i}} t\, A_{\mathrm{Ku}} &\leftrightarrow \\
\neg\neg\exists x\, (x \leq^{\mathrm{i}} t \wedge A_{\mathrm{Ku}}) &\leftrightarrow \\
\neg\neg\exists x\, \neg\neg(x \leq^{\mathrm{i}} t \wedge A_{\mathrm{Ku}}) &\leftrightarrow \\
\neg\neg\exists x\, (\neg\neg(x \leq^{\mathrm{i}} t) \wedge \neg\neg A_{\mathrm{Ku}}) &\leftrightarrow \\
\neg\neg\exists x\, (x \leq^{\mathrm{i}} t \wedge \neg\neg A_{\mathrm{Ku}}) &\equiv \\
\neg\neg\exists x\, (x \leq^{\mathrm{i}} t \wedge A^{\mathrm{Ku}}) &\leftrightarrow \\
\neg\neg\exists x\, (x \leq^{\mathrm{i}} t \wedge A^{\mathrm{GG}}) &\leftrightarrow \\
\neg\forall x\, \neg(x \leq^{\mathrm{i}} t \wedge A^{\mathrm{GG}}) &\leftrightarrow \\
\neg\forall x\, (x \leq^{\mathrm{i}} t \to \neg A^{\mathrm{GG}}) &\leftrightarrow \\
\neg\forall x \leq^{\mathrm{i}} t\, \neg A^{\mathrm{GG}} &\equiv \\
(\exists x \leq^{\mathrm{i}} t\, A)^{\mathrm{GG}}&.
\end{aligned}$$

Analogously for A_{at}, $\wedge$, $\vee$, $\to$, $\forall{\leq^{\mathrm{i}}}$, $\forall$ and $\exists$.

$\underline{A^{\mathrm{Ku}} \leftrightarrow A^{\mathrm{Ko}}}$ Using induction hypothesis in the last equivalence, we get

$$\begin{aligned}
(A \vee B)^{\mathrm{Ku}} &\equiv \\
\neg\neg(A_{\mathrm{Ku}} \vee B_{\mathrm{Ku}}) &\leftrightarrow \\
\neg\neg(\neg\neg A_{\mathrm{Ku}} \vee \neg\neg B_{\mathrm{Ku}}) &\equiv \\
\neg\neg(A^{\mathrm{Ku}} \vee B^{\mathrm{Ku}}) &\leftrightarrow \\
\neg\neg(A^{\mathrm{Ko}} \vee B^{\mathrm{Ko}}) &\equiv \\
(A \vee B)^{\mathrm{Ko}}&.
\end{aligned}$$

Analogously for A_{at}, $\wedge$, $\rightarrow$, $\exists\leq^{\mathrm{i}}$, $\forall\leq^{\mathrm{i}}$, $\forall$ and $\exists$.

$\underline{A^{\mathrm{GG}} \leftrightarrow A^{\mathrm{Kr}}}$ Using the induction hypothesis in the first equivalence, we get

$$\begin{aligned}
(A \rightarrow B)^{\mathrm{GG}} &\equiv \\
(A^{\mathrm{GG}} \rightarrow B^{\mathrm{GG}}) &\leftrightarrow \\
A^{\mathrm{Kr}} \rightarrow B^{\mathrm{Kr}} &\equiv \\
(\neg A_{\mathrm{Kr}} \rightarrow \neg B_{\mathrm{Kr}}) &\leftrightarrow \\
\neg(\neg A_{\mathrm{Kr}} \wedge B_{\mathrm{Kr}}) &\equiv \\
(A \rightarrow B)^{\mathrm{Kr}}&.
\end{aligned}$$

Analogously for A_{at}, $\wedge$, $\vee$, $\exists\leq^{\mathrm{i}}$, $\forall\leq^{\mathrm{i}}$, $\forall$ and $\exists$.

2.9 Theorem (soundness)**.** Let Γ be a set of formulas of CL. For the pair $\mathsf{CL} + \Gamma, \mathsf{IL} + \Gamma^{\mathrm{GG}}$ we have: if $\mathsf{CL} + \Gamma \vdash A$, then $\mathsf{IL} + \Gamma^{\mathrm{GG}} \vdash A^{\mathrm{GG}}$ [27] [29, theorem I] [23] [24, theorem III]. Analogously for the pairs

$$\begin{gathered}
\mathsf{PA}^\omega \pm \mathsf{QF\text{-}AC} + \Gamma,\ \mathsf{HA}^\omega \pm \mathsf{QF\text{-}AC} \pm \mathsf{QF\text{-}MP} + \Gamma^{\mathrm{GG}}, \\
\mathsf{WE\text{-}PA}^\omega \pm \mathsf{QF\text{-}AC} + \Gamma,\ \mathsf{WE\text{-}HA}^\omega \pm \mathsf{QF\text{-}AC} \pm \mathsf{QF\text{-}MP} + \Gamma^{\mathrm{GG}}, \\
\mathsf{E\text{-}PA}^\omega \pm \mathsf{QF\text{-}AC} + \Gamma,\ \mathsf{E\text{-}HA}^\omega \pm \mathsf{QF\text{-}AC} \pm \mathsf{QF\text{-}MP} + \Gamma^{\mathrm{GG}}, \\
\mathsf{PA}^\omega_{\mathrm{e}} \pm \mathsf{B\text{-}BAC} \pm \mathsf{B\text{-}BCC} \pm \mathsf{MAJ} + \Gamma,\ \mathsf{HA}^\omega_{\mathrm{e}} \pm \mathsf{B\text{-}BAC} \pm \tilde{\exists}\mathsf{F\text{-}BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BMP} + \Gamma^{\mathrm{GG}}, \\
\mathsf{PA}^\omega_{\mathrm{i}} \pm \mathsf{B\text{-}BAC} \pm \mathsf{B\text{-}BCC} \pm \mathsf{MAJ} + \Gamma,\ \mathsf{HA}^\omega_{\mathrm{i}} \pm \mathsf{B\text{-}BAC} \pm \tilde{\forall}\mathsf{\text{-}BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BMP} + \Gamma^{\mathrm{GG}},
\end{gathered}$$

where in each pair the sign $\pm$ is taken the same everywhere [15, proposition 5] [75, theorem 1.10.11(ii)], and analogously for Ko [54] [53, section 3] and Kr [2, theorem 0.1.2] [69, proposition 2.1] and Ku [57, page 46].

2.10 Proof. By proposition 2.7 it suffices to prove the theorem for Ku. The proof is by induction on the derivation of A. When translating an axiom or rule, if it is easy to see that its translation is provable, we simply state the translation without comments.

$\underline{A \vee \neg A}$ We have

$$(A \vee \neg A)^{\mathrm{Ku}} \equiv \neg\neg(A_{\mathrm{Ku}} \vee \neg A_{\mathrm{Ku}}).$$

Analogously for $A \rightarrow A \wedge A$, $A \vee A \rightarrow A$, $A \wedge B \rightarrow A$, $A \rightarrow A \vee B$, $A \wedge B \rightarrow B \wedge A$, $A \vee B \rightarrow B \vee A$ and $\bot \rightarrow A$.

$\underline{\forall x\, A \to A[t/x]}$ We have

$$(\forall x\, A \to A[t/x])^{\mathrm{Ku}} \equiv \neg\neg(\forall x\, \neg\neg A_{\mathrm{Ku}} \to A[t/x]_{\mathrm{Ku}}).$$

Here we use $A_{\mathrm{Ku}}[t/x] \equiv A[t/x]_{\mathrm{Ku}}$. Analogously for $A[t/x] \to \exists x\, A$.

$\underline{A \to B \,/\, C \vee A \to C \vee B}$ We have

$$(A \to B)^{\mathrm{Ku}} \equiv \neg\neg(A_{\mathrm{Ku}} \to B_{\mathrm{Ku}}),$$
$$(C \vee A \to C \vee B)^{\mathrm{Ku}} \equiv \neg\neg(C_{\mathrm{Ku}} \vee A_{\mathrm{Ku}} \to C_{\mathrm{Ku}} \vee B_{\mathrm{Ku}}).$$

From $\neg\neg(A_{\mathrm{Ku}} \to B_{\mathrm{Ku}})$ we get $A_{\mathrm{Ku}} \to \neg\neg B_{\mathrm{Ku}}$, so $C_{\mathrm{Ku}} \vee A_{\mathrm{Ku}} \to C_{\mathrm{Ku}} \vee \neg\neg B_{\mathrm{Ku}}$, thus $C_{\mathrm{Ku}} \vee A_{\mathrm{Ku}} \to \neg\neg C_{\mathrm{Ku}} \vee \neg\neg B_{\mathrm{Ku}}$, therefore $C_{\mathrm{Ku}} \vee A_{\mathrm{Ku}} \to \neg\neg(C_{\mathrm{Ku}} \vee B_{\mathrm{Ku}})$, concluding $\neg\neg(C_{\mathrm{Ku}} \vee A_{\mathrm{Ku}} \to C_{\mathrm{Ku}} \vee B_{\mathrm{Ku}})$. Analogously for A, $A \to B \,/\, B$, $A \to B$, $B \to C \,/\, A \to C$, $A \wedge B \to C \,/\, A \to (B \to C)$ and $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, A \to \forall x\, B}$ We have

$$(A \to B)^{\mathrm{Ku}} \equiv \neg\neg(A_{\mathrm{Ku}} \to B_{\mathrm{Ku}}),$$
$$(A \to \forall x\, B)^{\mathrm{Ku}} \equiv \neg\neg(A_{\mathrm{Ku}} \to \forall x\, \neg\neg A_{\mathrm{Ku}}).$$

Here we use $x \notin \mathrm{FV}(A) = \mathrm{FV}(A_{\mathrm{Ku}})$. Analogously for $A \to B \,/\, \exists x\, A \to B$.

$\underline{\text{Axioms of } =_0, \text{S}, \Pi, \Sigma \text{ and } \underline{\text{R}}, \text{ and } x \leq^{\mathrm{i}}_0 y \leftrightarrow x \leq_0 y}$ Their translation is their double negation, which follows from the axioms themselves.

$\underline{A_{\mathrm{at}} \to s = t \,/\, A_{\mathrm{at}} \to r(s) =_0 r(t)}$ We have

$$(A_{\mathrm{at}} \to s = t)^{\mathrm{Ku}} \equiv \neg\neg(A_{\mathrm{at}} \to (s = t)_{\mathrm{Ku}}),$$
$$(r(s) =_0 r(t))^{\mathrm{Ku}} \equiv \neg\neg(A_{\mathrm{at}} \to r(s) =_0 r(t)).$$

Here we use $(s = t)_{\mathrm{Ku}} \leftrightarrow s = t$ (because if $s =_\rho t \equiv \forall x_1, \ldots, x_n\, (s x_1 \cdots x_n =_0 t x_1 \cdots x_n)$, then $(s =_\rho t)_{\mathrm{Ku}} \equiv \forall x_1 \neg\neg \cdots \forall x_n \neg\neg(s x_1 \cdots x_n =_0 t x_1 \cdots x_n)$ and $\neg\neg\forall x\, \neg\neg A \leftrightarrow \forall x\, \neg\neg A$ holds intuitionistically). Analogously for the induction rule, the extensionality axioms, the axioms of the bounded quantifications and $x \leq^{\mathrm{i}} y \to \forall u \leq^{\mathrm{i}} v\, (xu \leq^{\mathrm{i}} yv \wedge yu \leq^{\mathrm{i}} yv)$.

$\underline{A_{\mathrm{b}} \wedge x \leq^{\mathrm{i}} y \to sx \leq^{\mathrm{i}} ty \wedge tx \leq^{\mathrm{i}} ty \,/\, A_{\mathrm{b}} \to s \leq^{\mathrm{i}} t}$ We have

$$(A_{\mathrm{b}} \wedge x \leq^{\mathrm{i}} y \to sx \leq^{\mathrm{i}} ty \wedge tx \leq^{\mathrm{i}} ty)^{\mathrm{Ku}} \equiv$$
$$\neg\neg((A_{\mathrm{b}})_{\mathrm{Ku}} \wedge x \leq^{\mathrm{i}} y \to sx \leq^{\mathrm{i}} ty \wedge tx \leq^{\mathrm{i}} ty)$$
$$(A_{\mathrm{b}} \to s \leq^{\mathrm{i}} t)^{\mathrm{Ku}} \equiv \neg\neg((A_{\mathrm{b}})_{\mathrm{Ku}} \to s \leq^{\mathrm{i}} t).$$

Note that $(A_{\mathrm{b}})_{\mathrm{Ku}}$ is a bounded formula. From $\neg\neg((A_{\mathrm{b}})_{\mathrm{Ku}} \wedge x \leq^{\mathrm{i}} y \to sx \leq^{\mathrm{i}} ty \wedge tx \leq^{\mathrm{i}} ty)$ we get $(A_{\mathrm{b}})_{\mathrm{Ku}} \wedge x \leq^{\mathrm{i}} y \to \neg\neg(sx \leq^{\mathrm{i}} ty) \wedge \neg\neg(tx \leq^{\mathrm{i}} ty)$, that is $(A_{\mathrm{b}})_{\mathrm{Ku}} \wedge x \leq^{\mathrm{i}} y \to sx \leq^{\mathrm{i}} ty \wedge tx \leq^{\mathrm{i}} ty$ by (2.4), thus $(A_{\mathrm{b}})_{\mathrm{Ku}} \to s \leq^{\mathrm{i}} t$ by the rule of $\leq^{\mathrm{i}}$, concluding $\neg\neg((A_{\mathrm{b}})_{\mathrm{Ku}} \to s \leq^{\mathrm{i}} t)$.

$\underline{\mathsf{QF\text{-}AC}}$ Say $\underline{x} \equiv x_1, \ldots, x_n$. We have

$$\mathsf{QF\text{-}AC}^{\mathrm{Ku}} \equiv$$
$$\neg\neg(\forall x_1 \neg\neg \cdots \forall x_n \neg\neg\exists \underline{y}\, A_{\mathrm{qf}}(\underline{x}, \underline{y}) \to \exists \underline{Y}\, \forall x_1 \neg\neg \cdots \forall x_n \neg\neg A_{\mathrm{qf}}(\underline{x}, \underline{Y}\underline{x})).$$

This formula is equivalent to $\neg\neg(\forall \underline{x} \neg\neg\exists \underline{y}\, A_{\mathrm{qf}}(\underline{x}, \underline{y}) \to \exists \underline{Y}\, \forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}, \underline{Y}\underline{x}))$, which by $\mathsf{QF\text{-}MP}$ is equivalent to $\neg\neg\mathsf{QF\text{-}AC}$.

$\underline{\mathsf{B\text{-}BAC}}$ We have

$$\mathsf{B\text{-}BAC}^{\mathrm{Ku}} \equiv$$
$$\neg\neg(\forall x \neg\neg\exists \underline{y}\, (A_{\mathrm{b}})_{\mathrm{Ku}} \to \tilde{\exists} \underline{v}\, \forall u \neg\neg(u \leq^{\mathrm{i}} u \to \forall x \leq^{\mathrm{e}} u \neg\neg\exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, (A_{\mathrm{b}})_{\mathrm{Ku}}).$$

We have $\neg\neg\exists \underline{z}\, B_{\mathrm{b}} \to \tilde{\exists} \underline{z}' \neg\neg\exists \underline{z} \leq^{\mathrm{i}} \underline{z}'\, B_{\mathrm{b}}$ by $\mathsf{B\text{-}BMP}$ [15, section 4.1]. So from the premise of $\mathsf{B\text{-}BAC}^{\mathrm{Ku}}$ we get $\forall x\, \tilde{\exists} \underline{y}' \neg\neg\exists \underline{y} \leq^{\mathrm{i}} \underline{y}'\, (A_{\mathrm{b}})_{\mathrm{Ku}}$, thus $\exists \underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u\, \tilde{\exists} \underline{y}' \leq^{\mathrm{i}} \underline{v}u \neg\neg\exists \underline{y} \leq^{\mathrm{i}} \underline{y}'\, (A_{\mathrm{b}})_{\mathrm{Ku}}$, therefore $\exists \underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u \neg\neg\exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, (A_{\mathrm{b}})_{\mathrm{Ku}}$, getting the conclusion of $\mathsf{B\text{-}BAC}^{\mathrm{Ku}}$. Analogously for MAJ.

$\underline{\mathsf{B\text{-}BCC}}$ It is difficult to prove $\mathsf{B\text{-}BCC}^{\mathrm{Ku}}$ directly, so instead we prove $\mathsf{B\text{-}BC}^{\mathrm{Ku}}$ because $\mathsf{B\text{-}BC}$ (generalised to tuples) and $\mathsf{B\text{-}BCC}$ are the contrapositive of each other. The proof is analogously to the case of $\mathsf{B\text{-}BAC}$, using $\mathsf{B\text{-}BC}$ itself to prove $\mathsf{B\text{-}BC}^{\mathrm{Ku}}$ by point 1 of proposition 1.66.

2.4 Characterisation

2.11. Now we prove the characterisation theorems of GG, Ko, Kr and Ku. Informally, these theorems say that, as far as CL is concerned, GG, Ko, Kr and Ku do not change the meaning of formulas.

2.12 Theorem (characterisation)**.** We have $\mathsf{CL} \vdash A \leftrightarrow A^{\mathrm{GG}}$ [23] [24, theorem V]. Analogously for Ko [75, section 1.10.1], Kr [2, theorem 0.1.1] [69, proposition 2.1] and Ku [75, section 1.10.1], PA^ω, $\mathsf{WE\text{-}PA}^\omega$, $\mathsf{E\text{-}PA}^\omega$ [75, theorem 1.10.11(i)], $\mathsf{PA}^\omega_{\mathrm{e}}$ and $\mathsf{PA}^\omega_{\mathrm{i}}$.

2.13 Proof. By proposition 2.7 it suffices to prove the theorem for Ku. The formula A^{Ku} is obtained from A by adding double negations in subformulas of A, so it is equivalent to A.

2.5 Applications

2.14. Now we give two applications of negative translations.

<u>Conservation for formulas without $\vee$ and $\exists$</u> The first application says (essentially) that CL and IL prove exactly the same formulas without $\vee$ and $\exists$. This makes sense if we recall that (informally) the difference between CL and IL is that IL attaches a stronger meaning to $\vee$ and $\exists$:

1. $A \vee B$ means "A or B, and we can point to one that holds true";
2. $\exists x\, A(x)$ means "there exists an x such that $A(x)$, and we know such an x".

So, if we drop $\vee$ and $\exists$, then the difference between CL and IL disappears, and that is what our first application says.

Equiconsistency of CL and IL We can think of CL as a logical framework for the usual mathematics, and IL as a logical framework for constructivism. So, for the sake of our argument, let us identify CL with the usual mathematics, and IL with constructivism.

One of the motivations for constructivism is that constructivism is sounder than the usual mathematics, that is constructivism is less likely to produce contradictions than the usual mathematics. Our second application denies this: CL and IL are equiconsistent, that is the usual mathematics and constructivism are equally sound.

2.15 Definition. The *negative fragment* is the set NF of formulas of $\mathsf{HA}^\omega_\mathrm{e}$ generated recursively by:

1. $\perp \in \mathrm{NF}$;
2. $\neg\neg A_\mathrm{at} \in \mathrm{NF}$;
3. if $A, B \in \mathrm{NF}$, then $A \wedge B, A \to B, \forall x \leq^\mathrm{e} t\, A, \forall x\, A \in \mathrm{NF}$.

Analogously for CL, HA^ω, $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{HA}^\omega_\mathrm{i}$.

2.16. In other words, NF is the set of formulas without disjunctions and existential (bounded and unbounded) quantifications, and with all atomic subformulas negated (possibly with the exception of $\perp$).

2.17 Theorem (conservation and relative consistency)**.**

1. Let $A \in \mathrm{NF}$ and $\Gamma \subseteq \mathrm{NF}$. For the pair $\mathsf{CL} + \Gamma, \mathsf{IL} + \Gamma$ we have: if $\mathsf{CL} + \Gamma \vdash A$, then $\mathsf{IL} + \Gamma \vdash A$ [23] [24, theorem IV].
2. For the pair $\mathsf{CL} + \Gamma, \mathsf{IL} + \Gamma$ we have: if $\mathsf{CL} + \Gamma \vdash \perp$, then $\mathsf{IL} + \Gamma \vdash \perp$ [23] [24, theorem VI] [27] [29, page 295].

Analogously for the pairs

$$\mathsf{PA}^\omega \pm \mathsf{QF\text{-}AC} + \Gamma,\ \mathsf{HA}^\omega \pm \mathsf{QF\text{-}AC} \pm \mathsf{QF\text{-}MP} + \Gamma^\mathrm{GG},$$
$$\mathsf{WE\text{-}PA}^\omega \pm \mathsf{QF\text{-}AC} + \Gamma,\ \mathsf{WE\text{-}HA}^\omega \pm \mathsf{QF\text{-}AC} \pm \mathsf{QF\text{-}MP} + \Gamma^\mathrm{GG},$$
$$\mathsf{E\text{-}PA}^\omega \pm \mathsf{QF\text{-}AC} + \Gamma,\ \mathsf{E\text{-}HA}^\omega \pm \mathsf{QF\text{-}AC} \pm \mathsf{QF\text{-}MP} + \Gamma^\mathrm{GG},$$
$$\mathsf{PA}^\omega_\mathrm{e} \pm \mathsf{B\text{-}BAC} \pm \mathsf{B\text{-}BCC} \pm \mathsf{MAJ} + \Gamma,\ \mathsf{HA}^\omega_\mathrm{e} \pm \mathsf{B\text{-}BAC} \pm \tilde{\exists}\mathsf{F\text{-}BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BMP} + \Gamma^\mathrm{GG},$$
$$\mathsf{PA}^\omega_\mathrm{i} \pm \mathsf{B\text{-}BAC} \pm \mathsf{B\text{-}BCC} \pm \mathsf{MAJ} + \Gamma,\ \mathsf{HA}^\omega_\mathrm{i} \pm \mathsf{B\text{-}BAC} \pm \forall\mathsf{\text{-}BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BMP} + \Gamma^\mathrm{GG},$$

where in each pair the sign $\pm$ is taken the same everywhere.

2.18 Proof. We only do the proof for the pair $\mathsf{CL}+\Gamma, \mathsf{IL}+\Gamma$; the cases of the other pairs are analogous.

1. We can prove $\mathsf{IL} \vdash B^{\mathrm{GG}} \leftrightarrow B$ for all $B \in \mathrm{NF}$ by induction on the structure of B. So we have $\mathsf{IL} \vdash A^{\mathrm{GG}} \leftrightarrow A$ and $\mathsf{IL} + \Gamma^{\mathrm{GG}} = \mathsf{IL} + \Gamma$. Assuming the premise of the theorem, by the soundness theorem of GG we get the conclusion of the theorem.

2. Follows from the previous point.

2.6 Conclusion

2.19. We introduced four negative translations GG, Ko, Kr and Ku as embeddings of CL into IL. The main results about these negative translations are the following.

Soundness and characterisation theorems Negative translations embed CL in IL.

Applications We used negative translations to do applications on:

1. conservation;
2. relative consistency.

Chapter 3

Modified realisability

3.1 Introduction

3.1. In this chapter we introduce a proof interpretation called modified realisability. Maybe the best way to motivate modified realisability is by means of the BHK (Brouwer-Heyting-Kolmogorov) interpretation [33, pages 14 and 17] [50, section 3.1]. This interpretation explains the constructive meaning of the symbols $\wedge$, $\vee$, $\to$, $\forall$ and $\exists$ by saying, by recursion on the structure of A, what it takes to constructively prove A, that is what is the meaning of "a is a (constructive) proof of A". For example,

$$\text{a proof of } A \wedge B \text{ is a pair } a, b \text{ where } a \text{ is a proof of } A \text{ and } b \text{ is a proof of } B.$$

(Note that we are collecting two proofs a and b in a pair a, b; if we iterate this, we end up with tuples $a, b, c, \ldots$ of proofs, so below we use tuples.) Here is the complete BHK interpretation:

$$\begin{aligned}
\text{a proof of } A \wedge B &\text{ is a tuple } \underline{a}, \underline{b} \text{ where } \underline{a} \text{ is a proof of } A \text{ and } \underline{b} \text{ is a proof of } B,\\
\text{a proof of } A \vee B &\text{ is a tuple } c, \underline{a}, \underline{b} \text{ where if } c = 0 \text{ then } \underline{a} \text{ is a proof of } A,\\
&\text{ and if } c \neq 0 \text{ then } \underline{b} \text{ is a proof of } B,\\
\text{a proof of } A \to B &\text{ is a tuple of functions } \underline{B} \text{ that map each proof } \underline{a} \text{ of } A\\
&\text{ to a proof } \underline{B}\underline{a} \text{ of } B,\\
\text{a proof of } \forall x\, A(x) &\text{ is a tuple of functions } \underline{A} \text{ that for all } x \text{ give a proof } \underline{A}x \text{ of } A(x),\\
\text{a proof of } \exists x\, A(x) &\text{ is a tuple } x, \underline{a} \text{ where } \underline{a} \text{ is a proof of } A(x).
\end{aligned}$$

Let us note the following two aspects of the BHK interpretation.

1. In the clause of $\vee$, the BHK interpretation asks for a c that decides if we proved A or if we proved B. So the BHK interpretation tries to decide disjunctions.

2. In the clause of $\exists$, the BHK interpretation asks for an x such that we proved $A(x)$. So the BHK interpretation tries to witness existential quantifications.

These two aspects give to the BHK interpretation a constructive nature.

Let us denote "$\underline{a}$ is a proof of A" by $A_{\mathrm{mr}}(\underline{a})$; then the BHK interpretation becomes

$$\begin{aligned}
(A \wedge B)_{\mathrm{mr}}(\underline{a}, \underline{b}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \wedge B_{\mathrm{mr}}(\underline{b}), \\
(A \vee B)_{\mathrm{mr}}(c, \underline{a}, \underline{b}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \vee_c B_{\mathrm{mr}}(\underline{b}), \\
(A \to B)_{\mathrm{mr}}(\underline{B}) &:\equiv \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \to B_{\mathrm{mr}}(\underline{B}\underline{a})), \\
(\forall x\, A(x))_{\mathrm{mr}}(\underline{A}) &:\equiv \forall x\, A(x)_{\mathrm{mr}}(\underline{A}x), \\
(\exists x\, A(x))_{\mathrm{mr}}(x, \underline{a}) &:\equiv A(x)_{\mathrm{mr}}(\underline{a}).
\end{aligned}$$

This is modified realisability mr.

3.2. We also present two variants with truth mrq and mrt of modified realisability mr; let us motivate these variants. Modified realisability mr maps an original formula A to the interpreted formula $A_{\mathrm{mr}}(\underline{a})$ and gives us information about the latter. However, we usually want information about the former. If the formula A belongs to a certain class of formulas Γ, then holds the so-called truth property $(*)$ $A_{\mathrm{mr}}(\underline{a}) \to A$, so we can transfer the information from $A_{\mathrm{mr}}(\underline{a})$ to A. But in general we lose a connection between $A_{\mathrm{mr}}(\underline{a})$ and A. The variants with truth change mr in such a way that $(*)$ holds for larger classes Γ: mrq enlarges Γ to include disjunctive and existential formulas, and mrt further enlarges Γ to include all formulas. This is pictured in figure 3.1.

$$\begin{array}{lcccccc}
\mathrm{mr}: & A & \xrightarrow{\text{information}} & A_{\mathrm{mr}}(\underline{a}) & \xrightarrow[\text{for } A \in \Gamma]{\text{transference}} & A \\
\mathrm{mrq}: & A & \xrightarrow{\text{information}} & A_{\mathrm{mrq}}(\underline{a}) & \xrightarrow[\text{for } \Gamma, \vee, \exists]{\text{transference}} & A \\
\mathrm{mrt}: & A & \xrightarrow{\text{information}} & A_{\mathrm{mrt}}(\underline{a}) & \xrightarrow[\text{for all } A]{\text{transference}} & A
\end{array}$$

Figure 3.1: transference of information by mr, mrq and mrt.

3.3. Our main contribution to this topic is the characterisation theorems for mrq and mrt [22, theorem 2.6] (theorem 3.14).

3.2 Definition

3.4 Definition.

1. *Modified realisability* mr [55, paragraph 3.52] assigns to each formula A of HA^ω the formula $A^{\mathrm{mr}} :\equiv \exists \underline{a}\, A_{\mathrm{mr}}(\underline{a})$, where $A_{\mathrm{mr}}(\underline{a})$ is defined by induction on the

structure of A by

$$
\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{mr}}() &:\equiv A_{\mathrm{at}}, \\
(A \wedge B)_{\mathrm{mr}}(\underline{a}, \underline{b}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \wedge B_{\mathrm{mr}}(\underline{b}), \\
(A \vee B)_{\mathrm{mr}}(c^0, \underline{a}, \underline{b}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \vee_c B_{\mathrm{mr}}(\underline{b}), \\
(A \to B)_{\mathrm{mr}}(\underline{B}) &:\equiv \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \to B_{\mathrm{mr}}(\underline{B}\underline{a})), \\
(\forall x\, A)_{\mathrm{mr}}(\underline{A}) &:\equiv \forall x\, A_{\mathrm{mr}}(\underline{A}x), \\
(\exists x\, A)_{\mathrm{mr}}(x, \underline{a}) &:\equiv A_{\mathrm{mr}}(\underline{a}).
\end{aligned}
$$

By $(A_{\mathrm{at}})_{\mathrm{mr}}()$ we mean $(A_{\mathrm{at}})_{\mathrm{mr}}(\underline{a})$ with the tuple $\underline{a}$ empty.

2. *Modified realisability with q-truth* mrq [45] [75, definition 3.4.2] is defined analogously to mr except for

$$
\begin{aligned}
(A \vee B)_{\mathrm{mrq}}(c^0, \underline{a}, \underline{b}) &:\equiv (A_{\mathrm{mrq}}(\underline{a}) \wedge A) \vee_c (B_{\mathrm{mrq}}(\underline{b}) \wedge B), \\
(A \to B)_{\mathrm{mrq}}(\underline{B}) &:\equiv \forall \underline{a}\, (A_{\mathrm{mrq}}(\underline{a}) \wedge A \to B_{\mathrm{mrq}}(\underline{B}\underline{a})), \\
(\exists x\, A)_{\mathrm{mrq}}(x, \underline{a}) &:\equiv A_{\mathrm{mrq}}(\underline{a}) \wedge A.
\end{aligned}
$$

3. *Modified realisability with t-truth* mrt [31] [78, exercise 9.7.11 in chapter 9] is defined analogously to mr except for

$$(A \to B)_{\mathrm{mrt}}(\underline{B}) :\equiv \forall \underline{a}\, (A_{\mathrm{mrt}}(\underline{a}) \to B_{\mathrm{mrt}}(\underline{B}\underline{a})) \wedge (A \to B).$$

Analogously for $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$.

3.5. Modified realisability has the word "modified" in its name likely because it is a modification of Kleene's recursive realisability [43, section 5].

3.6 Remark.

1. Modified realisability with q-truth mrq has truth in the sense of: for all disjunctive and existential formulas A of HA^ω we have $\mathsf{HA}^\omega \vdash A_{\mathrm{mrq}}(\underline{a}) \to A$ [22, remark 2.2].

2. Modified realisability with t-truth mrt has truth in the sense of: for all formulas A of HA^ω we have $\mathsf{HA}^\omega \vdash A_{\mathrm{mrt}}(\underline{a}) \to A$ [78, exercise 9.7.11 in chapter 9].

Modified realisability with t-truth mrt is a $(*_1)$ strengthening of mrq which $(*_2)$ has truth for all formulas. This can be given a rigorous meaning: $(*_3)$ $\mathsf{HA}^\omega \vdash A_{\mathrm{mrt}}(\underline{a}) \leftrightarrow A_{\mathrm{mrq}}(\underline{a}) \wedge A$ for all formulas A of HA^ω [22, theorem 2.5]. From $(*_3)$ we get: $\mathsf{HA}^\omega \vdash A_{\mathrm{mrt}}(\underline{a}) \to A_{\mathrm{mrq}}(\underline{a})$, that is $(*_1)$; $\mathsf{HA}^\omega \vdash A_{\mathrm{mrt}}(\underline{a}) \to A$, that is $(*_2)$. It follows the analogous statements for $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$.

3.7 Remark. The formulas $A_{\mathrm{mr}}(\underline{a})$ are $\exists$-free.

3.8 Remark.

1. Modified realisability mr acts as the identity on $\exists$-free formulas of HA^ω in the sense of: $(A_{\exists\mathrm{f}})_{\mathrm{mr}}() \equiv A_{\exists\mathrm{f}}$ for all $\exists$-free formulas $A_{\exists\mathrm{f}}$ of HA^ω [75, remark 3.4.4].

2. Modified realisability with q-truth mrq acts as the identity on $\exists$-free formulas of HA^ω in the sense of: $\mathsf{HA}^\omega \vdash (A_{\exists\mathrm{f}})_{\mathrm{mrq}}() \leftrightarrow A_{\exists\mathrm{f}}$ for all $\exists$-free formulas $A_{\exists\mathrm{f}}$ of HA^ω [75, remark 3.4.4]. Analogously for mrt. (For mrt, this even holds true for negated formulas [50, proposition 5.7].)

It follows the analogous statements for $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$.

3.3 Soundness

3.9. Now we are going to prove the main theorem about mr: the soundness theorem. This theorem says that if $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F\text{-}IP} \vdash A$, then we can effectively (with an algorithm given by the proof of the soundness theorem) extract from a proof of A terms $\underline{t}$ such that $A_{\mathrm{mr}}(\underline{t})$. These terms $\underline{t}$ encapsulate computational content from the proof of A; for example:

1. if A is a disjunction $B \vee C$, then $\underline{t}$ decide between B and C;
2. if A is an existential statement $\exists x\, B(x)$, then $\underline{t}$ witness x;
3. if A is of the form $\forall x\, \exists y\, B(x, y)$, then $\underline{t}$ give y as a function of x.

(Actually, mr decides between $B_{\mathrm{mr}}(\underline{b})$ and $C_{\mathrm{mr}}(\underline{c})$, witnesses x in $B_{\mathrm{mr}}(\underline{b})$, and gives y as a function of x in $B(x, y)_{\mathrm{mr}}(\underline{b})$. If B and C are $\exists$-free, or if we use mrq and mrt instead, or if we move to the theory of the characterisation theorem below, then the information on $B_{\mathrm{mr}}(\underline{b})$ and $C_{\mathrm{mr}}(\underline{c})$ transfers to B and C.)

3.10. In the next theorem, the sentence "if $\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F\text{-}IP} + \Gamma \vdash A$, then ... $\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F\text{-}IP} + \Gamma \vdash A_{\mathrm{mrq}}(\underline{t})$" abbreviates the following four possibilities:

1. "if $\mathsf{HA}^\omega + \Gamma \vdash A$, then ... $\mathsf{HA}^\omega + \Gamma \vdash A_{\mathrm{mrq}}(\underline{t})$";
2. "if $\mathsf{HA}^\omega + \mathsf{AC} + \Gamma \vdash A$, then ... $\mathsf{HA}^\omega + \mathsf{AC} + \Gamma \vdash A_{\mathrm{mrq}}(\underline{t})$";
3. "if $\mathsf{HA}^\omega + \exists\mathsf{F\text{-}IP} + \Gamma \vdash A$, then ... $\mathsf{HA}^\omega + \exists\mathsf{F\text{-}IP} + \Gamma \vdash A_{\mathrm{mrq}}(\underline{t})$";
4. "if $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F\text{-}IP} + \Gamma \vdash A$, then ... $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F\text{-}IP} + \Gamma \vdash A_{\mathrm{mrq}}(\underline{t})$".

Analogously for $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}HA}^\omega$ and mrt.

3.11 Theorem (soundness)**.** Let A be a formula of HA^ω and let Γ be a set of $\exists$-free formulas of HA^ω.

1. If $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F\text{-}IP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega + \Gamma \vdash A_{\mathrm{mr}}(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$ [75, theorem 3.4.5] [50, theorem 5.13].
2. If $\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F\text{-}IP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F\text{-}IP} + \Gamma \vdash A_{\mathrm{mrq}}(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$ [75, theorem 3.4.5].
3. If $\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F\text{-}IP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F\text{-}IP} + \Gamma \vdash A_{\mathrm{mrt}}(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$ [78, exercise 9.7.11 in chapter 9] [50, theorem 5.23].

The terms constructed in the following proof for the three points above are the same. Analogously for $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$.

3.12 Proof. Let us make some remarks. We do the remarks only for HA^ω, but they also work for $\mathsf{WE\text{-}HA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$.

1. We will treat mr, mrq and mrt in a unified manner in the following way. Let id and $\top$ be functions, mapping formulas of HA^ω to formulas of HA^ω, defined by $A^{\mathrm{id}} :\equiv A$ and $A^\top :\equiv 0 =_0 0$. Let $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$. We redefine mr by changing some of its clauses to

$$\begin{aligned}(A \vee B)_{\mathrm{mr}}(c^0, \underline{a}, \underline{b}) &:\equiv (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}}) \vee_c (B_{\mathrm{mr}}(\underline{b}) \wedge B^{\mathrm{q}}),\\ (A \to B)_{\mathrm{mr}}(\underline{B}) &:\equiv \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to B_{\mathrm{mr}}(\underline{B}\underline{a})) \wedge (A \to B)^{\mathrm{t}},\\ (\exists x\, A)_{\mathrm{mr}}(x, \underline{a}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}}.\end{aligned}$$

   This redefined mr reduces:

   (a) to the old mr when $\mathrm{q} = \top$ and $\mathrm{t} = \top$;

   (b) to mrq when $\mathrm{q} = \mathrm{id}$ and $\mathrm{t} = \top$;

   (c) to mrt when $\mathrm{q} = \top$ and $\mathrm{t} = \mathrm{id}$ (or when $\mathrm{q} = \mathrm{id}$ and $\mathrm{t} = \mathrm{id}$).

   By reducing we mean, for example, $\mathsf{HA}^\omega \vdash A_{\mathrm{mr}}(\underline{a}) \leftrightarrow A_{\mathrm{mrq}}(\underline{a})$ in second case. We prove the soundness theorem for the redefined mr, hence proving the theorem for the old mr, for mrq and for mrt. Moreover, the terms working for them will not depend on q and t, so they are the same.

2. The interpretation of a formula of the form $A \to B$ is of the form $\ldots \wedge (A \to B)^{\mathrm{t}}$. So to prove that $A \to B$ is interpretable we have in particular to prove $(A \to B)^{\mathrm{t}}$: if $\mathrm{t} = \top$ it is trivial, and if $\mathrm{t} = \mathrm{id}$ it follows from the formula itself. Since the argument is always the same, we will systematically omit $(A \to B)^{\mathrm{t}}$. When we do it, we write "$\equiv$" instead of $\equiv$.

   We also use "$\equiv$" if, following remark 3.8, we replace $(A_{\exists\mathrm{f}})_{\mathrm{mrq}}()$ or $(A_{\exists\mathrm{f}})_{\mathrm{mrt}}()$ by $A_{\exists\mathrm{f}}$.

3. When interpreting a rule $A,\, B \,/\, C$,

   (a) we denote by $\underline{r}_{\underline{x}}$ the terms that by induction hypothesis exist witnessing some variables $\underline{x}$ in the interpretation of A;

   (b) we denote by $\underline{s}_{\underline{y}}$ the terms that by induction hypothesis exist witnessing some variables $\underline{y}$ in the interpretation of B;

   (c) we denote by $\underline{t}_{\underline{z}}$ the terms that we will construct witnessing some variables $\underline{z}$ in the interpretation of C.

4. The claim of the theorem asks for terms $\underline{t}$ such that $\mathsf{HA}^\omega \vdash A_{\mathrm{mr}}(\underline{t})$ and $(*)$ $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$. We do not worry with $(*)$ because if the terms $\underline{t}$ do not satisfy $(*)$, then we can replace them by the terms $\underline{t}' :\equiv \underline{t}[\underline{\mathcal{O}}/\underline{x}]$, where $\mathrm{FV}(\underline{t}) \setminus \mathrm{FV}(A) = \{\underline{x}\}$, satisfying $\mathsf{HA}^\omega \vdash A_{\mathrm{mr}}(\underline{t}')$ and $\mathrm{FV}(\underline{t}') \subseteq \mathrm{FV}(A)$.

5. When interpreting an axiom or rule, if it is easy to see that the terms work, we simply state the interpretation and the terms without comments.

Let us prove the theorem by induction on the derivation of A.

$\underline{A \vee A \to A}$ We have

$$(A \vee A \to A)_{\mathrm{mr}}(\underline{C}) \text{ “}\equiv\text{”}$$
$$\forall d, \underline{a}, \underline{b} \left(\left((A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}}) \vee_d (A_{\mathrm{mr}}(\underline{b}) \wedge A^{\mathrm{q}}) \right) \wedge (A \vee A)^{\mathrm{q}} \to A_{\mathrm{mr}}(\underline{C} d \underline{a} \underline{b}) \right),$$

$$\underline{t_C} :\equiv \lambda d, \underline{a}, \underline{b} \,.\, (\underline{a} \vee_d \underline{b}).$$

Analogously for $A \to A \wedge A$.

$\underline{A \to A \vee B}$ We have

$$(A \to A \vee B)_{\mathrm{mr}}(D, \underline{B}, \underline{C}) \text{ “}\equiv\text{”}$$
$$\forall \underline{a} \left(A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to (A_{\mathrm{mr}}(\underline{B} \underline{a}) \wedge A^{\mathrm{q}}) \vee_{D \underline{a}} (B_{\mathrm{mr}}(\underline{C} \underline{a}) \wedge B^{\mathrm{q}}) \right),$$

$$t_D :\equiv \mathcal{O}, \quad \underline{t_B} :\equiv \lambda \underline{a} \,.\, \underline{a}, \quad \underline{t_C} :\equiv \underline{\mathcal{O}}.$$

Analogously for $A \wedge B \to A$ and $\perp \to A$.

$\underline{A \vee B \to B \vee A}$ We have

$$(A \vee B \to B \vee A)_{\mathrm{mr}}(F, \underline{C}, \underline{D}) \text{ “}\equiv\text{”}$$
$$\forall e, \underline{a}, \underline{b} \left(\left((A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}}) \vee_e (B_{\mathrm{mr}}(\underline{b}) \wedge B^{\mathrm{q}}) \right) \wedge (A \vee B)^{\mathrm{q}} \to \right.$$
$$\left. \left((B_{\mathrm{mr}}(\underline{C} e \underline{a} \underline{b}) \wedge B^{\mathrm{q}}) \vee_{F e \underline{a} \underline{b}} (A_{\mathrm{mr}}(\underline{D} e \underline{a} \underline{b}) \wedge A^{\mathrm{q}}) \right) \right),$$

$$t_F :\equiv \lambda e, \underline{a}, \underline{b} \,.\, \overline{\mathrm{sg}}\, e, \quad \underline{t_C} :\equiv \lambda e, \underline{a}, \underline{b} \,.\, \underline{b}, \quad \underline{t_D} :\equiv \lambda e, \underline{a}, \underline{b} \,.\, \underline{a}.$$

Analogously for $A \wedge B \to B \wedge A$.

$\underline{A[t/x] \to \exists x\, A}$ We have

$$(A[t/x] \to \exists x\, A)_{\mathrm{mr}}(X, \underline{B}) \text{ “}\equiv\text{”}$$
$$\forall \underline{a} \, (A[t/x]_{\mathrm{mr}}(\underline{a}) \wedge A[t/x]^{\mathrm{q}} \to A_{\mathrm{mr}}(\underline{b})[X \underline{a}, \underline{B} \underline{a}/x, \underline{b}] \wedge A^{\mathrm{q}}[X \underline{a}/x]),$$

$$t_X :\equiv \lambda \underline{a} \,.\, t, \quad \underline{t_B} :\equiv \lambda \underline{a} \,.\, \underline{a}.$$

Here we use $A_{\mathrm{mr}}(\underline{b})[t, \underline{a}/x, \underline{b}] \equiv A[t/x]_{\mathrm{mr}}(\underline{a})$. Analogously for $\forall x\, A \to A[t/x]$.

$\underline{A \to B,\ B \to C \,/\, A \to C}$ We have

$$(A \to B)_{\mathrm{mr}}(\underline{B}) \text{ “}\equiv\text{” } \forall \underline{a} \, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to B_{\mathrm{mr}}(\underline{B} \underline{a})), \tag{3.1}$$
$$(B \to C)_{\mathrm{mr}}(\underline{C}) \text{ “}\equiv\text{” } \forall \underline{b} \, (B_{\mathrm{mr}}(\underline{b}) \wedge B^{\mathrm{q}} \to C_{\mathrm{mr}}(\underline{C} \underline{b})), \tag{3.2}$$
$$(A \to C)_{\mathrm{mr}}(\underline{C}) \text{ “}\equiv\text{” } \forall \underline{a} \, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to C_{\mathrm{mr}}(\underline{C} \underline{a})),$$
$$\underline{t_C} :\equiv \lambda \underline{a} \,.\, \underline{s_C}(\underline{r_B} \underline{a}).$$

If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved $A \to B$, so that the part A^{q} in (3.1) implies the part B^{q} in (3.2). Analogously for $A,\ A \to B \,/\, B$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ We have

$$\begin{gathered}(A \wedge B \to C)_{\mathrm{mr}}(\underline{C}) \equiv \\ \forall \underline{a}, \underline{b}\, (A_{\mathrm{mr}}(\underline{a}) \wedge B_{\mathrm{mr}}(\underline{b}) \wedge (A \wedge B)^{\mathrm{q}} \to C_{\mathrm{mr}}(\underline{C}\underline{a}\underline{b})) \wedge (A \wedge B \to C)^{\mathrm{t}},\end{gathered} \tag{3.3}$$

$$\begin{gathered}(A \to (B \to C))_{\mathrm{mr}}(\underline{\mathrm{C}}) \equiv \\ \forall \underline{a}\, \big(A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to \forall \underline{b}\, (B_{\mathrm{mr}}(\underline{b}) \wedge B^{\mathrm{q}} \to C_{\mathrm{mr}}(\underline{\mathrm{C}}\underline{a}\underline{b})) \wedge (B \to C)^{\mathrm{t}}\big) \wedge \\ (A \to (B \to C))^{\mathrm{t}},\end{gathered} \tag{3.4}$$

$$\underline{t}_{\underline{\mathrm{C}}} :\equiv \underline{s}_{\underline{C}}.$$

If t = id, then we use $A_{\mathrm{mr}}(\underline{a}) \to A$, so that the parts $(A \wedge B \to C)^{\mathrm{t}}$ in (3.3) and $A_{\mathrm{mr}}(\underline{a})$ in (3.4) together imply the part $(B \to C)^{\mathrm{t}}$ in (3.4). Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, C \vee A \to C \vee B}$ We have

$$(A \to B)_{\mathrm{mr}}(\underline{B}) \text{ “}{\equiv}\text{” } \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to B_{\mathrm{mr}}(\underline{B}\underline{a})),$$

$$\begin{gathered}(C \vee A \to C \vee B)_{\mathrm{mr}}(F, \underline{D}, \underline{B}) \text{ “}{\equiv}\text{”} \\ \forall e, \underline{c}, \underline{a}\, \big((C_{\mathrm{mr}}(\underline{c}) \wedge C^{\mathrm{q}}) \vee_e (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}}) \to \\ (C_{\mathrm{mr}}(\underline{D}e\underline{c}\underline{a}) \wedge C^{\mathrm{q}}) \vee_{Fe\underline{c}\underline{a}} (B_{\mathrm{mr}}(\underline{B}e\underline{c}\underline{a}) \wedge B^{\mathrm{q}})\big),\end{gathered} \tag{3.5}$$

$$t_F :\equiv \lambda e, \underline{c}, \underline{a} \,.\, e, \quad \underline{t}_{\underline{D}} :\equiv \lambda e, \underline{c}, \underline{a} \,.\, \underline{c}, \quad \underline{t}_{\underline{B}} :\equiv \lambda e, \underline{c}, \underline{a} \,.\, \underline{s}_{\underline{B}}\underline{a}.$$

If q = id, then we use the assumption that we proved $A \to B$, so that the part A^{q} in (3.5) implies the part B^{q} in (3.5).

$\underline{A \to B \,/\, A \to \forall x\, B}$ We have

$$\begin{gathered}(A \to B)_{\mathrm{mr}}(\underline{B}) \text{ “}{\equiv}\text{” } \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to B_{\mathrm{mr}}(\underline{B}\underline{a})), \\ (A \to \forall x\, B)_{\mathrm{mr}}(\underline{\mathrm{B}}) \text{ “}{\equiv}\text{” } \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to \forall x\, B_{\mathrm{mr}}(\underline{\mathrm{B}}x\underline{a})), \\ \underline{t}_{\underline{\mathrm{B}}} :\equiv \lambda x, \underline{a} \,.\, \underline{s}_{\underline{B}}\underline{a}.\end{gathered}$$

Here we use $x \notin \mathrm{FV}(A) \cup \{\underline{a}\} = \mathrm{FV}(A_{\mathrm{mr}}(\underline{a}))$. Analogously for $A \to B \,/\, \exists x\, A \to B$.

Axioms of $=_0$, S, Π, Σ and $\underline{\mathrm{R}}$, and extensionality rule and axioms Their formulas are ∃-free, so they are equivalent to their own interpretation.

$\underline{A[0/x],\ A \to A[\mathrm{S}x/x] \,/\, A}$ We have

$$\begin{gathered}A[0/x]_{\mathrm{mr}}(\underline{a}), \\ (A \to A[\mathrm{S}x/x])_{\mathrm{mr}}(\underline{B}) \text{ “}{\equiv}\text{” } \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to A[\mathrm{S}x/x]_{\mathrm{mr}}(\underline{B}\underline{a})), \\ A_{\mathrm{mr}}(\underline{a}), \\ \underline{t}_{\underline{a}}(x) :\equiv \underline{\mathrm{R}}x\underline{r}_{\underline{a}}\, \lambda \underline{a}, x \,.\, \underline{s}_{\underline{B}}(x)\underline{a}.\end{gathered}$$

By induction hypothesis we have

$$A[0/x]_{\mathrm{mr}}(\underline{r_a}), \tag{3.6}$$

$$\forall \underline{a} \left(A_{\mathrm{mr}}(\underline{a}) \wedge A^{\mathrm{q}} \to A[\mathrm{S}x/x]_{\mathrm{mr}}(\underline{s_B}(x)\underline{a}) \right). \tag{3.7}$$

Let us prove $\forall x\, A_{\mathrm{mr}}(\underline{t_a}(x))$ by induction on x.

Base case The formula $A_{\mathrm{mr}}(\underline{t_a}(x))[0/x]$ is equivalent to (3.6).

Induction step By induction hypothesis we assume $A_{\mathrm{mr}}(\underline{t_a}(x))$. Taking $\underline{a} = \underline{t_a}(x)$ in (3.7) we get $A[\mathrm{S}x/x]_{\mathrm{mr}}(\underline{s_B}(x)\underline{t_a}(x))$, that is $A_{\mathrm{mr}}(\underline{t_a}(x))[\mathrm{S}x/x]$. If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved A, so as to have the part A^{q} in (3.7).

AC To keep the notation simple, we denote $A(x,y)_{\mathrm{mr}}(\underline{a})[Yx/x]$ and $A(x,Yx)_{\mathrm{mr}}(\underline{a})$ by $A_{\mathrm{mr}}(\underline{A}; x, Yx)$. We have

$$\begin{gathered} \mathsf{AC}_{\mathrm{mr}}(\mathrm{Y}, \underline{\mathrm{B}}) \text{ “}\equiv\text{”} \\ \forall Y, \underline{A} \left(\forall x\, (A_{\mathrm{mr}}(\underline{A}x; x, Yx) \wedge A(x,Yx)^{\mathrm{q}}) \wedge (\forall x\, \exists y\, A(x,y))^{\mathrm{q}} \to \right. \\ \left. \forall x\, A_{\mathrm{mr}}(\underline{\mathrm{B}}Y\underline{A}x; x, \mathrm{Y}\underline{A}Yx) \wedge (\forall x\, A(x, \mathrm{Y}Y\underline{A}x))^{\mathrm{q}} \right), \\ t_{\mathrm{Y}} :\equiv \lambda Y, \underline{A}, x\,.\, Yx, \quad \underline{t_{\mathrm{B}}} :\equiv \lambda Y, \underline{A}, x\,.\, \underline{A}x. \end{gathered}$$

$\exists$F-IP We have

$$\begin{gathered} \exists\mathsf{F\text{-}IP}_{\mathrm{mr}}(X, \underline{B}) \text{ “}\equiv\text{”} \\ \forall x, \underline{a} \left((A_{\exists\mathrm{f}} \wedge A^{\mathrm{q}}_{\exists\mathrm{f}} \to B_{\mathrm{mr}}(\underline{a}; x) \wedge B(x)^{\mathrm{q}}) \wedge \right. \\ (A_{\exists\mathrm{f}} \to \exists x\, B(x))^{\mathrm{t}} \wedge (A_{\exists\mathrm{f}} \to \exists x\, B(x))^{\mathrm{q}} \to \\ \left. (A_{\exists\mathrm{f}} \wedge A^{\mathrm{q}}_{\exists\mathrm{f}} \to B_{\mathrm{mr}}(\underline{B}x\underline{a}; Xx\underline{a})) \wedge (A_{\exists\mathrm{f}} \to B(Xx\underline{a}))^{\mathrm{t}} \wedge (A_{\exists\mathrm{f}} \to B(Xx\underline{a}))^{\mathrm{q}} \right), \\ t_X :\equiv \lambda x, \underline{a}\,.\, x, \quad \underline{t_{\mathrm{B}}} :\equiv \lambda x, \underline{a}\,.\, \underline{a}. \end{gathered}$$

If $\mathrm{t} = \mathrm{id}$, then we use $B_{\mathrm{mr}}(\underline{a}; x) \to B(x)$, so that the part $A_{\exists\mathrm{f}} \wedge A^{\mathrm{q}}_{\exists\mathrm{f}} \to B_{\mathrm{mr}}(\underline{a}; x) \wedge B(x)^{\mathrm{q}}$ in the premise implies the part $(A_{\exists\mathrm{f}} \to B(Xx\underline{a}))^{\mathrm{t}}$ in the conclusion.

Γ We have

$$(A_{\exists\mathrm{f}})_{\mathrm{mr}}() \text{ “}\equiv\text{”} A_{\exists\mathrm{f}}.$$

3.4 Characterisation

3.13. Now we prove the so-called characterisation theorem, saying $\mathsf{HA}^{\omega} + \mathsf{AC} + \exists\mathsf{F\text{-}IP} \vdash A \leftrightarrow A^{\mathrm{mr}}$. There are several ways of reading this theorem.

Distance We can think of the characterisation theorem as measuring the “displacement” created by mr, that is the “distance” between A and A^{mr}: the strongest the theory needed to prove $A \leftrightarrow A^{\mathrm{mr}}$, the greater the “distance” between A and A^{mr}.

<u>Construction</u> We can think of the characterisation theorem (and its proof) as showing us how A^{mr} is constructed from A. In particular, as identifying the principles used in that construction: AC and $\exists\mathsf{F}\text{-}\mathsf{IP}$.

<u>Optimality</u> We can use the characterisation theorem to show that the theory interpreted in the soundness theorem is optimal, that is it cannot be strengthened (we do this in remark 3.16).

<u>Transference</u> We can use the characterisation theorem to transfer information from the interpreted formula $A_{\mathrm{mr}}(\underline{a})$ to the original formula A. For example, if we proved $\exists x\, A(x)$ in $\mathsf{T} := \mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$, then the soundness theorem gives us terms $t, \underline{s}$ witnessing $x, \underline{a}$ in the interpreted formula $(\exists x\, A(x))_{\mathrm{mr}}(x, \underline{a}) \equiv A(x)_{\mathrm{mr}}(\underline{a})$, and the characterisation theorem allows us to transfer t to the original formula $A(x)$, as pictured in figure 3.2.

$$\mathsf{T} \vdash \exists x\, A(x) \quad \xRightarrow[t,\underline{s}]{\text{soundness}} \quad \mathsf{T} \vdash A(t)_{\mathrm{mr}}(\underline{s}) \quad \xRightarrow[A(t)_{\mathrm{mr}}(\underline{s}) \to A(t)^{\mathrm{mr}} \leftrightarrow A(t)]{\text{characterisation}} \quad \mathsf{T} \vdash A(t)$$

Figure 3.2: transference of t from $A(t)_{\mathrm{mr}}(\underline{s})$ to $A(t)$ by the characterisation theorem.

3.14 Theorem (characterisation)**.** Let us consider the theory $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$.

1. This theory proves $A \leftrightarrow A^{\mathrm{mr}}$ for all formulas A of HA^ω [75, theorem 3.4.8].
2. This theory is the least theory, containing HA^ω, satisfying the previous point.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$, $\mathsf{E}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$, mrq and mrt [22, theorem 2.6].

3.15 Proof. We only do the proof for $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$; the cases of the other theories are analogous. Let us prove the claim of the theorem for mr.

1. The proof is by induction on the structure of A.

   $\underline{\vee}$ Using HA^ω in the first equivalence, and induction hypothesis in the second equivalence, we get

$$\begin{aligned}
&A \vee B \leftrightarrow \\
&\exists c\, ((c =_0 0 \to A) \wedge (c \neq_0 0 \to B)) \leftrightarrow \\
&\exists c\, ((c =_0 0 \to A^{\mathrm{mr}}) \wedge (c \neq_0 0 \to B^{\mathrm{mr}})) \equiv \\
&\exists c\, \big((c =_0 0 \to \exists \underline{a}\, A_{\mathrm{mr}}(\underline{a})) \wedge (c \neq_0 0 \to \exists \underline{b}\, B_{\mathrm{mr}}(\underline{b}))\big) \leftrightarrow \\
&\exists c, \underline{a}, \underline{b}\, \big((c =_0 0 \to A_{\mathrm{mr}}(\underline{a})) \wedge (c \neq_0 0 \to B_{\mathrm{mr}}(\underline{b}))\big) \leftrightarrow \\
&(A \wedge B)^{\mathrm{mr}}.
\end{aligned}$$

   Analogously for A_{at}, $\vee$ and $\exists$.

$\underline{\to}$ Using induction hypothesis in the first equivalence, $\exists$F-IP in the third equivalence, and AC in the last equivalence, we get

$$
\begin{aligned}
(A \to B) &\leftrightarrow \\
A^{\mathrm{mr}} \to B^{\mathrm{mr}} &\equiv \\
(\exists \underline{a}\, A_{\mathrm{mr}}(\underline{a}) \to \exists \underline{b}\, A_{\mathrm{mr}}(\underline{b})) &\leftrightarrow \\
\forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \to \exists \underline{b}\, A_{\mathrm{mr}}(\underline{b})) &\leftrightarrow \\
\forall \underline{a}\, \exists \underline{b}\, (A_{\mathrm{mr}}(\underline{a}) \to A_{\mathrm{mr}}(\underline{b})) &\leftrightarrow \\
\exists \underline{B}\, \forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \to A_{\mathrm{mr}}(\underline{B}\underline{a})) &\equiv \\
(A \to B)^{\mathrm{mr}}&.
\end{aligned}
$$

Analogously for $\forall$.

2. Let T be a theory, containing HA^ω, that proves the equivalences $A \leftrightarrow A^{\mathrm{mr}}$ for all formulas A. Let P be one of the principles AC and $\exists$F-IP. Let us show $\mathsf{T} \vdash \mathsf{P}$. By the soundness theorem of mr we have $\mathsf{HA}^\omega \vdash \mathsf{P}^{\mathrm{mr}}$, so $\mathsf{T} \vdash \mathsf{P}^{\mathrm{mr}}$, thus $\mathsf{T} \vdash \mathsf{P}$.

Now let us prove the claim of the theorem for mrq and mrt.

1. The point 1 of the theorem for mrq and mrt follows from the point 1 for mr by proving by induction on the structure of A that $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F\text{-}IP}$ proves $(*_1)$ $A_{\mathrm{mr}}(\underline{a}) \leftrightarrow A_{\mathrm{mrq}}(\underline{a})$ and $(*_2)$ $A_{\mathrm{mr}}(\underline{a}) \leftrightarrow A_{\mathrm{mrt}}(\underline{a})$.

<u>Proof of $(*_1)$</u> Let us only see the case of $\vee$; the cases of A_{at}, $\wedge$, $\to$, $\forall$ and $\exists$ are analogous. Using $A_{\mathrm{mr}}(\underline{a}) \to A^{\mathrm{mr}} \leftrightarrow A$ and $B_{\mathrm{mr}}(\underline{b}) \to B^{\mathrm{mr}} \leftrightarrow B$ in the first equivalence, and induction hypothesis in the second equivalence, we get

$$
\begin{aligned}
(A \vee B)_{\mathrm{mr}}(c, \underline{a}, \underline{b}) &\equiv \\
A_{\mathrm{mr}}(\underline{a}) \vee_c A_{\mathrm{mr}}(\underline{b}) &\leftrightarrow \\
(A_{\mathrm{mr}}(\underline{a}) \wedge A) \vee_c (A_{\mathrm{mr}}(\underline{b}) \wedge B) &\leftrightarrow \\
(A_{\mathrm{mrq}}(\underline{a}) \wedge A) \vee_c (A_{\mathrm{mrq}}(\underline{b}) \wedge B) &\equiv \\
(A \vee B)_{\mathrm{mrq}}(c, \underline{a}, \underline{b})&.
\end{aligned}
$$

<u>Proof of $(*_2)$</u> Let us only see the case of $\to$; the cases of A_{at}, $\wedge$, $\vee$, $\forall$ and $\exists$ are analogous. Using $(A \to B)_{\mathrm{mr}}(\underline{B}) \to (A \to B)^{\mathrm{mr}} \leftrightarrow (A \to B)$ in the first equivalence, and induction hypothesis in the second equivalence, we get

$$
\begin{aligned}
(A \to B)_{\mathrm{mr}}(\underline{B}) &\equiv \\
\forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \to A_{\mathrm{mr}}(\underline{B}\underline{a})) &\leftrightarrow \\
\forall \underline{a}\, (A_{\mathrm{mr}}(\underline{a}) \to A_{\mathrm{mr}}(\underline{B}\underline{a})) \wedge (A \to B) &\leftrightarrow \\
\forall \underline{a}\, (A_{\mathrm{mrt}}(\underline{a}) \to A_{\mathrm{mrt}}(\underline{B}\underline{a})) \wedge (A \to B) &\equiv \\
(A \to B)_{\mathrm{mrt}}(\underline{B})&.
\end{aligned}
$$

2. We adopt here the remarks made in the beginning of proof 3.12. Let $\mathrm{q} = \mathrm{id}$ or $\mathrm{t} = \mathrm{id}$. Let T be a theory, containing HA^ω, that proves the equivalences $(*)$ $A \leftrightarrow A^{\mathrm{mr}}$ for all formulas A. Let us show $\mathsf{T} \vdash \mathsf{AC}$ and $\mathsf{T} \vdash \exists\mathsf{F}\text{-}\mathsf{IP}$.

   $\underline{\mathsf{T} \vdash \mathsf{AC}}$ Using $(*)$ in the first implication, and $A(x, Yx)_{\mathrm{mr}}(\underline{A}x) \to A(x, Yx)$ in the second implication if $\mathrm{t} = \mathrm{id}$, we get

$$\begin{aligned}\forall x\, \exists y\, A(x,y) &\to \\ (\forall x\, \exists y\, A(x,y))^{\mathrm{mr}} &\equiv \\ \exists Y, \underline{A}\, \forall x\, (A(x,Yx)_{\mathrm{mr}}(\underline{A}x) \wedge A(x,Yx)^{\mathrm{q}}) &\to \\ \exists Y\, \forall x\, A(x, Yx)&.\end{aligned}$$

   $\underline{\mathsf{T} \vdash \exists\mathsf{F}\text{-}\mathsf{IP}}$ Using $(*)$ in the first implication, and $B_{\mathrm{mr}}(\underline{a}) \to B$ in the second implication if $\mathrm{t} = \mathrm{id}$, we get

$$\begin{aligned}(A_{\exists\mathrm{f}} \to \exists x\, B) &\to \\ (A_{\exists\mathrm{f}} \to \exists x\, B)^{\mathrm{mr}} &\text{ “}\equiv\text{”} \\ \exists x, \underline{a}\, \big((A_{\exists\mathrm{f}} \wedge A^{\mathrm{q}}_{\exists\mathrm{f}} \to B_{\mathrm{mr}}(\underline{a}) \wedge B^{\mathrm{q}}) \wedge (A_{\exists\mathrm{f}} \to \exists x\, B)^{\mathrm{t}}\big) &\to \\ \exists x\, (A_{\exists\mathrm{f}} \to B)&.\end{aligned}$$

3.16 Remark. The characterisation theorem of mr ensures that the soundness theorem of mr is optimal, in the sense that the theory $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP} + \Gamma$ there considered is the strongest theory T such that $(*)$ $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^\omega + \Gamma \vdash A^{\mathrm{mr}}$ (because $(*)$ implies $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP} + \Gamma \vdash A$). Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega$ and $\mathsf{E}\text{-}\mathsf{HA}^\omega$.

3.5 Applications

3.17. We finish with applications of mr, mrq and mrt. They illustrate the use of proof interpretations in general (as most proof interpretations have similar applications), and of modified realisability in particular. These applications are optimised for simplicity, not generality, since we intend them to be “short, simple and sweet” illustrations of what can be done with proof interpretations.

3.18 Theorem (disjunction property, existence property and program extraction)**.** Let $\mathsf{T} := \mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F}\text{-}\mathsf{IP}$.

1. Let $A \vee B$ be a sentence of T. If $\mathsf{T} \vdash A \vee B$, then $\mathsf{T} \vdash A$ or $\mathsf{T} \vdash B$.

2. If $\mathsf{T} \vdash \exists \underline{x}\, A(\underline{x})$, then we can extract from such a proof terms $\underline{t}$ of T such that $\mathsf{T} \vdash A(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A)$.

3. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A(\underline{x}, \underline{y})$, then we can extract from such a proof terms $\underline{t}(\underline{x})$ of T such that $\mathsf{T} \vdash \forall \underline{x}\, A(\underline{x}, \underline{t}(\underline{x}))$ and $\mathrm{FV}(\underline{t}(\underline{x})) \subseteq \mathrm{FV}(\exists \underline{y}\, A(\underline{x}, \underline{y}))$.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F}\text{-}\mathsf{IP}$ and $\mathsf{E}\text{-}\mathsf{HA}^\omega \pm \mathsf{AC} \pm \exists\mathsf{F}\text{-}\mathsf{IP}$ [75, theorem 3.7.2] [50, corollary 5.24].

3.19 Proof.

1. We have $(A \vee B)_{\mathrm{mrt}}(c^0, \underline{a}, \underline{b}) \equiv A_{\mathrm{mrt}}(\underline{a}) \vee_c B_{\mathrm{mrt}}(\underline{b})$. Assuming the premise of the theorem, by the soundness theorem of mrt we can extract closed terms $t^0, \underline{r}, \underline{s}$ of T such that $\mathsf{T} \vdash A_{\mathrm{mrt}}(\underline{r}) \vee_t B_{\mathrm{mrt}}(\underline{s})$. By truth we get $\mathsf{T} \vdash A \vee_t B$. By point 3 of theorem 1.30 we have $t \equiv \bar{n}$ for some $n \in \mathbb{N}$. If $n = 0$, then $\mathsf{T} \vdash A$; if $n \neq 0$, then $\mathsf{T} \vdash B$.

2. We have $(\exists \underline{x}\, A(\underline{x}))_{\mathrm{mrt}}(\underline{x}, \underline{a}) \equiv A(\underline{x})_{\mathrm{mrt}}(\underline{a})$. Assuming the premise of the theorem, by the soundness theorem of mrt we can extract terms $\underline{s}, \underline{t}$ of T such that $\mathsf{T} \vdash A(\underline{t})_{\mathrm{mrt}}(\underline{s})$ and $\mathrm{FV}(\underline{s}, \underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A(\underline{x}))$. By truth we get $\mathsf{T} \vdash A(\underline{t})$.

3. Follows from the previous point.

3.20 Theorem (conservation and relative consistency)**.**

1. If $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP} \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\exists\mathrm{f}}$, then $\mathsf{HA}^\omega \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\exists\mathrm{f}}$.

2. If $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP} \vdash \bot$, then $\mathsf{HA}^\omega \vdash \bot$.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega$ and $\mathsf{E}\text{-}\mathsf{HA}^\omega$ [75, theorem 3.6.6(ii)] [50, corollary 5.21].

3.21 Proof.

1. We have $(\forall \underline{x}\, \exists \underline{y}\, A_{\exists\mathrm{f}}(\underline{x}, \underline{y}))_{\mathrm{mr}}(\underline{Y}) \equiv \forall \underline{x}\, A_{\exists\mathrm{f}}(\underline{x}, \underline{Y}\underline{x})$. Assuming the premise of the theorem, by the soundness theorem of mr we can extract terms $\underline{t}$ of HA^ω such that $\mathsf{HA}^\omega \vdash \forall \underline{x}\, A_{\exists\mathrm{f}}(\underline{x}, \underline{t}\underline{x})$. So we get the conclusion of the theorem.

2. Follows from the previous point.

3.22 Theorem (independence)**.** Let $\mathsf{T} := \mathsf{E}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$.

1. We have $\mathsf{T} \nvdash \mathsf{QF}\text{-}\mathsf{MP}$ and $\mathsf{T} \nvdash \neg\mathsf{QF}\text{-}\mathsf{MP}$ [55, paragraph 3.52].

2. We have $\mathsf{T} \nvdash \mathsf{LEM}$ and $\mathsf{T} \nvdash \neg\mathsf{LEM}$ (already for Σ^0_1 and Π^0_1 formulas).

It follows the analogous statements for $\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$ and $\mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \exists\mathsf{F}\text{-}\mathsf{IP}$.

3.23 Proof. It suffices to show that $\mathsf{QF}\text{-}\mathsf{MP}$ and LEM are unprovable in T, since their negations cannot be proved in T (otherwise $\mathsf{PA}^\omega + \mathsf{AC}$ would be inconsistent). Using definition 1.37 we can define an atomic formula $\mathrm{T}x^0y^0z^0$ of T representing Kleene's T predicate asserting that the Turing machine (coded by) x, when given the input (coded by) y, halts with computation history (coded by) z.

1. We have $(\neg\neg\exists y\, \mathrm{T}xxy \to \exists z\, \mathrm{T}xxz)_{\mathrm{mr}}(z) \equiv \neg\forall y\, \neg\mathrm{T}xxy \to \mathrm{T}xxz$. By contradiction, we assume $\mathsf{T} \vdash \mathsf{QF}\text{-}\mathsf{MP}$, thus $\mathsf{T} \vdash \neg\neg\exists y\, \mathrm{T}xxy \to \exists z\, \mathrm{T}xxz$. So by the soundness theorem of mr we can extract a term $t(x)$ of $\mathsf{E}\text{-}\mathsf{HA}^\omega$ such that $\mathsf{E}\text{-}\mathsf{HA}^\omega \vdash \neg\forall y\, \neg\mathrm{T}xxy \to \mathrm{T}xxt(x)$. This term $t(x)$ induces a computable function (also denoted by) $t(x)$ such that $\mathbb{N} \models \exists y\, \mathrm{T}xxy \to \mathrm{T}xxt(x)$. So $t(x)$ solves the halting problem, a contradiction.

2. We have $(\exists y\, \mathrm{T}xxy \vee \neg\exists z\, \mathrm{T}xxz)_{\mathrm{mr}}(a, y) \equiv \mathrm{T}xxy \vee_a \forall z\, \neg\mathrm{T}xxz$. By contradiction, we assume $\mathsf{T} \vdash \mathsf{LEM}$, thus $(*)$ $\mathsf{T} \vdash \exists y\, \mathrm{T}xxy \vee \neg\exists z\, \mathrm{T}xxz$, an instance of LEM for Σ^0_1 formulas. So by the soundness theorem of mr we can extract terms $s(x)$ and $t(x)$ of $\mathsf{E\text{-}HA}^\omega$ such that $\mathsf{E\text{-}HA}^\omega \vdash \mathrm{T}xxs(x) \vee_{t(x)} \forall z\, \neg\mathrm{T}xxz$. Thus $\mathbb{N} \models \exists y\, \mathrm{T}xxy \vee_{t(x)} \forall z\, \neg\mathrm{T}xxz$. So $t(x)$ solves the halting problem, a contradiction. Analogously for Π^0_1 formulas using $\mathsf{T} \vdash \forall y\, \neg\mathrm{T}xxy \vee \neg\forall z\, \neg\mathrm{T}xxz$ instead of $(*)$.

3.6 Conclusion

3.24. We introduced modified realisability, motivated by the BHK interpretation. The main results about modified realisability are the following.

Soundness theorem This theorem says that we can use modified realisability to extract computational content from proofs in $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \exists\mathsf{F\text{-}IP}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used modified realisability to do applications on:

1. disjunction property;
2. existence property;
3. program extraction;
4. conservation;
5. relative consistency;
6. independence.

Chapter 4

Bounded modified realisability

4.1 Introduction

4.1. The modified realisability mr (essentially) extracts exact witnesses for existential statements: given a theorem $\exists x\, A(x)$, extracts a term t such that $A(t)$. Now we introduce the bounded modified realisability br that (essentially) extracts bounds instead of exact witnesses: given a theorem $\exists x\, A(x)$, extracts a term t such that $\exists x \leq^{\mathrm{e}} t\, A(x)$. This change from exact witnesses to bounds is mainly obtained by changing the clause of $\exists x$ from asking for x to asking for a bound b on x:

$$(\exists x\, A)_{\mathrm{mr}}(x, \underline{a}) :\equiv A_{\mathrm{mr}}(\underline{a}),$$
$$(\exists x\, A)_{\mathrm{br}}(b, \underline{a}) :\equiv \exists x \leq^{\mathrm{e}} b\, A_{\mathrm{br}}(\underline{a}).$$

We also introduce two variants with truth of br: the bounded modified realisability with q-truth brq and the bounded modified realisability with t-truth brt.

4.2. Our main contributions to this topic are the following.

1. The bounded modified realisabilities with q-truth brq and with t-truth brt and their soundness and characterisation theorems [22, section 5] (definition 4.3 and theorems 4.10 and 4.12).

2. The bounded existence property and the bounded program extraction (theorem 4.15).

4.2 Definition

4.3 Definition.

1. The *bounded modified realisability* br [14, definition 4] assigns to each formula A of $\mathsf{HA}^{\omega}_{\mathrm{e}}$ the formula $A^{\mathrm{br}} :\equiv \tilde{\exists}\underline{a}\, A_{\mathrm{br}}(\underline{a})$, where $A_{\mathrm{br}}(\underline{a})$ is defined by recursion

on the structure of A by

$$\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{br}}() &:\equiv A_{\mathrm{at}},\\
(A \wedge B)_{\mathrm{br}}(\underline{a}, \underline{b}) &:\equiv A_{\mathrm{br}}(\underline{a}) \wedge B_{\mathrm{br}}(\underline{b}),\\
(A \vee B)_{\mathrm{br}}(\underline{a}, \underline{b}) &:\equiv A_{\mathrm{br}}(\underline{a}) \vee B_{\mathrm{br}}(\underline{b}),\\
(A \to B)_{\mathrm{br}}(\underline{B}) &:\equiv \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \to B_{\mathrm{br}}(\underline{B}\underline{a})),\\
(\forall x \leq^{\mathrm{e}} t\, A)_{\mathrm{br}}(\underline{a}) &:\equiv \forall x \leq^{\mathrm{e}} t\, A_{\mathrm{br}}(\underline{a}),\\
(\exists x \leq^{\mathrm{e}} t\, A)_{\mathrm{br}}(\underline{a}) &:\equiv \exists x \leq^{\mathrm{e}} t\, A_{\mathrm{br}}(\underline{a}),\\
(\forall x\, A)_{\mathrm{br}}(\underline{A}) &:\equiv \tilde{\forall} b\, \forall x \leq^{\mathrm{e}} b\, A_{\mathrm{br}}(\underline{A}b),\\
(\exists x\, A)_{\mathrm{br}}(b, \underline{a}) &:\equiv \exists x \leq^{\mathrm{e}} b\, A_{\mathrm{br}}(\underline{a}).
\end{aligned}$$

By $(A_{\mathrm{at}})_{\mathrm{br}}()$ we mean $(A_{\mathrm{at}})_{\mathrm{br}}(\underline{a})$ with the tuple $\underline{a}$ empty.

2. The *bounded modified realisability with q-truth* brq [22, definition 5.1] is defined analogously to br except for

$$\begin{aligned}
(A \vee B)_{\mathrm{brq}}(\underline{a}, \underline{b}) &:\equiv (A_{\mathrm{brq}}(\underline{a}) \wedge A) \vee (B_{\mathrm{brq}}(\underline{b}) \wedge B),\\
(A \to B)_{\mathrm{brq}}(\underline{B}) &:\equiv \tilde{\forall}\underline{a}\,(A_{\mathrm{brq}}(\underline{a}) \wedge A \to B_{\mathrm{brq}}(\underline{B}\underline{a})),\\
(\exists x \leq^{\mathrm{e}} t\, A)_{\mathrm{brq}}(\underline{a}) &:\equiv \exists x \leq^{\mathrm{e}} t\,(A_{\mathrm{brq}}(\underline{a}) \wedge A),\\
(\exists x\, A)_{\mathrm{brq}}(b, \underline{a}) &:\equiv \exists x \leq^{\mathrm{e}} b\,(A_{\mathrm{brq}}(\underline{a}) \wedge A).
\end{aligned}$$

3. The *bounded modified realisability with t-truth* brt [22, definition 5.3] is defined analogously to br except for

$$\begin{aligned}
(A \to B)_{\mathrm{brt}}(\underline{B}) &:\equiv \tilde{\forall}\underline{a}\,(A_{\mathrm{brt}}(\underline{a}) \to B_{\mathrm{brt}}(\underline{B}\underline{a})) \wedge (A \to B),\\
(\forall x\, A)_{\mathrm{brt}}(\underline{A}) &:\equiv \tilde{\forall} b\, \forall x \leq^{\mathrm{e}} b\, A_{\mathrm{brt}}(\underline{A}b) \wedge \forall x\, A.
\end{aligned}$$

4.4. Let us note that, contrarily to what is done for mrt, in brt we added "$\wedge\, \forall x\, A$" in the clause of $\forall$; this will be discussed later in chapter 13.

4.5 Remark.

1. The bounded modified realisability with q-truth brq has truth in the sense of: $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash A_{\mathrm{brq}}(\underline{a}) \to A$ for all disjunctive and (bounded and unbounded) existential formulas A of $\mathsf{HA}^{\omega}_{\mathrm{e}}$ [22, remark 5.2].

2. The bounded modified realisability with t-truth brt has truth in the sense of: $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash A_{\mathrm{brt}}(\underline{a}) \to A$ for all formulas A [22, remark 5.4].

The bounded modified realisability with t-truth brt is a $(*_1)$ strengthening of brq which $(*_2)$ has truth for all formulas. This can be given a rigorous meaning: $(*_3)$ $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash A_{\mathrm{brt}}(\underline{a}) \leftrightarrow A_{\mathrm{brq}}(\underline{a}) \wedge A$ for all formulas A of $\mathsf{HA}^{\omega}_{\mathrm{e}}$ [22, proposition 5.6]. From $(*_3)$ we get: $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash A_{\mathrm{brt}}(\underline{a}) \to A_{\mathrm{brq}}(\underline{a})$, that is $(*_1)$; $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash A_{\mathrm{brt}}(\underline{a}) \to A$, that is $(*_2)$.

4.6 Remark. The formulas $A_{\mathrm{br}}(\underline{a})$ are $\tilde{\exists}$-free.

4.7 Remark.

1. The bounded modified realisability br acts as the identity on $\tilde{\exists}$-free formulas of $\mathsf{HA}^\omega_\mathrm{e}$ in the sense of: $(A_{\tilde{\exists}\mathrm{f}})_{\mathrm{br}}() \equiv A_{\tilde{\exists}\mathrm{f}}$ for all $\tilde{\exists}$-free formulas $A_{\tilde{\exists}\mathrm{f}}$ of $\mathsf{HA}^\omega_\mathrm{e}$ [14, proposition 1].

2. The bounded modified realisability with q-truth brq acts as the identity on $\tilde{\exists}$-free formulas of $\mathsf{HA}^\omega_\mathrm{e}$ in the sense of: $\mathsf{HA}^\omega_\mathrm{e} \vdash (A_{\tilde{\exists}\mathrm{f}})_{\mathrm{brq}}() \leftrightarrow A_{\tilde{\exists}\mathrm{f}}$ for all $\tilde{\exists}$-free formulas $A_{\tilde{\exists}\mathrm{f}}$ of $\mathsf{HA}^\omega_\mathrm{e}$ [22, proof of theorem 5.5]. Analogously for brt.

4.3 Soundness

4.8 Lemma (monotonicity)**.** We have $\mathsf{HA}^\omega_\mathrm{e} \vdash \tilde{\forall}\underline{a}'\,\forall\underline{a} \leq^\mathrm{e} \underline{a}'\,(A_{\mathrm{br}}(\underline{a}) \to A_{\mathrm{br}}(\underline{a}'))$ [14, lemma 4]. Analogously for brq [22, proof of theorem 5.5] and brt.

4.9 Proof. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth, with q-truth and with t-truth, by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12. The proof is by induction on A. Let us only do the case of $\to$; the other cases are analogous. Let us take arbitrary monotone $\underline{B}'$ and arbitrary $\underline{B} \leq^\mathrm{e} \underline{B}'$. Using the induction hypothesis in the implication, we get

$$\begin{aligned} &(A \to B)_{\mathrm{br}}(\underline{B}) \equiv \\ \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \wedge A^\mathrm{q} &\to B_{\mathrm{br}}(\underline{B}\underline{a})) \wedge (A \to B)^\mathrm{t} \to \\ \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \wedge A^\mathrm{q} &\to B_{\mathrm{br}}(\underline{B}'\underline{a})) \wedge (A \to B)^\mathrm{t} \equiv \\ &(A \to B)_{\mathrm{br}}(\underline{B}'). \end{aligned}$$

4.10 Theorem (soundness)**.** Let A be a formula of $\mathsf{HA}^\omega_\mathrm{e}$ with $\mathrm{FV}(A) = \{\underline{\ell}\}$, and let Γ be a set of formulas of $\mathsf{HA}^\omega_\mathrm{e}$ of the form $\forall\underline{x}\,\exists\underline{y} \leq^\mathrm{e} \underline{s}\,\forall\underline{z}\,A_{\tilde{\exists}\mathrm{f}}$ where $\underline{s}$ are terms of $\mathsf{HA}^\omega_\mathrm{e}$.

1. If $\mathsf{HA}^\omega_\mathrm{e} + \mathsf{BAC} + \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} + \mathsf{MAJ} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}^\omega_\mathrm{e} + \Gamma \vdash \tilde{\forall}\underline{\ell}'\,\forall\underline{\ell} \leq^\mathrm{e} \underline{\ell}'\,A_{\mathrm{br}}(\underline{t}(\underline{\ell}'))$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(A)$ [14, theorem 4].

2. If $\mathsf{HA}^\omega_\mathrm{e} \pm \mathsf{BAC} \pm \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} \pm \mathsf{MAJ} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}^\omega_\mathrm{e} \pm \mathsf{BAC} \pm \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} \pm \mathsf{MAJ} + \Gamma \vdash \tilde{\forall}\underline{\ell}'\,\forall\underline{\ell} \leq^\mathrm{e} \underline{\ell}'\,A_{\mathrm{brq}}(\underline{t}(\underline{\ell}'))$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(A)$ [22, theorem 5.5].

3. If $\mathsf{HA}^\omega_\mathrm{e} \pm \mathsf{BAC} \pm \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} \pm \mathsf{MAJ} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}^\omega_\mathrm{e} \pm \mathsf{BAC} \pm \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} \pm \mathsf{MAJ} + \Gamma \vdash \tilde{\forall}\underline{\ell}'\,\forall\underline{\ell} \leq^\mathrm{e} \underline{\ell}'\,A_{\mathrm{brt}}(\underline{t}(\underline{\ell}'))$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(A)$ [22, theorem 5.7].

The terms constructed in the following proof for the three points above are the same.

4.11 Proof. Let us make some remarks.

1. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth, with q-truth and with t-truth, by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12.

2. We have

$$\begin{aligned}
&\mathsf{HA}^\omega_\mathrm{e} \vdash (\exists \underline{x}\, A)_\mathrm{br}(\underline{b}, \underline{a}) \leftrightarrow \exists \underline{x} \le^\mathrm{e} \underline{b}\, (A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q}),\\
&\mathsf{HA}^\omega_\mathrm{e} \vdash (\tilde{\exists} \underline{x}\, A)_\mathrm{br}(\underline{b}, \underline{a}) \leftrightarrow \tilde{\exists} \underline{x} \le^\mathrm{e} \underline{b}\, (A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q}),\\
&\mathsf{HA}^\omega_\mathrm{e} \vdash (\exists \underline{x} \le^\mathrm{e} \underline{t}\, A)_\mathrm{br}(\underline{a}) \leftrightarrow \exists \underline{x} \le^\mathrm{e} \underline{t}\, (A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q}),\\
&\mathsf{HA}^\omega_\mathrm{e} \vdash (\forall \underline{x}\, A)_\mathrm{br}(\underline{A}) \leftrightarrow \tilde{\forall} \underline{b}\, \forall \underline{x} \le^\mathrm{e} \underline{b}\, A_\mathrm{br}(\underline{A}\underline{b}) \wedge (\forall \underline{x}\, A)^\mathrm{t},
\end{aligned}$$

so below we replace the left sides of the equivalences by the right sides. When we do it, we use “$\equiv$” instead of $\equiv$.

Let us prove the theorem by induction on the derivation of A.

$\underline{A \vee A \to A}$ We have

$$\begin{gathered}
(A \vee A \to A)_\mathrm{br}(\underline{C}) \text{ “}\equiv\text{”}\\
\tilde{\forall} \underline{a}, \underline{b}\, \big(\big((A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q}) \vee (A_\mathrm{br}(\underline{b}) \wedge A^\mathrm{q})\big) \wedge (A \vee A)^\mathrm{q} \to A_\mathrm{br}(\underline{C}\underline{a}\underline{b})\big),\\
\underline{t_C} :\equiv \lambda \underline{a}, \underline{b}\, .\, \max(\underline{a}, \underline{b}).
\end{gathered}$$

Here we use monotonicity. Analogously for $A \to A \wedge A$, $A \wedge B \to A$, $A \to A \vee B$, $A \wedge B \to B \wedge A$, $A \vee B \to B \vee A$ and $\bot \to A$.

$\underline{A[t/x] \to \exists x\, A}$ We have

$$\begin{gathered}
(A[t/x] \to \exists x\, A)_\mathrm{br}(B, \underline{C}) \text{ “}\equiv\text{”}\\
\tilde{\forall} \underline{a}\, \big(A[t/x]_\mathrm{br}(\underline{a}) \wedge A[t/x]^\mathrm{q} \to \exists x \le^\mathrm{e} B\underline{a}\, (A_\mathrm{br}(\underline{C}\underline{a}) \wedge A^\mathrm{q})\big),\\
t_B(\underline{\ell}) :\equiv \lambda \underline{a}\, .\, t^\mathrm{m}(\underline{\ell}), \quad \underline{t_C} :\equiv \lambda \underline{a}\, .\, \underline{a}.
\end{gathered}$$

Let us see that the terms work, that is

$$\tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \le^\mathrm{e} \underline{\ell}'\, \tilde{\forall} \underline{a}\, \big(A[t/x]_\mathrm{br}(\underline{a}) \wedge A[t/x]^\mathrm{q} \to \exists x \le^\mathrm{e} t^\mathrm{m}(\underline{\ell}')\, (A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q})\big).$$

The premise, that is $A_\mathrm{br}(\underline{a})[t/x]$, implies the conclusion with $x = t(\underline{\ell})$ (which satisfies $x \le^\mathrm{e} t^\mathrm{m}(\underline{\ell}')$). Analogously for $\forall x\, A \to A[t/x]$.

$\underline{A \to B,\ B \to C \,/\, A \to C}$ We have

$$\begin{aligned}
(A \to B)_\mathrm{br}(\underline{B}) &\text{ “}\equiv\text{” } \tilde{\forall} \underline{a}\, (A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q} \to B_\mathrm{br}(\underline{B}\underline{a})), && (4.1)\\
(B \to C)_\mathrm{br}(\underline{C}) &\text{ “}\equiv\text{” } \tilde{\forall} \underline{b}\, (B_\mathrm{br}(\underline{b}) \wedge B^\mathrm{q} \to C_\mathrm{br}(\underline{C}\underline{b})), && (4.2)\\
(A \to C)_\mathrm{br}(\underline{C}) &\text{ “}\equiv\text{” } \tilde{\forall} \underline{a}\, (A_\mathrm{br}(\underline{a}) \wedge A^\mathrm{q} \to C_\mathrm{br}(\underline{C}\underline{a})),\\
&\underline{t_C} :\equiv \lambda \underline{a}\, .\, \underline{s_c}(\underline{r_B}\underline{a}).
\end{aligned}$$

If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved $A \to B$, so that the part A^q in (4.1) implies the part B^q in (4.2). Analogously for $A,\ A \to B \,/\, B$ and $A \to B \,/\, C \vee A \to C \vee B$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ We have

$$
\begin{gathered}
(A \wedge B \to C)_{\mathrm{br}}(\underline{C}) \equiv \\
\tilde{\forall} \underline{a}, \underline{b}\, (A_{\mathrm{br}}(\underline{a}) \wedge B_{\mathrm{br}}(\underline{b}) \wedge (A \wedge B)^{\mathrm{q}} \to C_{\mathrm{br}}(\underline{C}\underline{a}\underline{b})) \wedge (A \wedge B \to C)^{\mathrm{t}},
\end{gathered}
\tag{4.3}
$$

$$
\begin{gathered}
(A \to (B \to C))_{\mathrm{br}}(\underline{\mathrm{C}}) \equiv \tilde{\forall} \underline{a} \left(A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to \right. \\
\left. \tilde{\forall} \underline{b}\, (B_{\mathrm{br}}(\underline{B}) \wedge B^{\mathrm{q}} \to C_{\mathrm{br}}(\underline{\mathrm{C}}\underline{a}\underline{b})) \wedge (B \to C)^{\mathrm{t}}\right) \wedge (A \to (B \to C))^{\mathrm{t}},
\end{gathered}
\tag{4.4}
$$

$$
\underline{t}_{\underline{\mathrm{C}}} :\equiv \underline{s}_{\underline{C}}.
$$

If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{br}}(\underline{a}) \to A$, so that the parts $A_{\mathrm{br}}(\underline{a})$ in (4.4) and $(A \wedge B \to C)^{\mathrm{t}}$ in (4.3) together imply the part $(B \to C)^{\mathrm{t}}$ in (4.4). Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, A \to \forall x\, B}$ We have

$$
(A \to B)_{\mathrm{br}}(\underline{B}) \equiv \tilde{\forall} \underline{a}\, (A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to B_{\mathrm{br}}(\underline{B}\underline{a})) \wedge (A \to B)^{\mathrm{t}},
\tag{4.5}
$$

$$
\begin{gathered}
(A \to \forall x\, B)_{\mathrm{br}}(\underline{\mathrm{B}}) \equiv \\
\tilde{\forall} \underline{a}\, (A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to \tilde{\forall} c\, \forall x \leq^{\mathrm{e}} c\, B_{\mathrm{br}}(\underline{\mathrm{B}}\underline{a}c) \wedge (\forall x\, B)^{\mathrm{t}}) \wedge (A \to \forall x\, B)^{\mathrm{t}},
\end{gathered}
\tag{4.6}
$$

$$
\underline{t}_{\underline{\mathrm{B}}}(\underline{\ell}) :\equiv \lambda \underline{a}, c\, .\, \underline{s}_{\underline{B}}(\underline{\ell}, c)\underline{a}.
$$

Let us see that the terms work. By induction hypothesis we have (4.7) and we want to prove (4.8):

$$
\tilde{\forall} \underline{\ell}', c\, \forall \underline{\ell}, x \leq^{\mathrm{e}} \underline{\ell}', c\, \tilde{\forall} \underline{a} \left(A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to B_{\mathrm{br}}(\underline{s}_{\underline{B}}(\underline{\ell}', c)\underline{a})\right) \wedge (A \to B)^{\mathrm{t}},
\tag{4.7}
$$

$$
\begin{gathered}
\tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'\, \tilde{\forall} \underline{a} \left(A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to \tilde{\forall} c\, \forall x \leq^{\mathrm{e}} c\, B_{\mathrm{br}}(\underline{s}_{\underline{B}}(\underline{\ell}', c)\underline{a}) \wedge (\forall x\, B)^{\mathrm{t}}\right) \wedge \\
(A \to \forall x\, B)^{\mathrm{t}},
\end{gathered}
\tag{4.8}
$$

(actually, if $x \notin \mathrm{FV}(A \to B)$, then in (4.7) where is $\tilde{\forall} \underline{\ell}', c\, \forall \underline{\ell}, x \leq^{\mathrm{e}} \underline{\ell}', c$ should be $\tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'$). If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{br}}(\underline{a}) \to A$, so that the parts $(A \to B)^{\mathrm{t}}$ (that by induction hypothesis was proved, so we can upgrade it to $(A \to \forall x\, B)^{\mathrm{t}}$) in (4.7) and $A_{\mathrm{br}}(\underline{a})$ in (4.8) together imply the part $(\forall x\, B)^{\mathrm{t}}$ in (4.8). Analogously for $A \to B \,/\, \exists x\, A \to B$.

$\underline{\forall x \leq^{\mathrm{e}} t\, A \leftrightarrow \forall x\, (x \leq^{\mathrm{e}} t \to A)}$ To interpret $A \leftrightarrow B$ it suffices to interpret $A \to B$ and $B \to A$ separately.

$\underline{\to}$ We have

$$
\begin{gathered}
(\forall x \leq^{\mathrm{e}} t\, A \to \forall x\, (x \leq^{\mathrm{e}} t \to A))_{\mathrm{br}}(\underline{\mathrm{B}}) \text{ “}\equiv\text{”} \\
\tilde{\forall} \underline{a} \left(\forall x \leq^{\mathrm{e}} t\, A_{\mathrm{br}}(\underline{a}) \wedge (\forall x \leq^{\mathrm{e}} t\, A)^{\mathrm{q}} \to \right. \\
\tilde{\forall} c\, \forall x \leq^{\mathrm{e}} c \left((x \leq^{\mathrm{e}} t \wedge (x \leq^{\mathrm{e}} t)^{\mathrm{q}} \to A_{\mathrm{br}}(\underline{\mathrm{B}}\underline{a}c)) \wedge (x \leq^{\mathrm{e}} t \to A)^{\mathrm{t}}\right) \wedge \\
\left. (\forall x\, (x \leq^{\mathrm{e}} t \to A))^{\mathrm{t}}\right),
\end{gathered}
$$

$$
\underline{t}_{\underline{\mathrm{B}}} :\equiv \lambda \underline{a}, c\, .\, \underline{a}.
$$

If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{br}}(\underline{a}) \to A$, so that the part $\forall x \leq^{\mathrm{e}} t\, A_{\mathrm{br}}(\underline{a})$ in the premise implies the parts $(x \leq^{\mathrm{e}} t \to A)^{\mathrm{t}}$ and $(\forall x\, (x \leq^{\mathrm{e}} t \to A))^{\mathrm{t}}$ in the conclusion.

$\underline{\leftarrow}$ We have

$$
\begin{gathered}
(\forall x\, (x \leq^{\mathrm{e}} t \to A) \to \forall x \leq^{\mathrm{e}} t\, A)_{\mathrm{br}}(\underline{B}) \text{ “}\equiv\text{” } \\
\tilde{\forall}\underline{A}\, \big(\tilde{\forall} c\, \forall x \leq^{\mathrm{e}} c\, \big((x \leq^{\mathrm{e}} t \wedge (x \leq^{\mathrm{e}} t)^{\mathrm{q}} \to A_{\mathrm{br}}(\underline{A}c)) \wedge (x \leq^{\mathrm{e}} t \to A)^{\mathrm{t}}\big) \wedge \\
(\forall x\, (x \leq^{\mathrm{e}} t \to A))^{\mathrm{t}} \wedge (\forall x\, (x \leq^{\mathrm{e}} t \to A))^{\mathrm{q}} \to \forall x \leq^{\mathrm{e}} t\, A_{\mathrm{br}}(\underline{B}\underline{A})\big), \\
\underline{t}_{\underline{B}}(\underline{\ell}) :\equiv \lambda \underline{A}\,.\, \underline{A} t^{\mathrm{m}}(\underline{\ell}).
\end{gathered}
$$

To see that the terms work, in the premise we take $c = t^{\mathrm{m}}(\underline{\ell}')$ (which satisfies $c \leq^{\mathrm{e}} c$ if $\underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'$). Analogously for $\exists x \leq^{\mathrm{e}} t\, A \leftrightarrow \exists x\, (x \leq^{\mathrm{e}} t \wedge A)$.

Axioms of $=_0$, S, Π, Σ, and $\underline{\mathrm{R}}$ Their formulas are $\tilde{\exists}$-free, so they are equivalent to their own interpretation.

$A[0/x],\ A \to A[\mathrm{S}x/x]\,/\,A$ We can assume $x \in \mathrm{FV}(A)$, otherwise $A[0/x] \equiv A$ and so the terms working for $A[0/x]$ also work for A. We have

$$
\begin{gathered}
A[0/x]_{\mathrm{br}}(\underline{a}), \\
(A \to A[\mathrm{S}x/x])_{\mathrm{br}}(\underline{B}) \text{ “}\equiv\text{” } \tilde{\forall}\underline{a}\, (A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to A[\mathrm{S}x/x]_{\mathrm{br}}(\underline{B}\underline{a})), \\
A_{\mathrm{br}}(\underline{a}), \\
\underline{t}_{\underline{a}}(\underline{\ell}, x) :\equiv \underline{\mathrm{R}} x \underline{r}_{\underline{a}}(\underline{\ell})\, \lambda \underline{a}, x\,.\, \max(\underline{s}_{\underline{B}}(\underline{\ell}, x)\underline{a}, \underline{a}).
\end{gathered}
$$

By induction hypothesis we have (4.9) and (4.10), and we want to prove (4.11):

$$\tilde{\forall}\underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'\, A[0/x]_{\mathrm{br}}(\underline{r}_{\underline{a}}(\underline{\ell}')), \tag{4.9}$$

$$\tilde{\forall}\underline{\ell}', x'\, \forall \underline{\ell}, x \leq^{\mathrm{e}} \underline{\ell}', x'\, \tilde{\forall}\underline{a}\, \big(A_{\mathrm{br}}(\underline{a}) \wedge A^{\mathrm{q}} \to A[\mathrm{S}x/x]_{\mathrm{br}}(\underline{s}_{\underline{B}}(\underline{\ell}', x')\underline{a})\big), \tag{4.10}$$

$$\tilde{\forall}\underline{\ell}', x'\, \forall \underline{\ell}, x \leq^{\mathrm{e}} \underline{\ell}', x'\, A_{\mathrm{br}}(\underline{t}_{\underline{a}}(\underline{\ell}', x')). \tag{4.11}$$

First, let us prove that $\underline{t}_{\underline{a}}(\underline{\ell}, x)$ are monotone, that is $\forall \underline{\ell}', x'\, \forall \underline{\ell}, x \leq^{\mathrm{e}} \underline{\ell}', x'$ $(\underline{t}_{\underline{a}}(\underline{\ell}, x) \leq^{\mathrm{e}} \underline{t}_{\underline{a}}(\underline{\ell}', x'))$. We take arbitrary $\underline{\ell}'$ and $\underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'$ prove $B(x, x') :\equiv \underline{t}_{\underline{a}}(\underline{\ell}, x) \leq^{\mathrm{e}} \underline{t}_{\underline{a}}(\underline{\ell}', x')$ by double induction on x and x'.

$B(0, x')$ It is provable by induction on x'.

- $B(0, 0)$ It is equivalent to $\underline{r}_{\underline{a}}(\underline{\ell}) \leq^{\mathrm{e}} \underline{r}_{\underline{a}}(\underline{\ell}')$, which provable by the monotonicity of $\underline{r}_{\underline{a}}$.
- $B(0, x') \to B(0, \mathrm{S}x')$ It is equivalent to $\underline{r}_{\underline{a}}(\underline{\ell}) \leq^{\mathrm{e}} \underline{t}_{\underline{a}}(\underline{\ell}', x') \to \underline{r}_{\underline{a}}(\underline{\ell}) \leq^{\mathrm{e}} \max(\underline{s}_{\underline{B}}(\underline{\ell}', x')\underline{t}_{\underline{a}}(\underline{\ell}', x'), \underline{t}_{\underline{a}}(\underline{\ell}', x'))$, which is provable by the monotonicity of max.

$B(x, 0)$ It can only be $x =_0 0$, so we fall in the case $B(0, 0)$.

$B(x, x') \to B(\mathrm{S}x, \mathrm{S}x')$ This formula is equivalent to $\underline{t}_{\underline{a}}(\underline{\ell}, x) \leq^{\mathrm{e}} \underline{t}_{\underline{a}}(\underline{\ell}', x') \to \max(\underline{s}_{\underline{B}}(\underline{\ell}, x)\underline{t}_{\underline{a}}(\underline{\ell}, x), \underline{t}_{\underline{a}}(\underline{\ell}, x)) \leq^{\mathrm{e}} \max(\underline{s}_{\underline{B}}(\underline{\ell}', x')\underline{t}_{\underline{a}}(\underline{\ell}', x'), \underline{t}_{\underline{a}}(\underline{\ell}', x'))$, which is provable by the monotonicity of max.

Now, let us prove (4.11) by induction on x. We start by proving (4.11) with $x =_0 x'$. We take arbitrary monotone $\underline{\ell}'$ and $\underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'$, and prove $\forall x\, A_{\mathrm{br}}(\underline{t}_{\underline{a}}(\underline{\ell}', x))$ by induction on x.

Base case The formula $A_{\mathrm{br}}(\underline{t}_{\underline{a}}(\underline{\ell}', x))[0/x]$ is equivalent to $A[0/x]_{\mathrm{br}}(\underline{r}_{\underline{a}}(\underline{\ell}'))$, which is provable by (4.9).

Induction step By induction hypothesis we assume $A_{\mathrm{br}}(\underline{t}_{\underline{a}}(\underline{\ell}', x))$. Taking $x' = x$ and $\underline{a} = \underline{t}_{\underline{a}}(\underline{\ell}', x)$ (that satisfies $\underline{a} \leq^{\mathrm{e}} \underline{a}$) in formula (4.10) we get $A[\mathrm{S}x/x]_{\mathrm{br}}(\underline{s}_{\underline{B}}(\underline{\ell}', x)\underline{t}_{\underline{a}}(\underline{\ell}', x))$ (if $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved A, so as to have the part A^{q} in (4.10)). By monotonicity we get $A[\mathrm{S}x/x]_{\mathrm{br}}\big(\max(\underline{s}_{\underline{B}}(\underline{\ell}', x)\underline{t}_{\underline{a}}(\underline{\ell}', x), \underline{t}_{\underline{a}}(\underline{\ell}', x))\big)$, that is $A_{\mathrm{br}}(\underline{t}_{\underline{a}}(\underline{\ell}', x))[\mathrm{S}x/x]$.

From (4.11) with $x =_0 x'$ we get (4.11) with $x \leq^{\mathrm{e}} x'$ by the monotonicity of $A_{\mathrm{br}}(\underline{a})$ and $\underline{t}_{\underline{a}}(\underline{\ell}, x)$.

BAC We have

$$(\forall x\, \exists \underline{y}\, A)_{\mathrm{br}}(\underline{C}, \underline{A}) \equiv \tilde{\forall} d\, \forall x \leq^{\mathrm{e}} d\, \exists \underline{y} \leq^{\mathrm{e}} \underline{C}d\, (A_{\mathrm{br}}(\underline{A}d) \wedge A^{\mathrm{q}}) \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{t}},$$

$$(\tilde{\exists} \underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)_{\mathrm{br}}(\underline{f}, \underline{B}) \text{ “}\equiv\text{” } \tilde{\exists} \underline{v} \leq^{\mathrm{e}} \underline{f}$$
$$\Big(\tilde{\forall} e\, \forall u \leq^{\mathrm{e}} e\, \big((u \leq^{\mathrm{e}} u \wedge (u \leq^{\mathrm{e}} u)^{\mathrm{q}} \to \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, (A_{\mathrm{br}}(\underline{B}e) \wedge A^{\mathrm{q}})\big) \wedge$$
$$(u \leq^{\mathrm{e}} u \to \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{t}}\big) \wedge (\tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{t}} \wedge$$
$$(\underline{v} \leq^{\mathrm{e}} \underline{v} \wedge \tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{q}}\Big),$$

$$\mathsf{BAC}_{\mathrm{br}}(\underline{F}, \underline{\mathrm{B}}) \text{ “}\equiv\text{”}$$
$$\tilde{\forall} \underline{C}, \underline{A}\, \Big(\tilde{\forall} d\, \forall x \leq^{\mathrm{e}} d\, \exists \underline{y} \leq^{\mathrm{e}} \underline{C}d\, (A_{\mathrm{br}}(\underline{A}d) \wedge A^{\mathrm{q}}) \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{t}} \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{q}}$$
$$\downarrow$$
$$\tilde{\exists} \underline{v} \leq^{\mathrm{e}} \underline{F}\underline{C}\underline{A}$$
$$\Big(\tilde{\forall} e\, \forall u \leq^{\mathrm{e}} e\, \big((u \leq^{\mathrm{e}} u \wedge (u \leq^{\mathrm{e}} u)^{\mathrm{q}} \to \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, (A_{\mathrm{br}}(\underline{\mathrm{B}}\underline{C}\underline{A}e) \wedge A^{\mathrm{q}})\big) \wedge$$
$$(u \leq^{\mathrm{e}} u \to \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{t}}\big) \wedge (\tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{t}} \wedge$$
$$(\underline{v} \leq^{\mathrm{e}} \underline{v} \wedge \tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{q}}\Big)\Big),$$

$$\underline{t}_{\underline{F}} :\equiv \lambda \underline{C}, \underline{A}\, .\, \underline{C}, \quad \underline{t}_{\underline{\mathrm{B}}} :\equiv \lambda \underline{C}, \underline{A}, e\, .\, \underline{A}e.$$

To see that the terms work, we take $\underline{v} = \underline{C}$ (which satisfies $\underline{v} \leq^{\mathrm{e}} \underline{C}$ and $\underline{v} \leq^{\mathrm{e}} \underline{v}$) in the conclusion. If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{br}}(\underline{A}d) \to A$, so that the part $\tilde{\forall} d\, \forall x \leq^{\mathrm{e}} d\, \exists \underline{y} \leq^{\mathrm{e}} \underline{C}d\, A_{\mathrm{br}}(\underline{A}d)$ in the premise implies the parts $(u \leq^{\mathrm{e}} u \to \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{t}}$ and $(\tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A)^{\mathrm{t}}$ in the conclusion.

$\underline{\tilde{\exists}\mathsf{F\text{-}BIP}}$ We have

$$(A_{\tilde{\exists}\mathrm{f}} \to \exists x\, B)_{\mathrm{br}}(c, \underline{a}) \text{ “}{\equiv}\text{” }$$
$$\big(A_{\tilde{\exists}\mathrm{f}} \wedge A^{\mathrm{q}}_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} c\, (B_{\mathrm{br}}(\underline{a}) \wedge B^{\mathrm{q}})\big) \wedge (A_{\tilde{\exists}\mathrm{f}} \to \exists x\, B)^{\mathrm{t}},$$

$$(\tilde{\exists} y\, (A_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B))_{\mathrm{br}}(d, \underline{b}) \text{ “}{\equiv}\text{” }$$
$$\tilde{\exists} y \leq^{\mathrm{e}} d\, \big(\big(A_{\tilde{\exists}\mathrm{f}} \wedge A^{\mathrm{q}}_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, (B_{\mathrm{br}}(\underline{b}) \wedge B^{\mathrm{q}})\big) \wedge$$
$$(A_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B)^{\mathrm{t}} \wedge (y \leq^{\mathrm{e}} y \wedge (A_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B))^{\mathrm{q}}\big),$$

$$(\tilde{\exists}\mathsf{F\text{-}BIP})_{\mathrm{br}}(D, \underline{B}) \text{ “}{\equiv}\text{” }$$
$$\tilde{\forall} c, \underline{a}\, \Big(\big(A_{\tilde{\exists}\mathrm{f}} \wedge A^{\mathrm{q}}_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} c\, (B_{\mathrm{br}}(\underline{a}) \wedge B^{\mathrm{q}})\big) \wedge (A_{\tilde{\exists}\mathrm{f}} \to \exists x\, B)^{\mathrm{t}} \wedge (A_{\tilde{\exists}\mathrm{f}} \to \exists x\, B)^{\mathrm{q}}$$
$$\downarrow$$
$$\tilde{\exists} y \leq^{\mathrm{e}} Dc\underline{a}\, \big(\big(A_{\tilde{\exists}\mathrm{f}} \wedge A^{\mathrm{q}}_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, (B_{\mathrm{br}}(\underline{B}c\underline{a}) \wedge B^{\mathrm{q}})\big) \wedge$$
$$(A_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B)^{\mathrm{t}} \wedge (y \leq^{\mathrm{e}} y \wedge (A_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B))^{\mathrm{q}}\big)\Big),$$

$$t_D :\equiv \lambda c, \underline{a} \,.\, c, \quad \underline{t_B} :\equiv \lambda c, \underline{a} \,.\, \underline{a}.$$

If $\mathrm{t} = \mathrm{id}$, then we use $B_{\mathrm{br}}(\underline{a}) \to B$, so that the part $A_{\tilde{\exists}\mathrm{f}} \wedge A^{\mathrm{q}}_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} c\, B_{\mathrm{br}}(\underline{a})$ in the premise implies the part $(A_{\tilde{\exists}\mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B)^{\mathrm{t}}$ in the conclusion.

$\underline{\mathsf{MAJ}}$ We have

$$\mathsf{MAJ}_{\mathrm{br}}(A) \text{ “}{\equiv}\text{” } \tilde{\forall} b\, \forall x \leq^{\mathrm{e}} b\, \exists y \leq^{\mathrm{e}} Ab\, (x \leq^{\mathrm{e}} y \wedge (x \leq^{\mathrm{e}} y)^{\mathrm{q}}),$$
$$t_A :\equiv \lambda b \,.\, b.$$

$\underline{\Gamma}$ We have

$$(\forall \underline{x}\, \exists \underline{y} \leq^{\mathrm{e}} \underline{s}\, \forall \underline{z}\, A_{\tilde{\exists}\mathrm{f}})_{\mathrm{br}}() \text{ “}{\equiv}\text{” }$$
$$\tilde{\forall} \underline{b}\, \forall \underline{x} \leq^{\mathrm{e}} \underline{b}\, \exists \underline{y} \leq^{\mathrm{e}} \underline{s}\, (\tilde{\forall} \underline{a}\, \forall \underline{z} \leq^{\mathrm{e}} \underline{a}\, A_{\tilde{\exists}\mathrm{f}} \wedge (\forall \underline{z}\, A_{\tilde{\exists}\mathrm{f}})^{\mathrm{t}} \wedge (\forall \underline{z}\, A_{\tilde{\exists}\mathrm{f}})^{\mathrm{q}}).$$

4.4 Characterisation

4.12 Theorem (characterisation)**.** Let us consider the theory $\mathsf{HA}^{\omega}_{\mathrm{e}} + \mathsf{BAC} + \tilde{\exists}\mathsf{F\text{-}BIP} + \mathsf{MAJ}$.

1. This theory proves $A \leftrightarrow A^{\mathrm{br}}$ for all formulas A of $\mathsf{HA}^{\omega}_{\mathrm{e}}$ [14, theorem 2].

2. This theory is the least theory, containing $\mathsf{HA}^{\omega}_{\mathrm{e}}$, satisfying the previous point.

Analogously for brq and brt [22, theorem 5.8].

4.13 Proof. Let us prove the claim of the theorem for br.

1. The proof is by induction on the structure of A.

$\underline{\to}$ Using induction hypothesis in the first equivalence, $\tilde{\exists}$F-BIP in the third equivalence, monotonicity in the fourth equivalence, and MAC (see point 2 of proposition 1.66) in the last equivalence, we get

$$\begin{aligned}
&(A \to B) \leftrightarrow \\
&A^{\mathrm{br}} \to B^{\mathrm{br}} \equiv \\
&(\tilde{\exists}\underline{a}\, A_{\mathrm{br}}(\underline{a}) \to \tilde{\exists}\underline{b}\, B_{\mathrm{br}}(\underline{b})) \leftrightarrow \\
&\tilde{\forall}\underline{a}\, (A_{\mathrm{br}}(\underline{a}) \to \tilde{\exists}\underline{b}\, B_{\mathrm{br}}(\underline{b})) \leftrightarrow \\
&\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{b}\, (A_{\mathrm{br}}(\underline{a}) \to \tilde{\exists}\underline{b}' \leq^{\mathrm{e}} \underline{b}\, B_{\mathrm{br}}(\underline{b}')) \leftrightarrow \\
&\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{b}\, (A_{\mathrm{br}}(\underline{a}) \to B_{\mathrm{br}}(\underline{b})) \leftrightarrow \\
&\tilde{\exists}\underline{B}\, \tilde{\forall}\underline{a}\, (A_{\mathrm{br}}(\underline{a}) \to B_{\mathrm{br}}(\underline{B}\underline{a})) \equiv \\
&(A \to B)^{\mathrm{br}}.
\end{aligned}$$

Analogously for A_{at}, $\wedge$, $\vee$ and $\exists \leq^{\mathrm{e}}$.

$\underline{\forall \leq^{\mathrm{e}}}$ Using induction hypothesis in the first equivalence, BC (see point 1 of proposition 1.66) in the second equivalence, and monotonicity in the last equivalence, we get

$$\begin{aligned}
&\forall x \leq^{\mathrm{e}} t\, A \leftrightarrow \\
&\forall x \leq^{\mathrm{e}} t\, A^{\mathrm{br}} \equiv \\
&\forall x \leq^{\mathrm{e}} t\, \tilde{\exists}\underline{a}\, A_{\mathrm{br}}(\underline{a}) \leftrightarrow \\
&\tilde{\exists}\underline{a}\, \forall x \leq^{\mathrm{e}} t\, \tilde{\exists}\underline{a}' \leq^{\mathrm{e}} \underline{a}\, A_{\mathrm{br}}(\underline{a}') \leftrightarrow \\
&\tilde{\exists}\underline{a}\, \forall x \leq^{\mathrm{e}} t\, A_{\mathrm{br}}(\underline{a}) \equiv \\
&(\forall x \leq^{\mathrm{e}} t\, A)^{\mathrm{br}}.
\end{aligned}$$

$\underline{\forall}$ Using induction hypothesis in the first equivalence, MAJ in the second equivalence, BC in the third equivalence, monotonicity in the fourth equivalence, and MAC in the last equivalence, we get

$$\begin{aligned}
&\forall x\, A \leftrightarrow \\
&\forall x\, A^{\mathrm{br}} \equiv \\
&\forall x\, \tilde{\exists}\underline{a}\, A_{\mathrm{br}}(\underline{a}) \leftrightarrow \\
&\tilde{\forall} b\, \forall x \leq^{\mathrm{e}} b\, \tilde{\exists}\underline{a}\, A_{\mathrm{br}}(\underline{a}) \leftrightarrow \\
&\tilde{\forall} b\, \tilde{\exists}\underline{a}\, \forall x \leq^{\mathrm{e}} b\, \tilde{\exists}\underline{a}' \leq^{\mathrm{e}} \underline{a}\, A_{\mathrm{br}}(\underline{a}') \leftrightarrow \\
&\tilde{\forall} b\, \tilde{\exists}\underline{a}\, \forall x \leq^{\mathrm{e}} b\, A_{\mathrm{br}}(\underline{a}) \leftrightarrow \\
&\tilde{\exists}\underline{A}\, \tilde{\forall} b\, \forall x \leq^{\mathrm{e}} b\, A_{\mathrm{br}}(\underline{A} b) \equiv \\
&(\forall x\, A)^{\mathrm{br}}.
\end{aligned}$$

Analogously for $\exists$.

2. Analogous to point 2 of proof 3.15.

Now let us prove the claim of the theorem for brq and brt.

1. The point 1 of the theorem for brq and brt follows from the point 1 for br by proving by induction on the structure of A that $\mathsf{HA}^\omega_{\mathrm{e}} + \mathsf{BAC} + \tilde{\exists}\mathsf{F\text{-}BIP} + \mathsf{MAJ}$ proves $(*_1)\ \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \leftrightarrow A_{\mathrm{brq}}(\underline{a}))$ and $(*_2)\ \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \leftrightarrow A_{\mathrm{brt}}(\underline{a}))$.

   Proof of $(*_1)$ Let us only see the case of $\vee$; the cases of A_{at}, $\wedge$, $\to$, $\forall \leq^{\mathrm{e}}$, $\exists \leq^{\mathrm{e}}$, $\forall$ and $\exists$ are analogous. Let us assume $\underline{a}, \underline{b} \leq^{\mathrm{e}} \underline{a}, \underline{b}$. Using $A_{\mathrm{br}}(\underline{a}) \to A^{\mathrm{br}} \leftrightarrow A$ and $B_{\mathrm{br}}(\underline{b}) \to B^{\mathrm{br}} \leftrightarrow B$ in the first equivalence, and induction hypothesis in the second equivalence, we get

$$\begin{aligned}(A \vee B)_{\mathrm{br}}(\underline{a}, \underline{b}) &\equiv \\ A_{\mathrm{br}}(\underline{a}) \vee A_{\mathrm{br}}(\underline{b}) &\leftrightarrow \\ (A_{\mathrm{br}}(\underline{a}) \wedge A) \vee (A_{\mathrm{br}}(\underline{b}) \wedge B) &\leftrightarrow \\ (A_{\mathrm{brq}}(\underline{a}) \wedge A) \vee (A_{\mathrm{brq}}(\underline{b}) \wedge B) &\equiv \\ (A \vee B)_{\mathrm{brq}}(\underline{a}, \underline{b}).&\end{aligned}$$

   Proof of $(*_2)$ Let us only see the case of $\to$; the cases of A_{at}, $\wedge$, $\vee$, $\forall \leq^{\mathrm{e}}$, $\exists \leq^{\mathrm{e}}$, $\forall$ and $\exists$ are analogous. Let us assume $\underline{B} \leq^{\mathrm{e}} \underline{B}$. Using $(A \to B)_{\mathrm{br}}(\underline{B}) \to (A \to B)^{\mathrm{br}} \leftrightarrow (A \to B)$ in the first equivalence, and induction hypothesis in the second equivalence, we get

$$\begin{aligned}(A \to B)_{\mathrm{br}}(\underline{B}) &\equiv \\ \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \to A_{\mathrm{br}}(\underline{B}\underline{a})) &\leftrightarrow \\ \tilde{\forall}\underline{a}\,(A_{\mathrm{br}}(\underline{a}) \to A_{\mathrm{br}}(\underline{B}\underline{a})) \wedge (A \to B) &\leftrightarrow \\ \tilde{\forall}\underline{a}\,(A_{\mathrm{brt}}(\underline{a}) \to A_{\mathrm{brt}}(\underline{B}\underline{a})) \wedge (A \to B) &\equiv \\ (A \to B)_{\mathrm{brt}}(\underline{B}).&\end{aligned}$$

2. We adopt here (with the proper adaptations, including an analogous unified treatment of variants with q-truth and with t-truth by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proofs 3.12 and 4.11. Let $\mathrm{q} = \mathrm{id}$ or $\mathrm{t} = \mathrm{id}$. Let T be a theory, containing $\mathsf{HA}^\omega_{\mathrm{e}}$, that proves the equivalences $(*)\ A \leftrightarrow A^{\mathrm{br}}$ for all formulas A. Let us show $\mathsf{T} \vdash \mathsf{BAC}$, $\mathsf{T} \vdash \tilde{\exists}\mathsf{F\text{-}BIP}$ and $\mathsf{T} \vdash \mathsf{MAJ}$.

   $\mathsf{T} \vdash \mathsf{BAC}$ Using $(*)$ in the first implication, and $A_{\mathrm{br}}(\underline{A}u) \to A$ in the second implication if $\mathrm{t} = \mathrm{id}$, we get

$$\begin{aligned}\forall x\, \exists \underline{y}\, A &\to \\ (\forall x\, \exists \underline{y}\, A)^{\mathrm{br}} &\equiv \\ \tilde{\exists}\underline{v}, \underline{A}\left(\tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, (A_{\mathrm{br}}(\underline{A}u) \wedge A^{\mathrm{q}}) \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{t}}\right) &\to \\ \tilde{\exists}\underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{e}} u\, \exists \underline{y} \leq^{\mathrm{e}} \underline{v}u\, A.&\end{aligned}$$

   $\mathsf{T} \vdash \tilde{\exists}\mathsf{F\text{-}BIP}$ Using $(*)$ in the first implication, and $B_{\mathrm{br}}(\underline{a}) \to B$ in the second implication if t = id, we get

$$
\begin{aligned}
&(A_{\tilde\exists \mathrm{f}} \to \exists x\, B) \to \\
&(A_{\tilde\exists \mathrm{f}} \to \exists x\, B)^{\mathrm{br}} \text{ “}\equiv\text{”} \\
\tilde\exists y, \underline{a}\, \big(\big(A_{\tilde\exists \mathrm{f}} \wedge A^{\mathrm{q}}_{\tilde\exists \mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, (B_{\mathrm{br}}(\underline{a}) \wedge B^{\mathrm{q}})\big) \wedge (A_{\tilde\exists \mathrm{f}} \to \exists x\, B)^{\mathrm{t}}\big) &\to \\
&\tilde\exists y\, (A_{\tilde\exists \mathrm{f}} \to \exists x \leq^{\mathrm{e}} y\, B).
\end{aligned}
$$

$\mathsf{T} \vdash \mathsf{MAJ}$ Using $(*)$ in the first implication, and $x' \leq^{\mathrm{e}}_{\rho} y \wedge x =_{\rho} x' \to x \leq^{\mathrm{e}}_{\rho} y$ (which is provable by induction on the structure of ρ) [50, lemma 3.49(i)] in the second implication, we get

$$
\begin{aligned}
&\exists x'\, (x = x') \equiv \\
&\exists x'\, \forall \underline{z}\, (x\underline{z} =_0 x'\underline{z}) \to \\
&(\exists x'\, \forall \underline{z}\, (x\underline{z} =_0 x'\underline{z}))^{\mathrm{br}} \text{ “}\equiv\text{”} \\
\tilde\exists y\, \exists x' \leq^{\mathrm{e}} y\, (\tilde\forall \underline{a}\, \forall \underline{z} \leq^{\mathrm{e}} \underline{a}\, (x\underline{z} =_0 x'\underline{z}) \wedge (x = x')^{\mathrm{t}} \wedge (x = x')^{\mathrm{q}}) &\to \\
&\exists y\, (x \leq^{\mathrm{e}} y).
\end{aligned}
$$

We have $\mathsf{T} \vdash \exists x'\, (x = x')$, so $\mathsf{T} \vdash \exists y\, (x \leq^{\mathrm{e}} y)$, therefore $\mathsf{T} \vdash \mathsf{MAJ}$.

4.14 Remark. The characterisation theorem of br ensures that the soundness theorem of br is optimal, in the sense that the theory $\mathsf{HA}^{\omega}_{\mathrm{e}} + \mathsf{BAC} + \tilde\exists\mathsf{F}\text{-}\mathsf{BIP} + \mathsf{MAJ} + \Gamma$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^{\omega}_{\mathrm{e}} + \Gamma \vdash A^{\mathrm{br}}$ (analogously to remark 3.16).

4.5 Applications

4.15 Theorem (bounded existence property and bounded program extraction)**.** Let $\mathsf{T} := \mathsf{HA}^{\omega}_{\mathrm{e}} \pm \mathsf{BAC} \pm \tilde\exists\mathsf{F}\text{-}\mathsf{BIP} \pm \mathsf{MAJ}$.

1. Let $\mathrm{FV}(\exists \underline{x}\, A) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \exists \underline{x}\, A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ of T such that $\mathsf{T} \vdash \tilde\forall \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'\, \exists \underline{x} \leq^{\mathrm{e}} \underline{t}(\underline{\ell}')\, A$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(\exists \underline{x}\, A)$.

2. Let $\mathrm{FV}(\forall \underline{x}\, \exists \underline{y}\, A(\underline{x}, \underline{y})) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell}, \underline{x})$ of T such that $\mathsf{T} \vdash \tilde\forall \underline{\ell}, \underline{x}'\, \forall \underline{\ell}, \underline{x} \leq^{\mathrm{e}} \underline{x}', \underline{\ell}'\, \exists \underline{y} \leq^{\mathrm{e}} \underline{t}(\underline{\ell}', \underline{x}')\, A$ and $\mathrm{FV}(\underline{t}(\underline{\ell}, \underline{x})) = \mathrm{FV}(\exists \underline{y}\, A(\underline{x}, \underline{y}))$.

4.16 Proof.

1. We have $(\exists \underline{x}\, A)_{\mathrm{brt}}(\underline{b}, \underline{a}) \equiv \exists \underline{x} \leq^{\mathrm{e}} \underline{b}\, A_{\mathrm{brt}}(\underline{a})$. Assuming the premise of the theorem, by the soundness theorem of brt we can extract monotone terms $\underline{s}(\underline{\ell}), \underline{t}(\underline{\ell})$ of T such that $\mathsf{T} \vdash \forall \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'\, \exists \underline{x} \leq^{\mathrm{e}} \underline{t}(\underline{\ell}')\, A_{\mathrm{brt}}(\underline{s}(\underline{\ell}'))$ and $\mathrm{FV}(\underline{s}(\underline{\ell}), \underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(\exists \underline{x}\, A)$. By truth we get $\mathsf{T} \vdash \forall \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{e}} \underline{\ell}'\, \exists \underline{x} \leq^{\mathrm{e}} \underline{t}(\underline{\ell}')\, A$.

2. Follows from the previous point.

4.17 Theorem (conservation and relative consistency).

1. Let $\tilde{\forall}\underline{x}'\,\forall\underline{x} \leq^{\mathrm{e}} \underline{x}'\,\exists\underline{y}\,A_{\tilde{\exists}\mathrm{f}}$ be a sentence of $\mathsf{HA}^{\omega}_{\mathrm{e}}$. If $\mathsf{HA}^{\omega}_{\mathrm{e}} + \mathsf{BAC} + \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} + \mathsf{MAJ} \vdash \tilde{\forall}\underline{x}'\,\forall\underline{x} \leq^{\mathrm{e}} \underline{x}'\,\exists\underline{y}\,A_{\tilde{\exists}\mathrm{f}}$, then $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash \tilde{\forall}\underline{x}'\,\forall\underline{x} \leq^{\mathrm{e}} \underline{x}'\,\exists\underline{y}\,A_{\tilde{\exists}\mathrm{f}}$ [14, corollary 2].

2. If $\mathsf{HA}^{\omega}_{\mathrm{e}} + \mathsf{BAC} + \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} + \mathsf{MAJ} \vdash \bot$, then $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash \bot$ [14, corollary 4].

4.18 Proof.

1. We have $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash (\tilde{\forall}\underline{x}'\,\forall\underline{x} \leq^{\mathrm{e}} \underline{x}'\,\exists\underline{y}\,A_{\tilde{\exists}\mathrm{f}})_{\mathrm{br}}(\underline{A}) \leftrightarrow \tilde{\forall}\underline{b}\,\tilde{\forall}\underline{x}' \leq^{\mathrm{e}} \underline{b}\,\forall\underline{x} \leq^{\mathrm{e}} \underline{x}'\,\exists\underline{y} \leq^{\mathrm{e}} \underline{A}\underline{b}\,A_{\tilde{\exists}\mathrm{f}} \leftrightarrow \tilde{\forall}\underline{b}\,\tilde{\forall}\underline{x} \leq^{\mathrm{e}} \underline{b}\,\exists\underline{y} \leq^{\mathrm{e}} \underline{A}\underline{b}\,A_{\tilde{\exists}\mathrm{f}}$. Assuming the premise of the theorem, by the soundness theorem of br we can extract closed monotone terms $\underline{t}$ of $\mathsf{HA}^{\omega}_{\mathrm{e}}$ such that $\mathsf{HA}^{\omega}_{\mathrm{e}} \vdash \tilde{\forall}\underline{b}\,\tilde{\forall}\underline{x} \leq^{\mathrm{e}} \underline{b}\,\exists\underline{y} \leq^{\mathrm{e}} \underline{t}\underline{b}\,A_{\exists\mathrm{f}}$. So we get the conclusion of the theorem.

2. Follows from the previous point.

4.6 Conclusion

4.19. We introduced the bounded modified realisability as a variant of modified realisability that aims at bounds instead of exact witnesses. The main results about the bounded modified realisability are the following.

Soundness theorem This theorem says that we can use the bounded modified realisability to extract computational content from proofs in $\mathsf{HA}^{\omega}_{\mathrm{e}} + \mathsf{BAC} + \tilde{\exists}\mathsf{F}\text{-}\mathsf{BIP} + \mathsf{MAJ}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used the bounded modified realisability to do applications on:

1. bounded existence property;
2. bounded program extraction;
3. conservation;
4. relative consistency.

Chapter 5

Gödel's functional interpretation

5.1 Introduction

5.1. We saw that mr can be used to interpret HA^ω, but what about PA^ω? We can think about composing mr with a negative translation N to get an interpretation of PA^ω, as pictured in figure 5.1. But, for the negative translations GG, Ko, Kr and

$$\begin{tikzcd} \mathsf{PA}^\omega \arrow[r, "\mathrm{N}"] \arrow[rr, bend right, "\mathrm{mr} \circ \mathrm{N}"'] & \mathsf{HA}^\omega \arrow[r, "\mathrm{mr}"] & \mathsf{HA}^\omega \end{tikzcd}$$

Figure 5.1: the composition mr ∘ N.

Ku, the composition gives a trivial interpretation, that is the tuple $\underline{a}$ in $(A^{\mathrm{N}})_{\mathrm{mr}}(\underline{a})$ is empty (so the soundness theorem of mr gives an empty tuple of terms, which is of no interest). Indeed:

GG the formulas A^{GG} are $\exists$-free, so in $(A^{\mathrm{GG}})_{\mathrm{mr}}(\underline{a})$ the tuple $\underline{a}$ is empty;

Ko, Kr and Ku the formulas A^{Ko} are negated, so in $(A^{\mathrm{Ko}})_{\mathrm{mr}}(\underline{a})$ the tuple $\underline{a}$ is empty, and analogously for Kr and Ku.

To better see the problem, let us, for example, compute the interpretation of $\exists x\, A_{\mathrm{at}}$ by the composition mr ∘ Ku:

$$(\exists x\, A_{\mathrm{at}})^{\mathrm{Ku}} \equiv \neg\neg\exists x\, A_{\mathrm{at}},$$

$$(\exists x\, A_{\mathrm{at}})_{\mathrm{mr}}(\underbrace{x}_{(*_1)}) \equiv A_{\mathrm{at}}, \tag{5.1}$$

$$(\neg\exists x\, A_{\mathrm{at}})_{\mathrm{mr}}(\underbrace{\ \ }_{(*_2)}) \equiv \forall x\, \neg A_{\mathrm{at}}, \tag{5.2}$$

$$(\neg\neg\exists x\, A_{\mathrm{at}})_{\mathrm{mr}}(\underbrace{\ \ }_{(*_3)}) \equiv \neg\forall x\, \neg A_{\mathrm{at}}.$$

We see that in (5.1) modified realisability seems to be on the right track by "capturing" the variable x, that is x appears in $(*_1)$. But then in (5.2) modified realisability loses x forever, that is x is absent in $(*_2)$ and $(*_3)$. The problem is that once x becomes universally quantified, mr no longer has a hold on x.

So to get a non-trivial interpretation of PA^ω, we need to replace mr by some other proof interpretation that "captures" universally quantified variables. That will be Gödel's functional interpretation D that captures universally quantified variables and collects them in $\underline{b}$ in $A^{\mathrm{D}} \equiv \exists \underline{a}\, \forall \underline{b}\, A_{\mathrm{D}}(\underline{a}; \underline{b})$. In this chapter we only present D; the composition of D with a negative translation is presented in chapter 7.

5.2. There are no main contributions of our own to this topic. Almost all of the material here is known.

5.2 Definition

5.3 Definition. *Gödel's functional interpretation* D [28] [30, page 248] [50, definition 8.1] assigns to each formula A of HA^ω the formula $A^{\mathrm{D}} :\equiv \exists \underline{a}\, \forall \underline{b}\, A_{\mathrm{D}}(\underline{a}; \underline{b})$, where $A_{\mathrm{D}}(\underline{a}; \underline{b})$ is defined by recursion on the structure of A by

$$
\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{D}}(;) &:\equiv A_{\mathrm{at}}, \\
(A \wedge B)_{\mathrm{D}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{D}}(\underline{a}; \underline{b}) \wedge B_{\mathrm{D}}(\underline{c}; \underline{d}), \\
(A \vee B)_{\mathrm{D}}(e^0, \underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{D}}(\underline{a}; \underline{b}) \vee_e B_{\mathrm{D}}(\underline{c}; \underline{d}), \\
(A \to B)_{\mathrm{D}}(\underline{C}, \underline{B}; \underline{a}, \underline{d}) &:\equiv A_{\mathrm{D}}(\underline{a}; \underline{B\underline{a}\underline{d}}) \to B_{\mathrm{D}}(\underline{C\underline{a}}; \underline{d}), \\
(\forall x\, A)_{\mathrm{D}}(\underline{A}; x, \underline{b}) &:\equiv A_{\mathrm{D}}(\underline{Ax}; \underline{b}), \\
(\exists x\, A)_{\mathrm{D}}(x, \underline{a}; \underline{b}) &:\equiv A_{\mathrm{D}}(\underline{a}; \underline{b}).
\end{aligned}
$$

By $(A_{\mathrm{at}})_{\mathrm{D}}(;)$ we mean $(A_{\mathrm{at}})_{\mathrm{D}}(\underline{a}; \underline{b})$ with the tuples $\underline{a}$ and $\underline{b}$ empty. Analogously for $\mathsf{WE\text{-}HA}^\omega$.

5.4. The letter D in the symbol for Gödel's functional interpretation D likely comes from this interpretation also be called Dialectica interpretation since it was introduced in a paper [28] in a journal called Dialectica.

5.5 Remark. The formulas $A_{\mathrm{D}}(\underline{a}; \underline{b})$ are quantifier-free.

5.6 Remark. Gödel's functional interpretation D acts as the identity on quantifier-free formulas of HA^ω without disjunctions in the sense of: $(A_{\mathrm{qf}})_{\mathrm{D}}(;) \equiv A_{\mathrm{qf}}$ for all quantifier-free formulas A_{qf} of HA^ω without disjunctions.

5.3 Soundness

5.7 Theorem (soundness)**.** Let A be a formula of HA^ω and let Γ be a set of formulas of HA^ω of the form $\forall \underline{x}\, A_{\mathrm{qf}}$. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega + \Gamma \vdash \forall \underline{b}\, A_{\mathrm{D}}(\underline{t}; \underline{b})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$. Analogously for $\mathsf{WE\text{-}HA}^\omega$ [28] [75, theorem 3.5.10(ii)] [50, theorem 8.6].

5.8 Proof. This proof is analogous and sometimes simpler than the proof 6.9, so we prefer to do the more complicated proof 6.9 later on. There is one exception:

contrarily to proof 6.9, here the axiom $A \to A \vee A$ requires defining terms by quantifier-free cases. Let see this. We have

$$(A \to A \wedge A)_{\mathrm{D}}(\underline{C}, \underline{E}, \underline{B}; \underline{a}, \underline{d}, \underline{f}) \equiv A_{\mathrm{D}}(\underline{a}; \underline{B}\underline{a}\underline{d}\underline{f}) \to A_{\mathrm{D}}(\underline{C}\underline{a}; \underline{d}) \wedge A_{\mathrm{D}}(\underline{E}\underline{a}; \underline{f}),$$
$$\underline{t_C} :\equiv \lambda \underline{a} \,.\, \underline{a}, \quad \underline{t_E} :\equiv \lambda \underline{a} \,.\, \underline{a}, \quad \underline{t_B} :\equiv \lambda \underline{a}, \underline{d}, \underline{f} \,.\, \underline{f} \vee_{\chi_{A_{\mathrm{D}}(\underline{a};\underline{d})}} \underline{d}.$$

Informally,

$$\underline{t_B}\underline{a}\underline{d}\underline{f} = \begin{cases} \underline{f} & \text{if } \ A_{\mathrm{D}}(\underline{a}; \underline{d}) \\ \underline{d} & \text{if } \neg A_{\mathrm{D}}(\underline{a}; \underline{d}) \end{cases}.$$

To see that the terms work, that is

$$\forall \underline{a}, \underline{d}, \underline{f} \, (\underbrace{A_{\mathrm{D}}(\underline{a}; \underline{f} \vee_{\chi_{A_{\mathrm{D}}(\underline{a};\underline{d})}} \underline{d})}_{(*_1)} \to \underbrace{A_{\mathrm{D}}(\underline{a}; \underline{d})}_{(*_2)} \wedge \underbrace{A_{\mathrm{D}}(\underline{a}; \underline{f})}_{(*_3)}),$$

we argue by cases:

$\underline{\chi_{A_{\mathrm{D}}(\underline{a};\underline{d})} =_0 0}$ we have $(*_2)$ and $(*_1) \leftrightarrow (*_3)$, so we have $(*_1) \to (*_2) \wedge (*_3)$;

$\underline{\chi_{A_{\mathrm{D}}(\underline{a};\underline{d})} \neq_0 0}$ we have $\neg(*_2)$ and $(*_1) \leftrightarrow (*_2)$, so we have $(*_1) \to (*_2) \wedge (*_3)$.

5.9. There seems to be no sound Gödel's functional interpretation with truth. This is because, for example, if we add a copy $\forall x\, A$ to clause of $\forall$ of D, getting

$$(\forall x\, A)_{\mathrm{Dt}}(\underline{A}; x, \underline{b}) :\equiv A_{\mathrm{Dt}}(\underline{A}x; \underline{b}) \wedge \forall x\, A,$$

then the formulas $A_{\mathrm{Dt}}(\underline{a}; \underline{b})$ are no longer quantifier-free formulas, so they do not have the characteristic terms $\chi_{A_{\mathrm{Dt}}(\underline{a};\underline{b})}$ necessary in the proof of the soundness theorem [41, section 6.1]. In contrast, in chapter 6 we have a sound Diller-Nahm functional interpretation (a variant of D) with truth because its soundness theorem does not require characteristic terms.

5.4 Characterisation

5.10 Theorem (characterisation)**.** Let us consider the theory $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

1. This theory proves $A \leftrightarrow A^{\mathrm{D}}$ for all formulas A of HA^ω [75, theorem 3.5.10(i)] [50, proposition 8.12].

2. This theory is the least theory, containing HA^ω, satisfying the previous point.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

5.11 Proof. This proof is analogous and sometimes simpler than the proof 6.13, so we prefer to do the more complicated proof 6.13 later on.

5.12 Remark. The characterisation theorem of D ensures that the soundness theorem of D is optimal, in the sense that the theory $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP} + \Gamma$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \ \Rightarrow\ \mathsf{HA}^\omega + \Gamma \vdash A^{\mathrm{D}}$ (analogously to remark 3.16). Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega$.

5.5 Applications

5.13 Theorem (disjunction property, existence property and program extraction)**.** Let $\mathsf{T} := \mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

1. Let $A \vee B$ be a sentence of T. If $\mathsf{T} \vdash A \vee B$, then $\mathsf{T} \vdash A$ or $\mathsf{T} \vdash B$.
2. If $\mathsf{T} \vdash \exists \underline{x}\, A(\underline{x})$, then we can extract from such a proof terms $\underline{t}$ of T such that $\mathsf{T} \vdash A(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A)$.
3. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A(\underline{x}, \underline{y})$, then we can extract from such a proof terms $\underline{t}(\underline{x})$ of T such that $\mathsf{T} \vdash \forall \underline{x}\, A(\underline{x}, \underline{t}(\underline{x}))$ and $\mathrm{FV}(\underline{t}(\underline{x})) = \mathrm{FV}(\exists \underline{y}\, A(\underline{x}, \underline{y}))$.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$ [75, theorem 3.7.5] [50, corollary 8.14 and theorem 8.15].

5.14 Proof. Analogous to proof 6.17.

5.15 Theorem (conservation and relative consistency)**.**

1. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP} \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}$, then $\mathsf{HA}^\omega \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}$.
2. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP} \vdash \bot$, then $\mathsf{HA}^\omega \vdash \bot$.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega$ [50, corollary 8.12].

5.16 Proof. Analogous to proof 3.21.

5.17 Theorem (independence)**.** Let $\mathsf{T} := \mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$. We have $\mathsf{T} \nvdash \mathsf{LEM}$ and $\mathsf{T} \nvdash \neg\mathsf{LEM}$ (already for Σ^0_1 and Π^0_1 formulas). It follows the analogous statement for $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

5.18 Proof. Analogous to point 2 of proof 3.23.

5.6 Conclusion

5.19. We introduced Gödel's functional interpretation as being a proof interpretation that solves a problem of modified realisability (when composed with a negative translation) by "capturing" universally quantified variables. The main results about Gödel's functional interpretation are the following.

Soundness theorem This theorem says that we can use Gödel's functional interpretation to extract computational content from proofs in $\mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used Gödel's functional interpretation to do applications on:

1. disjunction property;

2. existence property;
3. program extraction;
4. conservation;
5. relative consistency;
6. independence.

Chapter 6

Diller-Nahm functional interpretation

6.1 Introduction

6.1. Gödel's functional interpretation almost breaks when it interprets the seemly innocuous axiom $A \to A \wedge A$: its interpretation (essentially) asks for terms $\underline{t}$ such that

$$A_{\mathrm{D}}(\underline{a}; \underline{t}) \to A_{\mathrm{D}}(\underline{a}; \underline{d}) \wedge A_{\mathrm{D}}(\underline{a}; \underline{f}), \tag{6.1}$$

like

$$\underline{t} := \begin{cases} \underline{f} & \text{if } \quad A_{\mathrm{D}}(\underline{a}; \underline{d}) \\ \underline{d} & \text{if } \neg A_{\mathrm{D}}(\underline{a}; \underline{d}) \end{cases}$$

(the exact details are given in proof 5.8). The difficulty is that in (6.1) from one A in the premise we need to get two As in the conclusion. This would be no problem if we could have two As in the premise:

$$A_{\mathrm{D}}(\underline{a}; \underline{s}) \wedge A_{\mathrm{D}}(\underline{a}; \underline{t}) \to A_{\mathrm{D}}(\underline{a}; \underline{d}) \wedge A_{\mathrm{D}}(\underline{a}; \underline{f})$$

(even better, finitely many As to interpret $A \to A \wedge \cdots \wedge A$). That is what the Diller-Nahm functional interpretation DN does: to change D by allowing in

$$(A \to B)_{\mathrm{D}}(\underline{C}, \underline{B}; \underline{a}, \underline{d}) :\equiv A_{\mathrm{D}}(\underline{a}; \underline{B}\underline{a}\underline{d}) \to B_{\mathrm{D}}(\underline{C}\underline{a}; \underline{d})$$

a family $\{A_{\mathrm{D}}(\underline{a}; \underline{B}\underline{a}\underline{d}f)\}_{f \leq_0 e\underline{a}\underline{d}}$ of As in the premise:

$$(A \to B)_{\mathrm{DN}}(\underline{C}, \underline{B}, e; \underline{a}, \underline{d}) :\equiv \forall f \leq_0 e\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{d}f) \to B_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d}).$$

This allows to interpret theories in which we cannot define terms by quantifier-free cases.

In addition to the Diller-Nahm functional interpretation DN, we also introduce two variants with truth of DN: the Diller-Nahm functional interpretation with q-truth DNq and the Diller-Nahm functional interpretation with t-truth DNt.

6.2. Our main contribution to this topic is the Diller-Nahm functional interpretation with t-truth DNt and its soundness theorem [22, section 6] (definition 6.3 and theorem 6.8).

6.2 Definition

6.3 Definition.

1. The *Diller-Nahm functional interpretation* DN [10, pages 54–55] assigns to each formula A of HA^ω the formula $A^{\mathrm{DN}} :\equiv \exists \underline{a}\, \forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b})$, where $A_{\mathrm{DN}}(\underline{a}; \underline{b})$ is defined by recursion on the structure of A by

$$\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{DN}}(;) &:\equiv A_{\mathrm{at}},\\
(A \wedge B)_{\mathrm{DN}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{DN}}(\underline{a}; \underline{b}) \wedge B_{\mathrm{DN}}(\underline{c}; \underline{d}),\\
(A \vee B)_{\mathrm{DN}}(e^0, \underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{DN}}(\underline{a}; \underline{b}) \vee_e B_{\mathrm{DN}}(\underline{c}; \underline{d}),\\
(A \to B)_{\mathrm{DN}}(\underline{C}, \underline{B}, e; \underline{a}, \underline{d}) &:\equiv \forall f \leq_0 e\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{d}f) \to B_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d}),\\
(\forall x\, A)_{\mathrm{DN}}(\underline{A}; x, \underline{b}) &:\equiv A_{\mathrm{DN}}(\underline{A}x; \underline{b}),\\
(\exists x\, A)_{\mathrm{DN}}(x, \underline{a}; \underline{b}) &:\equiv A_{\mathrm{DN}}(\underline{a}; \underline{b}).
\end{aligned}$$

   By $(A_{\mathrm{at}})_{\mathrm{DN}}(;)$ we mean $(A_{\mathrm{at}})_{\mathrm{DN}}(\underline{a}; \underline{b})$ with the tuples $\underline{a}$ and $\underline{b}$ empty.

2. The *Diller-Nahm functional interpretation with q-truth* DNq [67, *Definition* 0.3] [41, definition 6.2.1] is defined analogously to DN except for

$$\begin{aligned}
(A \vee B)_{\mathrm{DNq}}(e^0, \underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv (A_{\mathrm{DNq}}(\underline{a}; \underline{b}) \wedge A) \vee_e (B_{\mathrm{DNq}}(\underline{c}; \underline{d}) \wedge B),\\
(A \to B)_{\mathrm{DNq}}(\underline{C}, \underline{B}, e; \underline{a}, \underline{d}) &:\equiv \forall f \leq_0 e\underline{a}\underline{d}\, A_{\mathrm{DNq}}(\underline{a}; \underline{B}\underline{a}\underline{d}f) \wedge A \to B_{\mathrm{DNq}}(\underline{C}\underline{a}; \underline{d}),\\
(\exists x\, A)_{\mathrm{DNq}}(x, \underline{a}; \underline{b}) &:\equiv A_{\mathrm{DNq}}(\underline{a}; \underline{b}) \wedge A.
\end{aligned}$$

3. The *Diller-Nahm functional interpretation with t-truth* DNt [22, definition 6.3] is defined analogously to DN except for

$$\begin{aligned}
(A \to B)_{\mathrm{DNt}}(\underline{C}, \underline{B}, e; \underline{a}, \underline{d}) &:\equiv (\forall f \leq_0 e\underline{a}\underline{d}\, A_{\mathrm{DNt}}(\underline{a}; \underline{B}\underline{a}\underline{d}f) \to B_{\mathrm{DNt}}(\underline{C}\underline{a}; \underline{b})) \wedge\\
&\qquad (A \to B),\\
(\forall x\, A)_{\mathrm{DNt}}(\underline{A}; x, \underline{b}) &:\equiv A_{\mathrm{DNt}}(\underline{A}x; \underline{b}) \wedge \forall x\, A.
\end{aligned}$$

Analogously for $\mathsf{WE\text{-}HA}^\omega$.

6.4. Let us note that, contrarily to what is done for mrt, in DNt we added "$\wedge \forall x\, A$" in the clause of $\forall$; this will be discussed later in chapter 13. We can add here that it was known that does not suffice to change only the clause of $\to$ in DN. Indeed, HA^ω proves the formula $\forall y\, \neg \mathrm{T}xxy \to \neg \exists z\, \mathrm{T}xxz$ (T was introduced in proof 3.23), and the soundness theorem of DNt applied to this formula gives us terms $s(x, z)$ and $t(x, y, a)$ such that we have (6.2) without $(*)$ if we only change the clause of $\to$, and with $(*)$ if we also change the clause of $\forall$:

$$\mathsf{HA}^\omega \vdash \forall x, z\, (\forall a \leq s(x, z)\, \neg \mathrm{T}xxt(x, z, a) \wedge \underbrace{\forall y\, \neg \mathrm{T}xxy}_{(*)} \to \neg \mathrm{T}xxz \wedge \neg \exists z\, \mathrm{T}xxz). \quad (6.2)$$

If we change only the clause of $\to$, then we do not have $(*)$ in (6.2), so the terms solve the halting problem, thus DNt cannot have a soundness theorem [41, theorem 6.1.1]. But if we also change the clause of $\forall$, then we have $(*)$ in (6.2), so there is nothing wrong with (6.2): it holds true with $s(x, z) :\equiv 0$ and $t(x, z, a) :\equiv z$ [22, section 6.3].

6.5 Remark.

1. The Diller-Nahm functional interpretation with q-truth DNq has truth in the sense of: for all disjunctive and existential formulas A of HA^ω we have $\mathsf{HA}^\omega \vdash A_{\mathrm{DNq}}(\underline{a};\underline{b}) \to A$ [22, remark 6.2].

2. The Diller-Nahm functional interpretation with t-truth DNt has truth in the sense of: for all formulas A of HA^ω we have $\mathsf{HA}^\omega \vdash A_{\mathrm{DNt}}(\underline{a};\underline{b}) \to A$ [22, remark 6.4].

The Diller-Nahm functional interpretation with t-truth DNt is a $(*_1)$ strengthening of DNq which $(*_2)$ has truth for all formulas. This can be given a rigorous meaning: $(*_3)$ $\mathsf{HA}^\omega \vdash A_{\mathrm{DNt}}(\underline{a};\underline{b}) \leftrightarrow A_{\mathrm{DNq}}(\underline{a};\underline{b}) \wedge A$ for all formulas A of HA^ω [22, proposition 6.5]. From $(*_3)$ we get: $\mathsf{HA}^\omega \vdash A_{\mathrm{DNt}}(\underline{a};\underline{b}) \to A_{\mathrm{DNq}}(\underline{a};\underline{b})$, that is $(*_1)$; $\mathsf{HA}^\omega \vdash A_{\mathrm{DNt}}(\underline{a};\underline{b}) \to A$, that is $(*_2)$. It follows the analogous statements for $\mathsf{WE\text{-}HA}^\omega$.

6.6 Remark. The formulas $A_{\mathrm{DN}}(\underline{a};\underline{b})$ are equivalent in HA^ω to quantifier-free formulas: we can replace each bounded quantification $\forall x \leq_0 t\, B_{\mathrm{qf}}(x)$ (working inside out of $A_{\mathrm{DN}}(\underline{a};\underline{b})$, so that the matrix of the bounded quantification is quantifier-free) by the equivalent quantifier-free formula $r(t) =_0 0$ where $r(y) :\equiv \mathrm{R}y\chi_{B_{\mathrm{qf}}}(0)\lambda z, y\,.\,(z + \chi_{B_{\mathrm{qf}}}(\mathrm{S}y))$ (that is $r(y) = \chi_{B_{\mathrm{qf}}}(0) + \cdots + \chi_{B_{\mathrm{qf}}}(y)$).

6.7 Remark. The Diller-Nahm functional interpretation DN acts as the identity on quantifier-free formulas of HA^ω without disjunctions in the sense of: $\mathsf{HA}^\omega \vdash (A_{\mathrm{qf}})_{\mathrm{DN}}(\underline{a};\underline{b}) \leftrightarrow A_{\mathrm{qf}}$ for all quantifier-free formulas A_{qf} of HA^ω without disjunctions. (The variables $\underline{a}, \underline{b}$ are introduced by DN, for example, when interpreting implications $A_{\mathrm{at}} \to B_{\mathrm{at}}$, as a bound c on the dummy quantification in $(A_{\mathrm{at}} \to B_{\mathrm{at}})_{\mathrm{DN}}(c;) \equiv \forall d \leq_0 c\, A_{\mathrm{at}} \to B_{\mathrm{at}}$ where $d \notin \mathrm{FV}(A_{\mathrm{at}})$. So the variables $\underline{a}, \underline{b}$ are dummy.) Analogously for DNq and DNt. It follows the analogous statements for $\mathsf{WE\text{-}HA}^\omega$.

6.3 Soundness

6.8 Theorem (soundness)**.** Let A be a formula of HA^ω and let Γ be a set of formulas of HA^ω of the form $\forall \underline{x}\, A_{\mathrm{qf}}$.

1. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega + \Gamma \vdash \forall \underline{b}\, A_{\mathrm{DN}}(\underline{t};\underline{b})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$ [10, *Satz* 3].

2. If $\mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP} + \Gamma \vdash \forall \underline{b}\, A_{\mathrm{DNq}}(\underline{t};\underline{b})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$ [67, *Satz* 1.1] [41, theorem 6.2.3].

3. If $\mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP} + \Gamma \vdash \forall \underline{b}\, A_{\mathrm{DNt}}(\underline{t};\underline{b})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A)$ [22, theorem 6.6].

The terms constructed in the following proof for the three points above are the same. Analogously for $\mathsf{WE\text{-}HA}^\omega$.

6.9 Proof. Let us make some remarks.

1. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth, with q-truth and with t-truth, by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12.

2. We have

$$\mathsf{HA}^\omega \vdash (\exists \underline{x}\, A)_{\mathrm{DN}}(\underline{x}, \underline{a}; \underline{b}) \leftrightarrow A_{\mathrm{DN}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}},$$
$$\mathsf{HA}^\omega \vdash (\forall \underline{x}\, A)_{\mathrm{DN}}(\underline{A}; \underline{x}, \underline{b}) \leftrightarrow A_{\mathrm{DN}}(\underline{A}\underline{x}; \underline{b}) \wedge (\forall \underline{x}\, A)^{\mathrm{t}}.$$

We will replace the left sides of the equivalences by the right sides. When we do it, we use "$\equiv$" instead of $\equiv$.

$\underline{A \to A \wedge A}$ We have

$$(A \to A \wedge A)_{\mathrm{DN}}(\underline{C}, \underline{E}, \underline{B}, g; \underline{a}, \underline{d}, \underline{f}) \text{ ``}\equiv\text{''}$$
$$\forall h \leq_0 g\underline{a}\underline{d}\underline{f}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{d}\underline{f}h) \wedge A^{\mathrm{q}} \to A_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d}) \wedge A_{\mathrm{DN}}(\underline{E}\underline{a}; \underline{f}),$$

$$\underline{t_C} :\equiv \lambda \underline{a} \,.\, \underline{a}, \quad \underline{t_E} :\equiv \lambda \underline{a} \,.\, \underline{a}, \quad \underline{t_B} :\equiv \lambda \underline{a}, \underline{d}, \underline{f}, h \,.\, \underline{d} \vee_h \underline{f}, \quad t_g :\equiv \lambda \underline{a}, \underline{d}, \underline{f} \,.\, \mathrm{S}0.$$

Analogously for $A \vee A \to A$.

$\underline{A \vee B \to B \vee A}$ We have

$$(A \vee B \to B \vee A)_{\mathrm{DN}}(J, \underline{E}, \underline{G}, \underline{B}, \underline{D}, k; i, \underline{a}, \underline{c}, \underline{f}, \underline{h}) \text{ ``}\equiv\text{''}$$
$$\forall l \leq_0 ki\underline{a}\underline{c}\underline{f}\underline{h} \big((A_{\mathrm{DN}}(\underline{a}; \underline{B}i\underline{a}\underline{c}\underline{f}\underline{h}l) \wedge A^{\mathrm{q}}) \vee_i (B_{\mathrm{DN}}(\underline{c}; \underline{D}i\underline{a}\underline{c}\underline{f}\underline{h}l)) \wedge B^{\mathrm{q}})\big) \wedge$$
$$(A \vee B)^{\mathrm{q}} \to (B_{\mathrm{DN}}(\underline{E}i\underline{a}\underline{c}; \underline{f}) \wedge B^{\mathrm{q}}) \vee_{Ji\underline{a}\underline{c}} (A_{\mathrm{DN}}(\underline{G}i\underline{a}\underline{c}; \underline{h})) \wedge A^{\mathrm{q}}),$$

$$t_J :\equiv \lambda i, \underline{a}, \underline{c} \,.\, \overline{\mathrm{sg}}\, i, \quad \underline{t_E} :\equiv \lambda i, \underline{a}, \underline{c} \,.\, \underline{c}, \quad \underline{t_G} :\equiv \lambda i, \underline{a}, \underline{c} \,.\, \underline{a},$$
$$\underline{t_B} :\equiv \lambda i, \underline{a}, \underline{c}, \underline{f}, \underline{h}, l \,.\, \underline{h}, \quad \underline{t_D} :\equiv \lambda i, \underline{a}, \underline{c}, \underline{f}, \underline{h}, l \,.\, \underline{f}, \quad t_k :\equiv \mathcal{O}.$$

Analogously for $A \wedge B \to A$, $A \to A \vee B$, $A \wedge B \to B \wedge A$ and $\bot \to A$.

$\underline{\forall x\, A \to A[t/x]}$ We have

$$(\forall x\, A \to A[t/x])_{\mathrm{DN}}(\underline{C}, X, \underline{B}, e; \underline{A}, \underline{d}) \text{ ``}\equiv\text{''}$$
$$\forall f \leq_0 e\underline{A}\underline{d}\, (A_{\mathrm{DN}}(\underline{a}; \underline{b})[\underline{A}(X\underline{A}\underline{d}f), \underline{B}\underline{A}\underline{d}f, X\underline{A}\underline{d}f / \underline{a}, \underline{b}, x] \wedge (\forall x\, A)^{\mathrm{t}}) \wedge (\forall x\, A)^{\mathrm{q}} \to$$
$$A[t/x]_{\mathrm{DN}}(\underline{C}\underline{A}; \underline{d}),$$

$$\underline{t_C} :\equiv \lambda \underline{A} \,.\, \underline{A}t, \quad \underline{t_X} :\equiv \lambda \underline{A}, \underline{d}, f \,.\, t, \quad \underline{t_B} :\equiv \lambda \underline{A}, \underline{d}, f \,.\, \underline{d}, \quad t_e :\equiv \mathcal{O}.$$

Analogously for $A[t/x] \to \exists x\, A$.

$\underline{A \to B,\ B \to C \,/\, A \to C}$ We have

$$(A \to B)_{\mathrm{DN}}(\underline{C}, \underline{B}, g; \underline{a}, \underline{d}) \text{ ``}\equiv\text{'' } \forall h \leq_0 g\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{d}h) \wedge A^{\mathrm{q}} \to B_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d}),$$
$$(B \to C)_{\mathrm{DN}}(\underline{E}, \underline{D}, g; \underline{c}, \underline{f}) \text{ ``}\equiv\text{'' } \forall h \leq_0 g\underline{c}\underline{f}\, B_{\mathrm{DN}}(\underline{c}; \underline{D}\underline{c}\underline{f}h) \wedge B^{\mathrm{q}} \to C_{\mathrm{DN}}(\underline{E}\underline{c}; \underline{f}),$$
$$(A \to C)_{\mathrm{DN}}(\underline{E}, \underline{B}, g; \underline{a}, \underline{f}) \text{ ``}\equiv\text{'' } \forall h \leq_0 g\underline{a}\underline{f}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{f}h) \wedge A^{\mathrm{q}} \to C_{\mathrm{DN}}(\underline{E}\underline{a}; \underline{f}).$$

In a primitive recursive way we define terms $\underline{t_B}$ and t_h such that $\{\underline{t_B}\underline{a}\underline{f}h\}_{h\leq_0 t_g\underline{a}\underline{f}}$ enumerates $\{\underline{r_B}\underline{a}(\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)h'\}_{\substack{h\leq_0 s_g(\underline{r_C}\underline{a})\underline{f}\\ h'\leq_0 r_g\underline{a}(\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)}}$ and therefore

$$\begin{gathered}\forall h\leq_0 t_g\underline{a}\underline{f}\, E(\underline{t_B}\underline{a}\underline{f}h)\leftrightarrow\\ \forall h\leq_0 s_g(\underline{r_C}\underline{a})\underline{f}\,\forall h'\leq_0 r_g\underline{a}(\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)\, E\big(\underline{r_B}\underline{a}(\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)h'\big).\end{gathered}\tag{6.3}$$

Let us see that the terms

$$\underline{t_E}:\equiv\lambda\underline{a}\,.\,\underline{s_E}(\underline{r_C}\underline{a}),\quad \underline{t_B},\quad t_g$$

work. By induction hypothesis we have (6.4) and (6.5), and we want to prove (6.6):

$$\forall h'\leq_0 r_g\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a};\underline{r_B}\underline{a}\underline{d}h')\wedge A^{\mathrm{q}}\to B_{\mathrm{DN}}(\underline{r_C}\underline{a};\underline{d}),\tag{6.4}$$

$$\forall h\leq_0 s_g\underline{c}\underline{f}\, B_{\mathrm{DN}}(\underline{c};\underline{s_D}\underline{c}\underline{f}h)\wedge B^{\mathrm{q}}\to C_{\mathrm{DN}}(\underline{s_E}\underline{c};\underline{f}),\tag{6.5}$$

$$\forall h\leq_0 t_g\underline{a}\underline{f}\, A_{\mathrm{DN}}(\underline{a};\underline{t_B}\underline{a}\underline{f}h)\wedge A^{\mathrm{q}}\to C_{\mathrm{DN}}(\underline{s_E}(\underline{r_C}\underline{a});\underline{f}).\tag{6.6}$$

Taking $\underline{d}=\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h$ in (6.4) and $\underline{c}=\underline{r_C}\underline{a}$ in (6.5) we get, respectively,

$$\begin{gathered}\forall h'\leq_0 r_g\underline{a}(\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)\, A_{\mathrm{DN}}\big(\underline{a};\underline{r_B}\underline{a}(\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)h'\big)\wedge A^{\mathrm{q}}\to\\ B_{\mathrm{DN}}(\underline{r_C}\underline{a};\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h),\end{gathered}\tag{6.7}$$

$$\begin{gathered}\forall h\leq_0 s_g(\underline{r_C}\underline{a})\underline{f}\, B_{\mathrm{DN}}(\underline{r_C}\underline{a};\underline{s_D}(\underline{r_C}\underline{a})\underline{f}h)\wedge B^{\mathrm{q}}\to\\ C_{\mathrm{DN}}(\underline{s_E}(\underline{r_C}\underline{a});\underline{f}).\end{gathered}\tag{6.8}$$

By (6.3), the premise of (6.6) implies the premise of (6.7) for all $h\leq_0 s_g(\underline{r_C}\underline{a})\underline{f}$, which implies the conclusion of (6.7) for the same hs, that is the first conjunctive of the premise of (6.8), so we get the conclusion of (6.8), that is the conclusion of (6.3), as we wanted. If $\mathrm{q}=\mathrm{id}$, then we use the assumption that we proved B, so as to have the part B^{q} in (6.8). Analogously for $A,\ A\to B\,/\,B$.

$\underline{A\wedge B\to C\,/\,A\to(B\to C)}$ We have

$$\begin{gathered}(A\wedge B\to C)_{\mathrm{DN}}(\underline{E},\underline{B},\underline{D},g;\underline{a},\underline{c},\underline{f})\equiv\\ \big(\forall h\leq_0 g\underline{a}\underline{c}\underline{f}\,(A_{\mathrm{DN}}(\underline{a};\underline{B}\underline{a}\underline{c}\underline{f}h)\wedge B_{\mathrm{DN}}(\underline{c};\underline{D}\underline{a}\underline{c}\underline{f}h))\wedge(A\wedge B)^{\mathrm{q}}\to\\ C_{\mathrm{DN}}(\underline{E}\underline{a}\underline{c};\underline{f})\big)\wedge(A\wedge B\to C)^{\mathrm{t}},\end{gathered}\tag{6.9}$$

$$\begin{gathered}(A\to(B\to C))_{\mathrm{DN}}(\underline{\mathrm{E}},\underline{\mathrm{D}},G,\underline{B},i;\underline{a},\underline{c},\underline{f})\equiv\\ \big(\forall j\leq_0 i\underline{a}\underline{c}\underline{f}\,A_{\mathrm{DN}}(\underline{a};\underline{B}\underline{a}\underline{c}\underline{f}j)\wedge A^{\mathrm{q}}\to\\ \big((\forall h\leq_0 G\underline{a}\underline{c}\underline{f}\,B_{\mathrm{DN}}(\underline{c};\underline{\mathrm{D}}\underline{a}\underline{c}\underline{f}h)\wedge B^{\mathrm{q}}\to C_{\mathrm{DN}}(\underline{\mathrm{E}}\underline{a}\underline{c};\underline{f}))\wedge(B\to C)^{\mathrm{t}}\big)\big)\wedge\\ (A\to(B\to C))^{\mathrm{t}},\end{gathered}\tag{6.10}$$

$$\underline{t_{\mathrm{E}}}:\equiv\underline{s_E},\quad \underline{t_{\mathrm{D}}}:\equiv\underline{s_D},\quad t_G:\equiv s_g,\quad \underline{t_B}:\equiv\underline{s_B},\quad t_i:\equiv s_g.$$

If $\mathrm{t}=\mathrm{id}$, then we use $A_{\mathrm{DN}}(\underline{a};\underline{B}\underline{a}\underline{c}\underline{f}j)\to A$, so that the parts $(A\wedge B\to C)^{\mathrm{t}}$ in (6.9) and $A_{\mathrm{DN}}(\underline{a};\underline{B}\underline{a}\underline{c}\underline{f}j)$ in (6.10) together imply the part $(B\to C)^{\mathrm{t}}$ in (6.10). Analogously $A\to B\,/\,C\vee A\to C\vee B$.

$\underline{A \to (B \to C) / A \wedge B \to C}$ The interpretations were computed in (6.9) and (6.10).

$$\underline{t}_{\underline{E}} :\equiv \underline{s}_{\mathrm{E}}, \quad \underline{t}_{\underline{B}} :\equiv \underline{s}_{\underline{B}}, \quad \underline{t}_{\underline{D}} :\equiv \underline{s}_{\mathrm{D}}, \quad t_g :\equiv \lambda \underline{a}, \underline{c}, \underline{f} \,.\, \max(s_i \underline{a}\underline{c}\underline{f})(s_G \underline{a}\underline{c}\underline{f}).$$

$\underline{A \to B / A \to \forall x\, B}$ We have

$$\begin{gathered}(A \to B)_{\mathrm{DN}}(\underline{C}, \underline{B}, e; \underline{a}, \underline{d}) \equiv \\ (\forall f \leq_0 e\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{d}f) \wedge A^{\mathrm{q}} \to B_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d})) \wedge (A \to B)^{\mathrm{t}},\end{gathered} \tag{6.11}$$

$$\begin{gathered}(A \to \forall x\, B)_{\mathrm{DN}}(\underline{\mathrm{C}}, \underline{B}, e; \underline{a}, x, \underline{d}) \text{ “}\equiv\text{”} \\ \forall f \leq_0 e\underline{a}x\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}x\underline{d}f) \wedge A^{\mathrm{q}} \to B_{\mathrm{DN}}(\underline{\mathrm{C}}\underline{a}x; \underline{d}) \wedge (\forall x\, B)^{\mathrm{t}},\end{gathered} \tag{6.12}$$

$$\begin{gathered}\underline{t}_{\underline{\mathrm{C}}}(\underline{\ell}) :\equiv \lambda \underline{a}, x \,.\, \underline{s}_{\underline{C}}(\underline{\ell}, x)\underline{a}, \quad \underline{t}_{\underline{B}}(\underline{\ell}) :\equiv \lambda \underline{a}, x, \underline{d}, f \,.\, \underline{s}_{\underline{B}}(\underline{\ell}, x)\underline{a}\underline{d}f, \\ t_e(\underline{\ell}) :\equiv \lambda \underline{a}, x, \underline{d} \,.\, s_e(\underline{\ell}, x)\underline{a}\underline{d}.\end{gathered}$$

If $\mathrm{t} = \mathrm{id}$, the we use $A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}x\underline{d}f) \to A$, so that the parts $(A \to B)^{\mathrm{t}}$ in (6.11) and $A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}x\underline{d}e)$ in (6.12) together imply the part $(\forall x\, B)^{\mathrm{t}}$ in (6.12). Analogously for $A \to B / \exists x\, A \to B$.

Axioms of $=_0$, S, Π, Σ and R Their formulas are quantifier-free, so they are equivalent to their own interpretation.

$\underline{A[0/x],\ A \to A[\mathrm{S}x/x] / A}$ We can assume $x \in \mathrm{FV}(A)$, otherwise $A[0/x] \equiv A$ and so the terms working for $A[0/x]$ also work for A. We have

$$A[0/x]_{\mathrm{DN}}(\underline{a}; \underline{b}),$$

$$\begin{gathered}(A \to A[\mathrm{S}x/x])_{\mathrm{DN}}(\underline{C}, \underline{B}, e; \underline{a}, \underline{d}) \text{ “}\equiv\text{” } \forall f \leq_0 e\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{B}\underline{a}\underline{d}f) \wedge A^{\mathrm{q}} \to \\ A[\mathrm{S}x/x]_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d}),\end{gathered}$$

$$A_{\mathrm{DN}}(\underline{a}; \underline{b}),$$

$$\underline{t}_{\underline{a}}(\underline{\ell}, x) :\equiv \underline{\mathrm{R}}x\underline{r}_{\underline{a}}(\underline{\ell})\, \lambda \underline{a}, x \,.\, \underline{s}_{\underline{C}}(\underline{\ell}, x)\underline{a}.$$

By induction hypothesis we have (6.13) and (6.14), and we want to prove (6.15) by induction on x:

$$\forall \underline{b}\, A[0/x]_{\mathrm{DN}}(\underline{r}_{\underline{a}}(\underline{\ell}); \underline{b}), \tag{6.13}$$

$$\begin{gathered}\forall \underline{a}, \underline{d}\, \forall f \leq_0 s_e(\underline{\ell}, x)\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{s}_{\underline{B}}(\underline{\ell}, x)\underline{a}\underline{d}f) \wedge A^{\mathrm{q}} \to \\ A[\mathrm{S}x/x]_{\mathrm{DN}}(\underline{s}_{\underline{C}}(\underline{\ell}, x)\underline{a}; \underline{d}),\end{gathered} \tag{6.14}$$

$$\forall \underline{b}\, A_{\mathrm{DN}}(\underline{t}_{\underline{a}}(\underline{\ell}, x); \underline{b}) \tag{6.15}$$

Base case The formula $\forall \underline{b}\, A_{\mathrm{DN}}(\underline{t}_{\underline{a}}(\underline{\ell}, x); \underline{b})[0/x]$ is equivalent to (6.13).

Induction step By induction hypothesis we assume $\forall \underline{b}\, A_{\mathrm{DN}}(\underline{t}_{\underline{a}}(\underline{\ell}, x); \underline{b})$. Taking $\underline{a} = \underline{t}_{\underline{a}}(\underline{\ell}, x)$ in (6.14) we get $\forall \underline{d}\, A[\mathrm{S}x/x]_{\mathrm{DN}}(\underline{s}_{\underline{C}}(\underline{\ell}, x)\underline{t}_{\underline{a}}(\underline{\ell}, x); \underline{d})$, that is $\forall \underline{b}\, A_{\mathrm{DN}}(\underline{t}_{\underline{a}}(\underline{\ell}, x); \underline{b})[\mathrm{S}x/x]$. If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved A, so as to have the part A^{q} in (6.14).

$\underline{A_{\mathrm{at}} \to \forall \underline{x}\,(r\underline{x} =_0 s\underline{x})\,/\,A_{\mathrm{at}} \to t(r) =_0 t(s)}$ We have

$$\begin{gathered}(A_{\mathrm{at}} \to \forall \underline{x}\,(r\underline{x} =_0 s\underline{x}))_{\mathrm{DN}}(a;\underline{x}) \text{ ``}\equiv\text{''} \\ \forall b \leq_0 a\underline{x}\, A_{\mathrm{at}} \wedge A_{\mathrm{at}}^{\mathrm{q}} \to r\underline{x} =_0 s\underline{x} \wedge (\forall \underline{x}\,(r\underline{x} =_0 s\underline{x}))^{\mathrm{t}},\end{gathered} \tag{6.16}$$

$$\begin{gathered}(A_{\mathrm{at}} \to t(r) =_0 t(s))_{\mathrm{DN}}(a;) \text{ ``}\equiv\text{''} \\ \forall b \leq_0 a\underline{x}\, A_{\mathrm{at}} \wedge A_{\mathrm{at}}^{\mathrm{q}} \to t(r) =_0 t(s),\end{gathered} \tag{6.17}$$

$$t_a :\equiv s_a.$$

Note that $\forall b \leq_0 s_a\underline{x}\, A_{\mathrm{at}} \leftrightarrow A_{\mathrm{at}}$ because the quantification is dummy, so we can apply the extensionality rule to (6.16) getting (6.17).

$\underline{\mathsf{AC}}$ To keep the notation simple, we denote $A(x, Yx)_{\mathrm{DN}}(\underline{a};\underline{b})$ and $A(x,y)_{\mathrm{DN}}(\underline{a};\underline{b})[Yx/x]$ by $A_{\mathrm{DN}}(\underline{a};\underline{b};x,Yx)$. We have

$$\begin{gathered}(\forall x\, \exists y\, A(x,y))_{\mathrm{DN}}(Y,\underline{A};x,\underline{b}) \equiv \\ A_{\mathrm{DN}}(\underline{A}x;\underline{b};x,Yx) \wedge A(x,Yx)^{\mathrm{q}} \wedge (\forall x\, \exists y\, A(x,y))^{\mathrm{t}},\end{gathered}$$

$$\begin{gathered}(\exists Y\, \forall x\, A(x,Yx))_{\mathrm{DN}}(Y,\underline{C};x,\underline{d}) \equiv \\ A_{\mathrm{DN}}(\underline{C}x;\underline{d};x,Yx) \wedge (\forall x\, A(x,Yx))^{\mathrm{t}} \wedge (\forall x\, A(x,Yx))^{\mathrm{q}},\end{gathered}$$

$$\begin{gathered}\mathsf{AC}_{\mathrm{DN}}(\mathrm{Y},\underline{\mathrm{C}},X,\underline{B},e;Y,\underline{A},x,\underline{d}) \text{ ``}\equiv\text{''} \\ \forall f \leq_0 eY\underline{A}x\underline{d}\,\big(A_{\mathrm{DN}}(\underline{A}(XY\underline{A}x\underline{d}f);\underline{B}Y\underline{A}x\underline{d}f;XY\underline{A}x\underline{d}f,Y(XY\underline{A}x\underline{d}f)) \wedge \\ A(XY\underline{A}x\underline{d}f,Y(XY\underline{A}x\underline{d}f))^{\mathrm{q}} \wedge (\forall x\, \exists y\, A(x,y))^{\mathrm{t}}\big) \wedge (\forall x\, \exists y\, A(x,y))^{\mathrm{q}} \\ \downarrow \\ A_{\mathrm{DN}}(\underline{\mathrm{C}}Y\underline{A}x;\underline{d};x,\mathrm{Y}Y\underline{A}x) \wedge (\forall x\, A(x,\mathrm{Y}Y\underline{A}x))^{\mathrm{t}} \wedge (\forall x\, A(x,\mathrm{Y}Y\underline{A}x))^{\mathrm{q}},\end{gathered}$$

$$\begin{gathered}\underline{t}_{\mathrm{Y}} :\equiv \lambda Y,\underline{A},x\,.\,Yx, \quad \underline{t}_{\underline{\mathrm{C}}} :\equiv \lambda Y,\underline{A},x\,.\,\underline{A}x, \quad t_X :\equiv \lambda Y,\underline{A},x,\underline{d},f\,.\,x, \\ \underline{t}_{\underline{B}} :\equiv \lambda Y,\underline{A},x,\underline{d},f\,.\,\underline{d}, \quad t_e :\equiv \mathcal{O}.\end{gathered}$$

These terms only seem to work if $\mathrm{q} = \top$ and $\mathrm{t} = \top$.

$\underline{\forall\text{-}\mathsf{IP}}$ We have

$$\begin{gathered}(\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to \exists y\, B(y))_{\mathrm{DN}}(y,\underline{a},\underline{X},e;\underline{b}) \equiv \\ \big(\forall f \leq_0 e\underline{b}\,\big(A_{\mathrm{qf}}(\underline{X}\underline{b}f) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{t}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{q}} \to B_{\mathrm{DN}}(\underline{a};\underline{b};y) \wedge B(y)^{\mathrm{q}}\big) \wedge \\ (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to \exists y\, B(y))^{\mathrm{t}},\end{gathered}$$

$$\begin{gathered}\big(\exists y\,(\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to B(y))\big)_{\mathrm{DN}}(y,\underline{c},\underline{X},g;\underline{d}) \equiv \\ \big(\forall h \leq_0 g\underline{d}\,\big(A_{\mathrm{qf}}(\underline{X}\underline{d}h) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{t}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{q}} \to B_{\mathrm{DN}}(\underline{c};\underline{d};y)\big) \wedge \\ (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to B(y))^{\mathrm{t}} \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to B(y))^{\mathrm{q}},\end{gathered}$$

$$
\begin{gathered}
\forall\text{-}\mathsf{IP}_{\mathrm{DN}}(Y,\underline{C},\underline{\mathrm{X}},G,\underline{B},i;y,\underline{a},\underline{X},e,\underline{d}) \text{ “}\equiv\text{” } \forall j \leq_0 iy\underline{a}\underline{X}e\underline{d} \\
\Big(\big(\forall f \leq_0 e(\underline{B}y\underline{a}\underline{X}e\underline{d}j)\, \big(A_{\mathrm{qf}}(\underline{X}(\underline{B}y\underline{a}\underline{X}e\underline{d}j)f) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{t}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{q}} \to \\
B_{\mathrm{DN}}(\underline{a};\underline{B}y\underline{a}\underline{X}e\underline{d}j;y) \wedge B(y)^{\mathrm{q}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to \exists y\, B(y))^{\mathrm{t}}\Big) \wedge \\
(\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to \exists y\, B(y))^{\mathrm{q}} \\
\downarrow \\
\big(\forall h \leq_0 Gy\underline{a}\underline{X}e\underline{d}\, \big(A_{\mathrm{qf}}(\underline{\mathrm{X}}y\underline{a}\underline{X}e\underline{d}h) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{t}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{q}} \to \\
B_{\mathrm{DN}}(\underline{C}y\underline{a}\underline{X}e;\underline{d};Y\underline{a}y\underline{X}e)\big) \wedge \\
(\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to B(Yy\underline{a}\underline{X}e))^{\mathrm{t}} \wedge (\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to B(Yy\underline{a}\underline{X}e))^{\mathrm{q}},
\end{gathered}
$$

$$
\begin{gathered}
t_Y :\equiv \lambda \underline{a},y,\underline{X},e\,.\,y, \quad t_{\underline{C}} :\equiv \lambda y,\underline{a},\underline{X},e\,.\,\underline{a}, \quad t_{\underline{\mathrm{X}}} :\equiv \lambda y,\underline{a},\underline{X},\underline{d},g\,.\,\underline{X}\underline{d}h, \\
t_G :\equiv \lambda y,\underline{a},\underline{X},e,\underline{d}\,.\,e\underline{d}, \quad t_{\underline{B}} :\equiv \lambda y,\underline{a},\underline{X},e,\underline{d},j\,.\,\underline{d}, \quad t_i :\equiv \mathcal{O}.
\end{gathered}
$$

If t = id, then we use $B_{\mathrm{DN}}(\underline{a};\underline{B}y\underline{a}\underline{X}e\underline{d}j;y) \to B(y)$, so that the premise of the interpretation of $\forall$-IP implies the part $(\forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \to B(t_Y y\underline{a}\underline{X}e))^{\mathrm{t}}$ in the conclusion.

QF-MP It suffices to interpret $\neg\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}) \to \exists \underline{y}\, A_{\mathrm{at}}(\underline{y})$ because every quantifier-free formula is equivalent in HA^ω to an atomic formula by theorem 1.44.

$$
\begin{gathered}
(\neg\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))_{\mathrm{DN}}(\underline{X},c;a) \text{ “}\equiv\text{” } \neg\Big(\forall d \leq_0 ca \\
\big(\neg\big(\forall b \leq_0 a(\underline{X}ad)\, (A_{\mathrm{at}}(\underline{X}ad) \wedge A_{\mathrm{at}}(\underline{X}ad)^{\mathrm{q}}) \wedge (\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{q}}\big) \wedge (\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{t}}\big) \wedge \\
(\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{q}}\Big) \wedge (\neg\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{t}},
\end{gathered}
$$

$$
(\exists \underline{y}\, A_{\mathrm{at}}(\underline{y}))_{\mathrm{DN}}(\underline{y};\,) \text{ “}\equiv\text{” } A_{\mathrm{at}}(\underline{y}) \wedge A_{\mathrm{at}}(\underline{y})^{\mathrm{q}},
$$

$$
\begin{gathered}
\mathsf{QF\text{-}MP}_{\mathrm{DN}}(\underline{Y},A,e;\underline{X},c) \text{ “}\equiv\text{” } \forall f \leq_0 e\underline{X}c\, \Bigg(\neg\Big(\forall d \leq_0 c(A\underline{X}cf) \\
\Big(\neg\big(\forall b \leq_0 A\underline{X}cf(\underline{X}(A\underline{X}cf)d)\, \big(A_{\mathrm{at}}(\underline{X}(A\underline{X}cf)d) \wedge A_{\mathrm{at}}(\underline{X}(A\underline{X}cf)d)^{\mathrm{q}}\big) \wedge \\
(\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{q}}\big) \wedge (\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{t}}\Big) \wedge (\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{q}}\Big) \wedge (\neg\neg\exists \underline{x}\, A_{\mathrm{at}}(\underline{x}))^{\mathrm{t}}\Bigg) \wedge \\
(\neg\neg\exists \underline{x}\, A_{\mathrm{qf}}(\underline{x}))^{\mathrm{q}} \\
\downarrow \\
A_{\mathrm{at}}(\underline{Y}\underline{X}c) \wedge A_{\mathrm{at}}(\underline{Y}\underline{X}c))^{\mathrm{q}},
\end{gathered}
$$

$$
t_{\underline{Y}} :\equiv \lambda \underline{X},c\,.\,\underline{X}\mathcal{O}0, \quad t_A :\equiv \mathcal{O}, \quad t_e :\equiv \mathcal{O}.
$$

These terms only seem to work if q $= \top$ and t $= \top$.

$\underline{\Gamma}$ We have

$$
(\forall \underline{x}\, A_{\mathrm{qf}})_{\mathrm{DN}}(;\underline{x}) \text{ “}\equiv\text{” } A_{\mathrm{qf}}.
$$

6.10 Remark.

1. The Diller-Nahm functional interpretation with q- [41, remark 6.2.4] and t-truth do not seem to interpret AC. To interpret it we (essentially and in

particular) should present a term witnessing Y in $A(x, Yx) \wedge \forall x\, \exists y\, A(x, y) \rightarrow \forall x\, A(x, \mathrm{Y}Yx)$ and this does not seem possible.

2. The Diller-Nahm functional interpretation with q- [41, remark 6.2.4] and t-truth do not seem to interpret QF-MP. To interpret it we (essentially and in particular) should present terms witnessing $\underline{Y}$ in $\neg\neg\exists \underline{x}\, A(\underline{x}) \rightarrow A_{\mathrm{at}}(\underline{Y}\,\underline{X}c)$ and this does not seem possible.

6.11. Because DNq and DNt do not seem to interpret QF-MP, we may wonder what do they add to mrq and mrt. For example, in theorem 6.16, the disjunction property, existence property and program extraction of $\mathsf{HA}^\omega \pm \forall$-IP already can be obtained with mrq and mrt (observing that their soundness theorems also hold with $\forall$-IP instead of $\exists$F-IP) [41, section 6.3] [52]. It seems that for complex formulas (for example, of the form $\Pi_2 \rightarrow \Pi_2$) DNt is stronger than mrt [22, section 6.4], but despite this applications of DNq and DNt that cannot be obtained with mrq and mrt are unknown.

6.4 Characterisation

6.12 Theorem (characterisation)**.** Let us consider the theory $\mathsf{HA}^\omega + \mathsf{AC} + \forall$-IP + QF-MP.

1. This theory proves $A \leftrightarrow A^{\mathrm{DN}}$ for all formulas A of HA^ω.

2. This theory is the least theory, containing HA^ω, satisfying the previous point.

Analogously for $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \forall$-IP + QF-MP [10, *Satz* 2.1].

6.13 Proof. We do the proof for $\mathsf{HA}^\omega + \mathsf{AC} + \forall$-IP + QF-MP; the case of the other theory is analogous. By remark 6.6 we can treat the formulas $A_{\mathrm{DN}}(\underline{a}; \underline{b})$ as if they were quantifier-free.

1. The proof is by induction on the structure of A.

   $\underline{\vee}$ Using HA^ω in the first equivalence, induction hypothesis in the second equivalence, and $\forall$-IP in the third equivalence, we get

$$\begin{aligned}
&A \vee B \leftrightarrow \\
&\exists e\, ((e = 0 \rightarrow A) \wedge (e \neq 0 \rightarrow B)) \leftrightarrow \\
&\exists e\, ((e = 0 \rightarrow A^{\mathrm{DN}}) \wedge (e \neq 0 \rightarrow B^{\mathrm{DN}})) \equiv \\
&\exists e\, \big((e = 0 \rightarrow \exists \underline{a}\, \forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b})) \wedge (e \neq 0 \rightarrow \exists \underline{c}\, \forall \underline{d}\, B_{\mathrm{DN}}(\underline{c}; \underline{d}))\big) \leftrightarrow \\
&\exists e, \underline{a}, \underline{c}\, \big((e = 0 \rightarrow \forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b})) \wedge (e \neq 0 \rightarrow \forall \underline{d}\, B_{\mathrm{DN}}(\underline{c}; \underline{d}))\big) \leftrightarrow \\
&\exists e, \underline{a}, \underline{c}\, \forall \underline{b}, \underline{d}\, \big((e = 0 \rightarrow A_{\mathrm{DN}}(\underline{a}; \underline{b})) \wedge (e \neq 0 \rightarrow B_{\mathrm{DN}}(\underline{c}; \underline{d}))\big) \equiv \\
&(A \vee B)^{\mathrm{DN}}.
\end{aligned}$$

   Analogously for A_{at}, $\wedge$ and $\exists$.

$\underline{\rightarrow}$ We have (∗) $\mathsf{HA}^\omega \vdash (\forall \underline{x}\, A_{\mathrm{qf}} \rightarrow B_{\mathrm{qf}}) \leftrightarrow \neg\neg\exists \underline{x}\, (A_{\mathrm{qf}} \rightarrow B_{\mathrm{qf}})$ where $\underline{x} \notin \mathrm{FV}(B_{\mathrm{qf}})$: follows from point 1 of theorem 1.53 and $\mathsf{IL} + (\neg\neg C \rightarrow C) + (\neg\neg D \rightarrow D) \vdash (\forall \underline{y}\, C \rightarrow D) \leftrightarrow \neg\neg\exists \underline{y}\, (C \rightarrow D)$ where $\underline{y} \notin \mathrm{FV}(D)$ [50, exercise 5 of section 8.3]. Using induction hypothesis in the first equivalence, $\forall$-IP in the third equivalence, (∗) in the sixth equivalence, QF-MP in the seventh equivalence, and AC in the last equivalence, we get

$$\begin{aligned}
&(A \rightarrow B) \leftrightarrow \\
&A^{\mathrm{DN}} \rightarrow B^{\mathrm{DN}} \equiv \\
&(\exists \underline{a}\, \forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b}) \rightarrow \exists \underline{c}\, \forall \underline{d}\, B_{\mathrm{DN}}(\underline{c}; \underline{d}))) \leftrightarrow \\
&\forall \underline{a}\, (\forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b}) \rightarrow \exists \underline{c}\, \forall \underline{d}\, B_{\mathrm{DN}}(\underline{c}; \underline{d})) \leftrightarrow \\
&\forall \underline{a}\, \exists \underline{c}\, (\forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b}) \rightarrow \forall \underline{d}\, B_{\mathrm{DN}}(\underline{c}; \underline{d})) \leftrightarrow \\
&\forall \underline{a}\, \exists \underline{c}\, \forall \underline{d}\, (\forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b}) \rightarrow B_{\mathrm{DN}}(\underline{c}; \underline{d})) \leftrightarrow \\
&\forall \underline{a}\, \exists \underline{c}\, \forall \underline{d}\, (\forall \underline{B}, e\, \forall f \leq_0 e\, A_{\mathrm{DN}}(\underline{a}; \underline{B}f) \rightarrow B_{\mathrm{DN}}(\underline{c}; \underline{d})) \leftrightarrow \\
&\forall \underline{a}\, \exists \underline{c}\, \forall \underline{d}\, \neg\neg\exists \underline{B}, e\, (\forall f \leq_0 e\, A_{\mathrm{DN}}(\underline{a}; \underline{B}f) \rightarrow B_{\mathrm{DN}}(\underline{c}; \underline{d})) \leftrightarrow \\
&\forall \underline{a}\, \exists \underline{c}\, \forall \underline{d}\, \exists \underline{B}, e\, (\forall f \leq_0 e\, A_{\mathrm{DN}}(\underline{a}; \underline{B}f) \rightarrow B_{\mathrm{DN}}(\underline{c}; \underline{d})) \leftrightarrow \\
&\exists \underline{C}, \underline{\mathrm{B}}, E\, \forall \underline{a}, \underline{d}\, (\forall f \leq_0 E\underline{a}\underline{d}\, A_{\mathrm{DN}}(\underline{a}; \underline{\mathrm{B}}\underline{a}\underline{d}f) \rightarrow B_{\mathrm{DN}}(\underline{C}\underline{a}; \underline{d})) \equiv \\
&(A \rightarrow B)^{\mathrm{DN}}.
\end{aligned}$$

$\underline{\forall}$ Using induction hypothesis in the first equivalence and AC in the second equivalence, we get

$$\begin{aligned}
&\forall x\, A \leftrightarrow \\
&\forall x\, A^{\mathrm{DN}} \equiv \\
&\forall x\, \exists \underline{a}\, \forall \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b}) \leftrightarrow \\
&\exists \underline{A}\, \forall x, \underline{b}\, A_{\mathrm{DN}}(\underline{a}; \underline{b}) \equiv \\
&(\forall x\, A)^{\mathrm{DN}}.
\end{aligned}$$

2. Analogous to point 2 of proof 3.15.

6.14 Remark. The characterisation theorem of DN ensures that the soundness theorem of DN is optimal, in the sense that the theory $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP} + \Gamma$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^\omega + \Gamma \vdash A^{\mathrm{DN}}$ (analogously to remark 3.16). Analogously for $\mathsf{WE\text{-}HA}^\omega$.

6.15 Remark. Stein proved a characterisation theorem of DNq [67, *Satz* 4.8], but we prefer not to include it here in detail because of its use of non-standard principles and its indirect and complicated proof. Nevertheless, we briefly sketch Stein's characterisation theorem of DNq. Let $\exists$F-AC be AC restricted to $\exists$-free formulas, and let ac be the principle

$$\forall \underline{Y}\, \exists \underline{X}, z^0 \big(\forall x \leq_0 z\, A(\underline{X}z, \underline{Y}(\underline{X}z)) \rightarrow \exists \underline{Y}\, (\forall \underline{x}\, \exists \underline{y}\, A(\underline{x}, \underline{y}) \rightarrow \forall \underline{x}\, A(\underline{x}, \underline{Y}\underline{x}))\big).$$

Stein proved the following characterisation theorem for DNq: the theory $\mathsf{WE\text{-}HA}^\omega + \exists\mathsf{F\text{-}AC} + \mathsf{ac} + \exists\mathsf{F\text{-}IP}$ is the least theory, containing $\mathsf{WE\text{-}HA}^\omega$, that proves $A \leftrightarrow A^{\mathrm{DNq}}$ for all formulas A of $\mathsf{WE\text{-}HA}^\omega$. Stein's proof goes along these lines.

1. We prove $\mathsf{WE\text{-}HA}^\omega + \exists\mathsf{F\text{-}AC} + \mathsf{ac} + \exists\mathsf{F\text{-}IP} \vdash A^{\mathrm{DNq}} \to A$ by induction on the structure of A. So let us focus on the reciprocal implication.

2. The proof of the characterisation theorem of mr only uses AC for $\exists$-free formulas, so $\mathsf{WE\text{-}HA}^\omega + \exists\mathsf{F\text{-}AC} + \exists\mathsf{F\text{-}IP} \vdash A \leftrightarrow A^{\mathrm{mr}}$.

3. We can prove the following variant of the soundness theorem of DNq: if $\mathsf{WE\text{-}HA}^\omega + \exists\mathsf{F\text{-}AC} + \exists\mathsf{F\text{-}IP} \vdash A$, then $\mathsf{WE\text{-}HA}^\omega + \exists\mathsf{F\text{-}AC} + \mathsf{ac} + \exists\mathsf{F\text{-}IP} \vdash A^{\mathrm{DNq}}$.

4. Combining the previous two points, we get $\mathsf{WE\text{-}HA}^\omega + \exists\mathsf{F\text{-}AC} + \mathsf{ac} + \exists\mathsf{F\text{-}IP} \vdash (A^{\mathrm{mr}} \to A)^{\mathrm{DNq}}$. From $(A^{\mathrm{mr}} \to A)^{\mathrm{DNq}}$ we get $(A^{\mathrm{mr}})^{\mathrm{DNq}} \wedge A^{\mathrm{mr}} \to A^{\mathrm{DNq}}$, so $A^{\mathrm{mr}} \to A^{\mathrm{DNq}}$ (since we can prove $A^{\mathrm{mr}} \to (A^{\mathrm{mr}})^{\mathrm{DNq}}$), thus $A \to A^{\mathrm{DNq}}$ (by the characterisation theorem of mr).

From Stein's characterisation theorem of DNq and $\mathsf{WE\text{-}HA}^\omega \vdash A^{\mathrm{DNt}} \leftrightarrow A^{\mathrm{DNq}} \wedge A$ (by remark 6.5) we get an analogous characterisation theorem of DNt.

6.5 Applications

6.16 Theorem (disjunction property, existence property and program extraction)**.** Let $\mathsf{T} := \mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP}$.

1. Let $A \vee B$ be a sentence of T. If $\mathsf{T} \vdash A \vee B$, then $\mathsf{T} \vdash A$ or $\mathsf{T} \vdash B$.

2. If $\mathsf{T} \vdash \exists \underline{x}\, A(\underline{x})$, then we can extract from such a proof terms $\underline{t}$ of T such that $\mathsf{T} \vdash A(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A)$.

3. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A(\underline{x}, \underline{y})$, then we can extract from such a proof terms $\underline{t}(\underline{x})$ of T such that $\mathsf{T} \vdash \forall \underline{x}\, A(\underline{x}, \underline{t}(\underline{x}))$ and $\mathrm{FV}(\underline{t}(\underline{x})) = \mathrm{FV}(\exists \underline{y}\, A(\underline{x}, \underline{y}))$.

Analogously for $\mathsf{WE\text{-}HA}^\omega \pm \forall\text{-}\mathsf{IP}$ [41, theorem 6.3.1] [75, theorem 3.7.2] [50, corollary 5.24], $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$ and $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$ [75, theorem 3.7.5] [50, corollary 8.14 and theorem 8.15].

6.17 Proof. We do two slightly different proofs: one for $\mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP}$ and $\mathsf{WE\text{-}HA}^\omega \pm \forall\text{-}\mathsf{IP}$ using DNt and its truth, and another one for $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$ and $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$ using DN and its characterisation theorem.

$\underline{\mathsf{HA}^\omega \pm \forall\text{-}\mathsf{IP} \text{ and } \mathsf{WE\text{-}HA}^\omega \pm \forall\text{-}\mathsf{IP}}$ Analogous to proof 3.19.

$\underline{\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP} \text{ and } \mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}}$ Let $\mathsf{T} := \mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$; the case of the other theory is analogous.

1. We have $(A \vee B)_{\mathrm{DN}}(e^0, \underline{a}, \underline{c}; \underline{b}, \underline{d}) \equiv A_{\mathrm{DN}}(\underline{a}; \underline{b}) \vee_e B_{\mathrm{DN}}(\underline{c}; \underline{d})$. Assuming the premise of the theorem, by the soundness theorem of DN we can extract terms $\underline{r}, \underline{s}, t^0$ of T such that $\mathsf{T} \vdash \forall \underline{b}, \underline{d}\, (A_{\mathrm{DN}}(\underline{r}; \underline{b}) \vee_t B_{\mathrm{DN}}(\underline{s}; \underline{d}))$, so $\mathsf{T} \vdash \forall \underline{c}\, A_{\mathrm{DN}}(\underline{r}; \underline{b}) \vee_t \forall \underline{d}\, B_{\mathrm{DN}}(\underline{s}; \underline{d})$. By the characterisation theorem of DN we get $\mathsf{T} \vdash A \vee_t B$. By point 3 of theorem 1.30 we have $t \equiv \bar{n}$ for some $n \in \mathbb{N}$. If $n = 0$, then $\mathsf{T} \vdash A$; if $n \neq 0$, then $\mathsf{T} \vdash B$.

2. We have $(\exists \underline{x}\, A(\underline{x}))_{\mathrm{DN}}(\underline{x}, \underline{a}; \underline{b}) \equiv A(\underline{x})_{\mathrm{DN}}(\underline{a}; \underline{b})$. Assuming the premise of the theorem, by the soundness theorem of DN we can extract terms $\underline{s}, \underline{t}$ of T such that $\mathsf{T} \vdash \forall \underline{b}\, A(\underline{t})_{\mathrm{DN}}(\underline{s}; \underline{b})$ and $\mathrm{FV}(\underline{s}), \mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A(\underline{x}))$. By the characterisation theorem of DN we get $\mathsf{T} \vdash A(\underline{t})$.

3. Follows from the previous point.

6.18 Theorem (conservation and relative consistency)**.**

1. Let $\forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}(\underline{x}, \underline{y})$ be a sentence of HA^ω. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP} \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}$, then $\mathsf{HA}^\omega \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}$.

2. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP} \vdash \bot$, then $\mathsf{HA}^\omega \vdash \bot$.

Analogously for $\mathsf{WE}\text{-}\mathsf{HA}^\omega$ [50, corollary 8.12].

6.19 Proof. By theorem 1.44, without loss of generality we can assume that A_{qf} is atomic. Then we proceed as in proof 3.21.

6.20 Theorem (independence)**.** Let $\mathsf{T} := \mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$. We have $\mathsf{T} \nvdash \mathsf{LEM}$ and $\mathsf{T} \nvdash \neg\mathsf{LEM}$ (already for Σ^0_1 and Π^0_1 formulas). It follows the analogous statement for $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

6.21 Proof. Analogous to point 2 of proof 3.23.

6.6 Conclusion

6.22. We introduced the Diller-Nahm functional interpretation as a variant of Gödel's functional interpretation that deals with the contraction axiom $A \to A \wedge A$ by allowing finitely many As in the premise. The main results about the Diller-Nahm functional interpretation are the following.

Soundness theorem This theorem says that we can use the Diller-Nahm functional interpretation to extract computational content from proofs in $\mathsf{WE}\text{-}\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF}\text{-}\mathsf{MP}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used the Diller-Nahm functional interpretation to do applications on:

1. disjunction property;
2. existence property;
3. program extraction;
4. conservation;
5. relative consistency;
6. independence.

Chapter 7

Shoenfield functional interpretation

7.1 Introduction

7.1. Gödel's functional interpretation D interprets HA^ω. When composed with a negative translation N we get an interpretation $\mathrm{D} \circ \mathrm{N}$ of PA^ω. If we take $\mathrm{N} = \mathrm{Kr}$, then composition $\mathrm{D} \circ \mathrm{Kr}$ is a known interpretation of PA^ω called Shoenfield functional interpretation S. This is pictured in figure 7.1. In this chapter we introduce S.

$$\begin{tikzcd} \mathsf{PA}^\omega \arrow[r, "\mathrm{Kr}"] \arrow[rr, bend right, "\mathrm{S=D\circ Kr}"'] & \mathsf{HA}^\omega \arrow[r, "\mathrm{D}"] & \mathsf{HA}^\omega \end{tikzcd}$$

Figure 7.1: the composition $\mathrm{S} = \mathrm{D} \circ \mathrm{Kr}$.

7.2. There are no main contributions of our own to this topic. Almost all of the material here is known.

7.2 Definition

7.3. In this chapter we consider PA^ω and $\mathsf{WE\text{-}PA}^\omega$ based on $\neg$, $\vee$ and $\forall$, so $A \wedge B :\equiv \neg(\neg A \vee \neg B)$, $A \to B :\equiv \neg A \vee B$ and $\exists x\, A :\equiv \neg \forall x\, \neg A$.

7.4 Definition. The *Shoenfield functional interpretation* S [63, page 219] assigns to each formula A of PA^ω the formula $A^{\mathrm{S}} :\equiv \forall \underline{a}\, \exists \underline{b}\, A_{\mathrm{S}}(\underline{a}; \underline{b})$, where $A_{\mathrm{S}}(\underline{a}; \underline{b})$ is defined by recursion on the structure of A by

$$\begin{aligned} (A_{\mathrm{at}})_{\mathrm{S}}(;) &:\equiv A_{\mathrm{at}}, \\ (\neg A)_{\mathrm{S}}(\underline{B}; \underline{a}) &:\equiv \neg A_{\mathrm{S}}(\underline{a}; \underline{B}\underline{a}), \\ (A \vee B)_{\mathrm{S}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{S}}(\underline{a}; \underline{b}) \vee B_{\mathrm{S}}(\underline{c}; \underline{d}), \\ (\forall x\, A)_{\mathrm{S}}(x, \underline{a}; \underline{b}) &:\equiv A_{\mathrm{S}}(\underline{a}; \underline{b}). \end{aligned}$$

By $(A_{\mathrm{at}})_{\mathrm{S}}(;)$ we mean $(A_{\mathrm{at}})_{\mathrm{S}}(\underline{a}; \underline{b})$ with the tuples $\underline{a}$ and $\underline{b}$ empty. Analogously for $\mathsf{WE\text{-}PA}^\omega$.

7.5 Remark. The formulas $A_{\mathrm{S}}(\underline{a}; \underline{b})$ are quantifier-free.

7.6 Remark. The Shoenfield functional interpretation S acts as the identity on quantifier-free formulas of PA^ω in the sense of: $(A_{\mathrm{qf}})_{\mathrm{S}}(;) \equiv A_{\mathrm{qf}}$ for all quantifier-free formulas A_{qf} of PA^ω [63, page 219].

7.3 Factorisation

7.7 Theorem (factorisation $\mathrm{S} = \mathrm{D} \circ \mathrm{Kr}$)**.** For all formulas A of PA^ω we have:

1. $\mathsf{HA}^\omega \vdash A_{\mathrm{S}}(\underline{a}; \underline{Ba}) \leftrightarrow (A^{\mathrm{Kr}})_{\mathrm{D}}(\underline{B}; \underline{a})$ [69, theorem 3.1.2] [2, proposition 0.4];
2. $\mathsf{HA}^\omega + \mathsf{QF\text{-}AC} \vdash A^{\mathrm{S}} \leftrightarrow (A^{\mathrm{Kr}})^{\mathrm{D}}$.

Analogously for $\mathsf{WE\text{-}HA}^\omega$.

7.8 Proof.

1. (a) First we prove $(*)$ $\mathsf{HA}^\omega \vdash A_{\mathrm{S}}(\underline{a}; \underline{b}) \leftrightarrow \neg(A_{\mathrm{Kr}})_{\mathrm{D}}(\underline{a}; \underline{b})$ by induction on the structure of A. Let us only see the case of $\vee$; the cases of A_{at}, $\neg$ and $\forall$ are analogous. Using induction hypothesis in first equivalence, and point 1 of theorem 1.53 in the second equivalence, we get

$$\begin{aligned}(A \vee B)_{\mathrm{S}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &\equiv \\ A_{\mathrm{S}}(\underline{a}; \underline{b}) \vee B_{\mathrm{S}}(\underline{c}; \underline{d}) &\leftrightarrow \\ \neg(A_{\mathrm{Kr}})_{\mathrm{D}}(\underline{a}; \underline{b}) \vee \neg(B_{\mathrm{Kr}})_{\mathrm{D}}(\underline{c}; \underline{d}) &\leftrightarrow \\ \neg((A_{\mathrm{Kr}})_{\mathrm{D}}(\underline{a}; \underline{b}) \wedge (B_{\mathrm{Kr}})_{\mathrm{D}}(\underline{c}; \underline{d})) &\equiv \\ \neg((A \vee B)_{\mathrm{Kr}})_{\mathrm{D}}(\underline{a}, \underline{c}; \underline{b}, \underline{d})&.\end{aligned}$$

   (b) Now we prove $\mathsf{HA}^\omega \vdash A_{\mathrm{S}}(\underline{a}; \underline{Ba}) \leftrightarrow (A^{\mathrm{Kr}})_{\mathrm{D}}(\underline{B}; \underline{a})$. Using $(*)$ in the equivalence we get

$$\begin{aligned}A_{\mathrm{S}}(\underline{a}; \underline{Ba}) &\leftrightarrow \\ \neg(A_{\mathrm{Kr}})_{\mathrm{D}}(\underline{a}; \underline{Ba}) &\equiv \\ (\neg A_{\mathrm{Kr}})_{\mathrm{D}}(\underline{B}; \underline{a}) &\equiv \\ (A^{\mathrm{Kr}})_{\mathrm{D}}(\underline{B}; \underline{a})&.\end{aligned}$$

2. Using $\mathsf{QF\text{-}AC}$ in the first equivalence and point 1 in the second equivalence, we get

$$\begin{aligned}A^{\mathrm{S}} &\equiv \\ \forall \underline{a}\, \exists \underline{b}\, A_{\mathrm{S}}(\underline{a}; \underline{b}) &\leftrightarrow \\ \exists \underline{B}\, \forall \underline{a}\, A_{\mathrm{S}}(\underline{a}; \underline{Ba}) &\leftrightarrow \\ \exists \underline{B}\, \forall \underline{a}\, (A^{\mathrm{Kr}})_{\mathrm{D}}(\underline{B}; \underline{a}) &\equiv \\ (A^{\mathrm{Kr}})^{\mathrm{D}}&.\end{aligned}$$

7.4 Soundness

7.9 Theorem (soundness)**.** Let A be a formula of PA^ω and let Γ be a set of formulas of PA^ω of the form $\forall \underline{x}\, A_{\mathrm{qf}}$. If $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC} + \Gamma \vdash A$, then we can extract from such a proof terms $\underline{t}$ such that $\mathsf{HA}^\omega + \Gamma \vdash \forall \underline{a}\, A_{\mathrm{S}}(\underline{a}; \underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(A) \cup \{\underline{a}\}$. Analogously for $\mathsf{WE\text{-}HA}^\omega$ [63, page 220].

7.10 Proof. We can prove $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}}^{\mathrm{Kr}} \leftrightarrow A_{\mathrm{qf}}$ by induction on the structure of A_{qf} (using point 1 of theorem 1.53), thus $(\forall \underline{x}\, A_{\mathrm{qf}})^{\mathrm{Kr}} \equiv \neg \exists \underline{x}\, (A_{\mathrm{qf}})_{\mathrm{Kr}}$ is equivalent in HA^ω to $\forall \underline{x}\, \neg (A_{\mathrm{qf}})_{\mathrm{Kr}} \equiv \forall \underline{x}\, A_{\mathrm{qf}}^{\mathrm{Kr}}$, which is equivalent in HA^ω to $\forall \underline{x}\, A_{\mathrm{qf}}$. So $(*)$ $\mathsf{HA}^\omega + \Gamma^{\mathrm{Kr}} = \mathsf{HA}^\omega + \Gamma$.

If $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC} + \Gamma \vdash A$, then $\mathsf{HA}^\omega + \mathsf{QF\text{-}AC} + \Gamma^{\mathrm{Kr}} \vdash A^{\mathrm{Kr}}$ by the soundness theorem of Kr, that is $\mathsf{HA}^\omega + \mathsf{QF\text{-}AC} + \Gamma \vdash A^{\mathrm{Kr}}$ by $(*)$, so by the soundness theorem of D we can extract terms $\underline{t}'$ of HA^ω such that $\mathsf{HA}^\omega + \Gamma \vdash (A^{\mathrm{Kr}})_{\mathrm{D}}(\underline{t}'; \underline{a})$ and $\mathrm{FV}(\underline{t}') \subseteq \mathrm{FV}(A^{\mathrm{Kr}}) = \mathrm{FV}(A)$. By point 1 of the factorisation $\mathrm{S} = \mathrm{D} \circ \mathrm{Kr}$ we get $\mathsf{HA}^\omega + \Gamma \vdash A_{\mathrm{S}}(\underline{a}; \underline{t}'\underline{a})$. Take $\underline{t} :\equiv \underline{t}'\underline{a}$.

7.5 Characterisation

7.11 Theorem (characterisation)**.** Let us consider the theory $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC}$.

1. This theory proves $A \leftrightarrow A^{\mathrm{S}}$ for all formulas A of PA^ω [63, page 219].

2. This theory is the least theory, containing HA^ω, satisfying the previous point.

Analogously for $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC}$.

7.12 Proof.

1. The proof is by induction on the structure of A. Let us only see the case of $\neg$; the cases of A_{at}, $\vee$ and $\forall$ are analogous. Using induction hypothesis in the first equivalence and $\mathsf{QF\text{-}AC}$ in the second equivalence, we get

$$\begin{aligned} &\neg A \leftrightarrow \\ &\neg A^{\mathrm{S}} \equiv \\ &\neg \forall \underline{a}\, \exists \underline{b}\, A_{\mathrm{S}}(\underline{a}; \underline{b}) \leftrightarrow \\ &\neg \exists \underline{B}\, \forall \underline{a}\, A_{\mathrm{S}}(\underline{a}; \underline{B}\underline{a}) \leftrightarrow \\ &\forall \underline{B}\, \exists \underline{a}\, \neg A_{\mathrm{S}}(\underline{a}; \underline{B}\underline{a}) \equiv \\ &(\neg A)^{\mathrm{S}}. \end{aligned}$$

2. Analogous to point 2 of proof 3.15.

7.13 Remark. The characterisation theorem of S ensures that the soundness theorem of S is optimal, in the sense that the theory $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC} + \Gamma$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^\omega + \Gamma \vdash A^{\mathrm{S}}$ (analogously to remark 3.16). Analogously for $\mathsf{WE\text{-}HA}^\omega$.

7.6 Applications

7.14 Theorem (existence property for quantifier-free formulas and program extraction for quantifier-free formulas)**.** Let $\mathsf{T} := \mathsf{PA}^\omega + \mathsf{QF\text{-}AC}$.

1. If $\mathsf{T} \vdash \exists \underline{x}\, A_{\mathrm{qf}}(\underline{x})$, then we can extract from such a proof terms $\underline{t}$ of HA^ω such that $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}}(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A)$.

2. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}(\underline{x}, \underline{y})$, then we can extract from such a proof terms $\underline{t}(\underline{x})$ of T such that $\mathsf{HA}^\omega \vdash \forall \underline{x}\, A_{\mathrm{qf}}(\underline{x}, \underline{t}(\underline{x}))$ and $\mathrm{FV}(\underline{t}(\underline{x})) = \mathrm{FV}(\exists \underline{y}\, A(\underline{x}, \underline{y}))$.

Analogously for $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC}$ [3, theorem 3.2.2] [50, theorem 10.8].

7.15 Proof.

1. Say $\underline{x} \equiv x_1, \dots, x_n$. Recall $\exists \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \equiv \neg \forall x_1 \neg \cdots \neg \forall x_n \neg A_{\mathrm{qf}}(\underline{x})$ (in PA^ω based on $\neg$, $\vee$ and $\forall$). We have $\mathsf{HA}^\omega \vdash (\exists \underline{x}\, A_{\mathrm{qf}}(\underline{x}))_{\mathrm{S}}(; \underline{x}) \leftrightarrow A_{\mathrm{qf}}(\underline{x})$. Assuming the premise of the theorem, by the soundness theorem of S we can extract terms $\underline{t}$ of HA^ω such that $\mathsf{HA}^\omega \vdash A_{\mathrm{qf}}(\underline{t})$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists \underline{x}\, A(\underline{x}))$.

2. Follows from the previous point.

7.16 Theorem (conservation and relative consistency)**.**

1. If $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC} \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}$, then $\mathsf{HA}^\omega \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{qf}}$ [3, corollary 3.2.5].

2. If $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC} \vdash \bot$, then $\mathsf{HA}^\omega \vdash \bot$ [63, page 222].

Analogously for $\mathsf{WE\text{-}PA}^\omega$ [3, corollary 3.2.5].

7.17 Proof. Say $\underline{x} \equiv x_1, \dots, x_n$. Recall $\exists \underline{x}\, A_{\mathrm{qf}}(\underline{x}) \equiv \neg \forall x_1 \neg \cdots \neg \forall x_n \neg A_{\mathrm{qf}}(\underline{x})$ (in PA^ω based on $\neg$, $\vee$ and $\forall$). We have $\mathsf{HA}^\omega \vdash (\exists \underline{x}\, A_{\mathrm{qf}}(\underline{x}))_{\mathrm{S}}(; \underline{x}) \leftrightarrow A_{\mathrm{qf}}(\underline{x})$. Then we proceed as in proof 3.21.

7.7 Conclusion

7.18. We introduced the Shoenfield functional interpretation S and motivated it by the composition $\mathrm{S} = \mathrm{D} \circ \mathrm{Kr}$. The main results about the Shoenfield functional interpretation are the following.

Factorisation We proved $\mathrm{S} = \mathrm{D} \circ \mathrm{Kr}$.

Soundness theorem This theorem says that we can use the Shoenfield functional interpretation to extract computational content from proofs in $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used the Shoenfield functional interpretation to do applications on:

1. existence property for quantifier-free formulas;
2. program extraction for quantifier-free formulas;
3. conservation;
4. relative consistency.

Chapter 8

Monotone functional interpretation

8.1 Introduction

8.1. Monotone functional interpretation MD is a variant of Gödel's functional interpretation D that extracts bounds instead of exact witnesses. More precisely, we obtain A^{MD} from A^{D} in the following way:

$$\begin{aligned}
\forall \underline{\ell}\, A^{\mathrm{D}} &\equiv && (\mathrm{FV}(A) = \{\underline{\ell}\})\\
\forall \underline{\ell}\, \exists \underline{b}\, \forall \underline{c}\, A_{\mathrm{D}}(\underline{b}; \underline{c}) &\leftrightarrow && (\text{by } \mathsf{AC})\\
\exists \underline{B}\, \forall \underline{\ell}, \underline{c}\, A_{\mathrm{D}}(\underline{B}\underline{\ell}; \underline{c}) &\rightsquigarrow && (\text{bound } \underline{B} \text{ by } \underline{a})\\
\tilde{\exists} \underline{a}\, \exists \underline{B} \leq^{\mathrm{e}} \underline{a}\, \forall \underline{\ell}, \underline{c}\, A_{\mathrm{D}}(\underline{B}\underline{\ell}; \underline{c}) &\equiv: &&\\
A^{\mathrm{MD}}. &&&
\end{aligned}$$

It may seem that MD just weakens MD by asking for bounds instead of exact witnesses. However, there are advantages in this.

<u>More axioms</u> Gödel's functional interpretation D interprets $\mathsf{HA}^{\omega} + \Gamma$ where Γ is a set of axioms of the form $\forall \underline{x}\, A_{\mathrm{qf}}$. For MD we can enlarge Γ to include axioms of the form $\forall \underline{x}\, \exists \underline{y} \leq \underline{t}\underline{x}\, \forall \underline{z}\, A_{\mathrm{qf}}$ [50, theorem 9.1] (where the inequality is pointwise and the terms $\underline{t}$ are closed).

<u>Uniformity</u> If we proved a theorem $\forall \underline{x}^{00}\, \forall \underline{y} \leq \underline{t}\underline{x}\, \exists \underline{z}^{0(00)}\, A$ (where the terms $\underline{t}$ are closed), then MD extracts a bound on $\underline{z}$ that is uniform (that is does not depend) on $\underline{y}$ [50, theorem 9.3]. One application of this is that points in arbitrary compact metric spaces can be represented by ys bounded by a term, so the bound extracted by MD is uniform on the space.

<u>Simpler terms</u> Bounds can be simpler than exact witnesses. For example, the bound 1 on x in $\exists x\, (A_{\mathrm{qf}} \leftrightarrow x =_0 0)$ is simpler than the exact witness

$$\begin{cases} 0 & \text{if } \quad A_{\mathrm{qf}} \\ 1 & \text{if } \neg A_{\mathrm{qf}} \end{cases}.$$

8.2. There are no main contributions of our own to this topic. Almost all of the material here is known.

8.2 Definition

8.3 Definition. *Monotone functional interpretation* MD [46, page 231] [50, section 9.1] assigns to each formula A of HA^ω, with $\mathrm{FV}(A) = \{\underline{\ell}\}$, the formula $A^{\mathrm{MD}} :\equiv \tilde{\exists}\underline{a}\,\exists\underline{b} \leq^{\mathrm{e}} \underline{a}\,\forall\underline{\ell},\underline{c}\,A_{\mathrm{D}}(\underline{b}\underline{\ell};\underline{c})$. Analogously for $\mathsf{WE\text{-}HA}^\omega$.

8.3 Soundness

8.4 Theorem (soundness)**.** Let A be a formula of HA^ω with $\mathrm{FV}(A) = \{\underline{\ell}\}$, let Γ be a set of formulas of HA^ω of the form $\forall\underline{x}\,\exists\underline{y} \leq \underline{s}\underline{x}\,\forall\underline{z}\,A_{\mathrm{qf}}(\underline{y})$ where the terms $\underline{s}$ are closed, and Γ' be the set of the corresponding Skolem normal forms $\exists\underline{Y} \leq \underline{s}\,\forall\underline{x},\underline{z}\,A_{\mathrm{qf}}(\underline{Y}\underline{x})$. If $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP} + \Gamma \vdash A$, then we can extract from such a proof closed monotone terms $\underline{t}$ such that $\mathsf{HA}^\omega + \Gamma' \vdash \exists\underline{b} \leq^{\mathrm{e}} \underline{t}\,\forall\underline{\ell},\underline{c}\,A_{\mathrm{D}}(\underline{b}\underline{\ell};\underline{c})$. Analogously for $\mathsf{WE\text{-}HA}^\omega$ [50, theorem 9.1].

8.5 Proof. To prove the theorem with Γ, we would need to prove it by induction on the derivation of A. To avoid a long proof by induction, we prove the theorem without Γ in a simpler way. Assume the premise of the theorem. By the soundness theorem of D we can extract from such a proof terms $\underline{t}'$ such that $\mathsf{HA}^\omega \vdash \forall\underline{\ell},\underline{c}\,A_{\mathrm{D}}(\underline{t}';\underline{c})$ and $\mathrm{FV}(\underline{t}') \subseteq \{\underline{\ell}\}$. Letting $\underline{t}'' :\equiv \lambda\underline{\ell}\,.\,\underline{t}'$, we have $\mathsf{HA}^\omega \vdash \forall\underline{\ell},\underline{c}\,A_{\mathrm{D}}(\underline{t}''\underline{\ell};\underline{c})$ and $\underline{t}''$ are closed. Take $\underline{t} :\equiv \underline{t}''^{\mathrm{m}}$.

8.6. There seems to be no sound monotone functional interpretation with truth by the same reason explained in paragraph 5.9.

8.7. An (optimal) characterisation theorem of MD is unknown.

8.4 Applications

8.8 Theorem (monotone existence property and monotone program extraction)**.** Let $\mathsf{T} := \mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$.

1. Let $\mathrm{FV}(\exists\underline{x}\,A(\underline{x})) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \exists\underline{x}\,A(\underline{x})$, then we can extract from such a proof closed monotone terms $\underline{t}$ of T such that $\mathsf{T} \vdash \exists\underline{X} \leq^{\mathrm{e}} \underline{t}\,\forall\underline{\ell}\,A(\underline{X}\underline{\ell})$.

2. Let $\mathrm{FV}(\forall\underline{x}\,\exists\underline{y}\,A(\underline{x},\underline{y})) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \forall\underline{x}\,\exists\underline{y}\,A(\underline{x},\underline{y})$, then we can extract from such a proof closed monotone terms $\underline{t}$ of T such that $\mathsf{T} \vdash \exists\underline{Y} \leq^{\mathrm{e}} \underline{t}\,\forall\underline{\ell},\underline{x}\,A(\underline{x},\underline{Y}\underline{\ell}\underline{x})$.

Analogously for $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$ [50, theorem 8.15 and remark 9.5].

8.9 Proof.

1. We have $(\exists\underline{x}\,A(\underline{x}))^{\mathrm{MD}} \equiv \tilde{\exists}\underline{a},\underline{b}\,\exists\underline{X},\underline{c} \leq^{\mathrm{e}} \underline{a},\underline{b}\,\forall\underline{\ell},\underline{d}\,A(\underline{X}\underline{\ell})_{\mathrm{D}}(\underline{c}\underline{\ell};\underline{d})$. Assuming the premise of the theorem, by the soundness theorem of MD we can extract closed monotone terms $\underline{s},\underline{t}$ of T such that $\mathsf{T} \vdash \exists\underline{X},\underline{c} \leq^{\mathrm{e}} \underline{t},\underline{s}\,\forall\underline{\ell},\underline{d}\,A(\underline{X}\underline{\ell})_{\mathrm{D}}(\underline{c}\underline{\ell};\underline{d})$ and $\mathrm{FV}(\underline{s}), \mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\exists\underline{x}\,A(\underline{x}))$. By the characterisation theorem of D we get $\mathsf{T} \vdash \exists\underline{X} \leq^{\mathrm{e}} \underline{t}\,\forall\underline{\ell}\,A(\underline{X}\underline{\ell})$.

2. Follows from the previous point.

8.10. We can prove the following: if $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP} \vdash \forall \underline{x}^{00} \forall \underline{y} \leq_{\underline{\rho}} \underline{s}\underline{x}\, \exists \underline{z}^{0(00)} A$ (where the formula is a sentence, the terms $\underline{s}$ are closed and $\leq_\rho$ is pointwise inequality), then we can extract from such a proof monotone terms $\underline{t}(\underline{x})$ such that $\mathsf{HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP} \vdash \forall \underline{x}\, \forall \underline{y} \leq \underline{s}\underline{x}\, \exists \underline{z} \leq \underline{t}(\underline{x})\, A$ and $\mathrm{FV}(\underline{t}(\underline{x})) = \{\underline{x}\}$ [50, theorem 9.3]. The interest of this result it that the terms $\underline{t}(\underline{x})$ are uniform (that is do not depend) on $\underline{y}$. However, this theorem on program extraction does not compare nicely (because of the restriction on the types of $\underline{x}$ and $\underline{z}$) with our other theorems on program extraction, so we prefer to present point 2 of theorem 8.8.

8.5 Conclusion

8.11. We introduced monotone functional interpretation as a variant of Gödel's functional interpretation that extracts bounds instead of exact witnesses (allowing for more axioms, uniformity and simpler terms). The main results about monotone functional interpretation are the following.

Soundness theorem This theorem says that we can use monotone functional interpretation to extract computational content from proofs in $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \forall\text{-}\mathsf{IP} + \mathsf{QF\text{-}MP}$.

Applications We used monotone functional interpretation to do applications on:

1. monotone existence property;
2. monotone program extraction.

Chapter 9

Bounded functional interpretation

9.1 Introduction

9.1. Gödel's functional interpretation D extracts exact witnesses for existential statements: given a theorem $\exists x\, A(x)$, extracts a term t such that $A(t)$. Monotone functional interpretation MD extracts bounds instead of exact witnesses: given a theorem $\exists x\, A(x)$, extracts a term t such that (essentially) $\exists x \leq^{\mathrm{e}} t\, A(x)$. The way MD does this is by starting by targeting an exact witness and then in a last step changing the target to a bound. Now we introduce the bounded functional interpretation that also targets bounds, not in a last step but systematically from the very beginning. The price to pay for this is that B will not work in the setting $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{QF\text{-}MP} + \forall\text{-}\mathsf{IP}$ but in the more exotic setting $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD}$.

This change from exact witnesses to bounds is mainly obtained by changing the clause of $\exists x$ from asking for x to asking for a bound c on x (as done for br):

$$(\exists x\, A)_{\mathrm{D}}(x, \underline{a}; \underline{b}) :\equiv A_{\mathrm{D}}(\underline{a}; \underline{b}),$$
$$(\exists x\, A)_{\mathrm{B}}(c, \underline{a}; \underline{d}) :\equiv \exists x \leq^{\mathrm{i}} c\, \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}).$$

(Note the parallelism between the pair mr, br and the pair D, B.) We also introduce two variants with truth of B: the bounded functional interpretation with q-truth Bt and the bounded functional interpretation with t-truth Bt.

9.2. Our main contributions to this topic are the following.

1. The bounded functional interpretations with q-truth Bt and with t-truth Bt and their soundness theorems [22, section 7] (definition 9.3 and theorem 9.10).

2. The bounded existence property and the bounded program extraction (theorem 9.17).

9.2 Definition

9.3 Definition.

1. The *bounded functional interpretation* B [15, definition 4] assigns to each formula A of $\mathsf{HA}^\omega_{\mathrm{i}}$ the formula $A^{\mathrm{B}} :\equiv \tilde{\exists} \underline{a}\, \tilde{\forall} \underline{b}\, A_{\mathrm{B}}(\underline{a}; \underline{b})$, where $A_{\mathrm{B}}(\underline{a}; \underline{b})$ is defined by

recursion on the structure of A by

$$
\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{B}}(;) &:\equiv A_{\mathrm{at}},\\
(A \wedge B)_{\mathrm{B}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge B_{\mathrm{B}}(\underline{c}; \underline{d}),\\
(A \vee B)_{\mathrm{B}}(\underline{a}, \underline{c}; \underline{e}, \underline{f}) &:\equiv \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \vee \tilde{\forall} \underline{d} \leq^{\mathrm{i}} \underline{f}\, B_{\mathrm{B}}(\underline{c}; \underline{d}),\\
(A \to B)_{\mathrm{B}}(\underline{C}, \underline{e}; \underline{a}, \underline{d}) &:\equiv \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{e}\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \to B_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d}),\\
(\forall x \leq^{\mathrm{i}} t\, A)_{\mathrm{B}}(\underline{a}; \underline{b}) &:\equiv \forall x \leq^{\mathrm{i}} t\, A_{\mathrm{B}}(\underline{a}; \underline{b}),\\
(\exists x \leq^{\mathrm{i}} t\, A)_{\mathrm{B}}(\underline{a}; \underline{c}) &:\equiv \exists x \leq^{\mathrm{i}} t\, \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{c}\, A_{\mathrm{B}}(\underline{a}; \underline{b}),\\
(\forall x\, A)_{\mathrm{B}}(\underline{A}; c, \underline{b}) &:\equiv \forall x \leq^{\mathrm{i}} c\, A_{\mathrm{B}}(\underline{A}c; \underline{b}),\\
(\exists x\, A)_{\mathrm{B}}(c, \underline{a}; \underline{d}) &:\equiv \exists x \leq^{\mathrm{i}} c\, \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}).
\end{aligned}
$$

By $(A_{\mathrm{at}})_{\mathrm{B}}(;)$ we mean $(A_{\mathrm{at}})_{\mathrm{B}}(\underline{a}; \underline{b})$ with the tuples $\underline{a}$ and $\underline{b}$ empty.

2. The *bounded functional interpretation with q-truth* Bq [22, definition 7.1] is defined analogously to B except for

$$
\begin{aligned}
(A \vee B)_{\mathrm{Bq}}(\underline{a}, \underline{c}; \underline{e}, \underline{f}) &:\equiv (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge A) \vee (\tilde{\forall} \underline{d} \leq^{\mathrm{i}} \underline{f}\, B_{\mathrm{Bq}}(\underline{c}; \underline{d}) \wedge B),\\
(A \to B)_{\mathrm{Bq}}(\underline{C}, \underline{e}; \underline{a}, \underline{d}) &:\equiv \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{e}\underline{a}\underline{d}\, A_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge A \to B_{\mathrm{Bq}}(\underline{C}\underline{a}; \underline{d}),\\
(\exists x \leq^{\mathrm{i}} t\, A)_{\mathrm{Bq}}(\underline{a}; \underline{c}) &:\equiv \exists x \leq^{\mathrm{i}} t\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{c}\, A_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge A),\\
(\exists x\, A)_{\mathrm{Bq}}(c, \underline{a}; \underline{d}) &:\equiv \exists x \leq^{\mathrm{i}} c\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{d}\, A_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge A).
\end{aligned}
$$

3. The *bounded functional interpretation with t-truth* Bt [22, definition 7.3] is defined analogously to B except for

$$
\begin{aligned}
(A \to B)_{\mathrm{Bt}}(\underline{C}, \underline{e}; \underline{a}, \underline{d}) &:\equiv (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{e}\underline{a}\underline{d}\, A_{\mathrm{Bt}}(\underline{a}; \underline{b}) \to B_{\mathrm{Bt}}(\underline{C}\underline{a}; \underline{d})) \wedge (A \to B),\\
(\forall x\, A)_{\mathrm{Bt}}(\underline{A}; c, \underline{b}) &:\equiv \forall x \leq^{\mathrm{i}} c\, A_{\mathrm{Bt}}(\underline{A}c; \underline{b}) \wedge \forall x\, A.
\end{aligned}
$$

9.4. Let us note that, contrarily to what is done for mrt, in Bt we added "$\wedge\, \forall x\, A$" in the clause of $\forall$; this will be discussed later in chapter 13.

9.5 Remark.

1. The bounded functional interpretation with q-truth Bq has truth in the sense of: $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash A_{\mathrm{Bq}}(\underline{a}; \underline{b}) \to A$ for all disjunctive and (bounded and unbounded) existential formulas A of $\mathsf{HA}^{\omega}_{\mathrm{i}}$ [22, theorem 7.2].

2. The bounded functional interpretation with t-truth Bt has truth in the sense of: $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash A_{\mathrm{Bt}}(\underline{a}; \underline{b}) \to A$ for all formulas A of $\mathsf{HA}^{\omega}_{\mathrm{i}}$ [22, remark 7.4].

The bounded functional interpretation with t-truth Bt is a $(*_1)$ strengthening of Bq which $(*_2)$ has truth for all formulas. This can be given a rigorous meaning: $(*_3)$ $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash \tilde{\forall} \underline{a}, \underline{b}\, (A_{\mathrm{Bt}}(\underline{a}; \underline{b}) \leftrightarrow A_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge A)$ for all formulas A of $\mathsf{HA}^{\omega}_{\mathrm{i}}$ [22, proposition 7.6]. From $(*_3)$ we get: $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash \tilde{\forall} \underline{a}, \underline{b}\, (A_{\mathrm{Bt}}(\underline{a}; \underline{b}) \to A_{\mathrm{Bq}}(\underline{a}; \underline{b}))$, that is $(*_1)$ (restricted to monotone $\underline{a}, \underline{b}$); $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash \tilde{\forall} \underline{a}, \underline{b}\, (A_{\mathrm{Bt}}(\underline{a}; \underline{b}) \to A)$, that is $(*_2)$ (restricted to monotone $\underline{a}, \underline{b}$).

9.6 Remark. The formulas $A_\mathrm{B}(\underline{a};\underline{b})$ are bounded.

9.7 Remark.

1. The bounded functional interpretation B acts as the identity on bounded formulas of $\mathsf{HA}^\omega_\mathrm{i}$ in the sense of: $(A_\mathrm{b})_\mathrm{B}(;) \equiv A_\mathrm{b}$ for all bounded formulas A_b of $\mathsf{HA}^\omega_\mathrm{i}$ [15, definition 4].

2. The bounded functional interpretation with q-truth Bq acts as the identity on bounded formulas of $\mathsf{HA}^\omega_\mathrm{i}$ in the sense of: $\mathsf{HA}^\omega_\mathrm{e} \vdash (A_\mathrm{b})_\mathrm{Bq}(;) \leftrightarrow A_\mathrm{b}$ for all bounded formulas A_b of $\mathsf{HA}^\omega_\mathrm{i}$. Analogously for Bt.

9.3 Soundness

9.8 Lemma (monotonicity)**.** We have $\mathsf{HA}^\omega_\mathrm{i} \vdash \tilde{\forall}\underline{a}'\,\forall\underline{a} \leq^\mathrm{i} \underline{a}'\,\tilde{\forall}b\,(A_\mathrm{B}(\underline{a};\underline{b}) \to A_\mathrm{B}(\underline{a}';\underline{b}))$ [15, lemma 6]. Analogously for Bq [22, proof of theorem 7.5] and Bt.

9.9 Proof. Analogous to proof 4.9.

9.10 Theorem (soundness)**.** Let A be a formula of $\mathsf{HA}^\omega_\mathrm{i}$ with $\mathrm{FV}(A) = \{\underline{\ell}\}$, and let Γ be a set of formulas of $\mathsf{HA}^\omega_\mathrm{i}$ of the form $\forall\underline{x}\,\exists\underline{y} \leq^\mathrm{i} \underline{s}\,\forall\underline{z}\,A_\mathrm{b}$ where $\underline{s}$ are terms of $\mathsf{HA}^\omega_\mathrm{i}$.

1. If $\mathsf{HA}^\omega_\mathrm{i} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}^\omega_\mathrm{i} + \Gamma \vdash \tilde{\forall}\underline{\ell}'\,\forall\underline{\ell} \leq^\mathrm{i} \underline{\ell}'\,\tilde{\forall}\underline{b}\,A_\mathrm{B}(\underline{t}(\underline{\ell}');\underline{b})$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(A)$ [15, theorem 4].

2. If $\mathsf{HA}^\omega_\mathrm{i} \pm \mathsf{B\text{-}BCC} \pm \forall\text{-}\mathsf{BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BUD} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}^\omega_\mathrm{i} \pm \mathsf{B\text{-}BCC} \pm \forall\text{-}\mathsf{BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BUD} + \Gamma \vdash \tilde{\forall}\underline{\ell}'\,\forall\underline{\ell} \leq^\mathrm{i} \underline{\ell}'\,\tilde{\forall}\underline{b}\,A_\mathrm{Bq}(\underline{t}(\underline{\ell}');\underline{b})$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(A)$ [22, theorem 7.5].

3. If $\mathsf{HA}^\omega_\mathrm{i} \pm \mathsf{B\text{-}BCC} \pm \forall\text{-}\mathsf{BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BUD} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}^\omega_\mathrm{i} \pm \mathsf{B\text{-}BCC} \pm \forall\text{-}\mathsf{BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BUD} + \Gamma \vdash \tilde{\forall}\underline{\ell}'\,\forall\underline{\ell} \leq^\mathrm{i} \underline{\ell}'\,\tilde{\forall}\underline{b}\,A_\mathrm{Bt}(\underline{t}(\underline{\ell}');\underline{b})$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(A)$ [22, theorem 7.7].

The terms constructed in the following proof for the three points above are the same.

9.11 Proof. Let us make some remark.

1. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth, with q-truth and with t-truth, by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12.

2. We have

$$\begin{aligned}
&\mathsf{HA}^\omega_\mathrm{i} \vdash (\forall\underline{x}\,A)_\mathrm{B}(\underline{A};\underline{c},\underline{b}) \leftrightarrow \forall\underline{x} \leq^\mathrm{i} \underline{c}\,A_\mathrm{B}(\underline{Ac};\underline{b}) \wedge (\forall\underline{x}\,A)^\mathrm{t},\\
&\mathsf{HA}^\omega_\mathrm{i} \vdash (\tilde{\forall}\underline{x}\,A)_\mathrm{B}(\underline{A};\underline{c},\underline{b}) \leftrightarrow \tilde{\forall}\underline{x} \leq^\mathrm{i} \underline{c}\,A_\mathrm{B}(\underline{Ac};\underline{b}) \wedge (\tilde{\forall}\underline{x}\,A)^\mathrm{t},\\
&\mathsf{HA}^\omega_\mathrm{i} \vdash \tilde{\forall}\underline{c}\,\big((\exists\underline{x} \leq^\mathrm{i} \underline{t}\,A)_\mathrm{B}(\underline{a};\underline{c}) \leftrightarrow \exists\underline{x} \leq^\mathrm{i} \underline{t}\,(\tilde{\forall}\underline{b} \leq^\mathrm{i} \underline{c}\,A_\mathrm{B}(\underline{a};\underline{b}) \wedge A^\mathrm{q})\big),\\
&\mathsf{HA}^\omega_\mathrm{i} \vdash \tilde{\forall}\underline{d}\,\big((\exists\underline{x}\,A)_\mathrm{B}(\underline{c},\underline{a};\underline{d}) \leftrightarrow \exists\underline{x} \leq^\mathrm{i} \underline{c}\,(\tilde{\forall}\underline{b} \leq^\mathrm{i} \underline{d}\,A_\mathrm{B}(\underline{a};\underline{b}) \wedge A^\mathrm{q})\big),\\
&\mathsf{HA}^\omega_\mathrm{i} \vdash \tilde{\forall}\underline{d}\,\big((\tilde{\exists}\underline{x}\,A)_\mathrm{B}(\underline{c},\underline{a};\underline{d}) \leftrightarrow \tilde{\exists}\underline{x} \leq^\mathrm{i} \underline{c}\,(\tilde{\forall}\underline{b} \leq^\mathrm{i} \underline{d}\,A_\mathrm{B}(\underline{a};\underline{b}) \wedge A^\mathrm{q})\big).
\end{aligned}$$

We will replace the left sides of the equivalences by the right sides. When we do it, we use "$\equiv$" instead of $\equiv$.

Let us prove the theorem by induction on the derivation of A.

$\underline{A \to A \wedge A}$ We have

$$(A \to A \wedge A)_{\mathrm{B}}(\underline{C}, \underline{E}, \underline{g}; \underline{a}, \underline{d}, \underline{f}) \text{ “}\equiv\text{” } \\ \tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{g}\underline{a}\underline{d}\underline{f}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to A_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d}) \wedge A_{\mathrm{B}}(\underline{E}\underline{a}; \underline{f}),$$

$$\underline{t_C} :\equiv \lambda\underline{a} \,.\, \underline{a}, \quad \underline{t_E} :\equiv \lambda\underline{a} \,.\, \underline{a}, \quad \underline{t_g} :\equiv \lambda\underline{a}, \underline{d}, \underline{f} \,.\, \max(\underline{d}, \underline{f}).$$

Here we use point 6b of proposition 1.56.

$\underline{A \to A \vee B}$ We have

$$(A \to A \vee B)_{\mathrm{B}}(\underline{C}, \underline{E}, \underline{i}; \underline{a}, \underline{g}, \underline{h}) \text{ “}\equiv\text{” } \tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{i}\underline{a}\underline{g}\underline{h}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \\ (\tilde{\forall}\underline{d} \leq^{\mathrm{i}} \underline{g}\, A_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d}) \wedge A^{\mathrm{q}}) \vee (\tilde{\forall}\underline{f} \leq^{\mathrm{i}} \underline{h}\, B_{\mathrm{B}}(\underline{E}\underline{a}; \underline{f}) \wedge B^{\mathrm{q}}),$$

$$\underline{t_C} :\equiv \lambda\underline{a} \,.\, \underline{a}, \quad \underline{t_E} :\equiv \lambda\underline{a} \,.\, \underline{\mathcal{O}}, \quad \underline{t_i} :\equiv \lambda\underline{a}, \underline{g}, \underline{h} \,.\, \underline{g}.$$

To see that the terms $\underline{t_E}$ are monotone, we prove $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash \mathcal{O} \leq^{\mathrm{i}}_{\rho} \mathcal{O}$ by induction on the structure of ρ. Analogously for $A \vee A \to A$, $A \wedge B \to A$, $A \wedge B \to B \wedge A$, $A \vee B \to B \vee A$ and $\bot \to A$.

$\underline{\forall x\, A \to A[t/x]}$ We have

$$(\forall x\, A \to A[t/x])_{\mathrm{B}}(\underline{C}, f, \underline{g}; \underline{A}, \underline{d}) \text{ “}\equiv\text{” } \\ \tilde{\forall}e, \underline{b} \leq^{\mathrm{i}} f\underline{A}\underline{d}, \underline{g}\underline{A}\underline{d}\, (\forall x \leq^{\mathrm{i}} e\, A_{\mathrm{B}}(\underline{A}e; \underline{b}) \wedge (\forall x\, A)^{\mathrm{t}}) \wedge (\forall x\, A)^{\mathrm{q}} \to \\ A[t/x]_{\mathrm{B}}(\underline{C}\underline{A}; \underline{d}),$$

$$\underline{t_C}(\underline{\ell}) :\equiv \lambda\underline{A} \,.\, \underline{A}t^{\mathrm{m}}(\underline{\ell}), \quad t_f(\underline{\ell}) :\equiv \lambda\underline{A}, \underline{d} \,.\, t^{\mathrm{m}}(\underline{\ell}), \quad \underline{t_g} :\equiv \lambda\underline{A}, \underline{d} \,.\, \underline{d}.$$

Let us see that the terms work, that is

$$\tilde{\forall}\underline{\ell}'\, \forall\underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \tilde{\forall}\underline{A}, \underline{d}\, \big(\tilde{\forall}e, \underline{b} \leq^{\mathrm{i}} t^{\mathrm{m}}(\underline{\ell}'), \underline{d}\, (\forall x \leq^{\mathrm{i}} e\, A_{\mathrm{B}}(\underline{A}e; \underline{b}) \wedge (\forall x\, A)^{\mathrm{t}}) \wedge \\ (\forall x\, A)^{\mathrm{q}} \to A[t/x]_{\mathrm{B}}(\underline{A}t^{\mathrm{m}}(\underline{\ell}'); \underline{d})\big).$$

Taking $\underline{b} = \underline{d}$, $e = t^{\mathrm{m}}(\underline{\ell}')$ (which satisfies $e \leq^{\mathrm{i}} e$) and $x = t(\underline{\ell})$ (which satisfies $x \leq^{\mathrm{i}} e$) in the premise we get $A_{\mathrm{B}}(\underline{A}t^{\mathrm{m}}(\underline{\ell}'); \underline{d})[t/x]$. Analogously for $A[t/x] \to \exists x\, A$.

$\underline{A \to B,\, B \to C \,/\, A \to C}$ We have

$$(A \to B)_{\mathrm{B}}(\underline{C}, \underline{g}; \underline{a}, \underline{d}) \text{ “}\equiv\text{” } \tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{g}\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to B_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d}),$$
$$(B \to C)_{\mathrm{B}}(\underline{E}, \underline{g}; \underline{c}, \underline{f}) \text{ “}\equiv\text{” } \tilde{\forall}\underline{d} \leq^{\mathrm{i}} \underline{g}\underline{c}\underline{f}\, B_{\mathrm{B}}(\underline{c}; \underline{d}) \wedge B^{\mathrm{q}} \to C_{\mathrm{B}}(\underline{E}\underline{c}; \underline{f}),$$
$$(A \to C)_{\mathrm{B}}(\underline{E}, \underline{g}; \underline{a}, \underline{f}) \text{ “}\equiv\text{” } \tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{g}\underline{a}\underline{f}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to C_{\mathrm{B}}(\underline{E}\underline{a}; \underline{f}),$$
$$\underline{t_E} :\equiv \lambda\underline{a} \,.\, \underline{s_E}(\underline{r_C}\underline{a}), \quad \underline{t_g} :\equiv \lambda\underline{a}, \underline{f} \,.\, \underline{r_g}(\underline{s_g}(\underline{r_C}\underline{a}))\underline{f}.$$

Let us see that the terms work. By induction hypothesis (9.1) and (9.2), and we want to prove (9.3):

$$\tilde\forall \underline{\ell}' \, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}' \, \tilde\forall \underline{a}, \underline{d} \, (\tilde\forall \underline{b} \leq^{\mathrm{i}} \underline{r_g} \underline{a} \underline{d} \, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to B_{\mathrm{B}}(\underline{r_C} \underline{a}; \underline{d})), \tag{9.1}$$

$$\tilde\forall \underline{\ell}' \, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}' \, \tilde\forall \underline{c}, \underline{f} \, (\tilde\forall \underline{d} \leq^{\mathrm{i}} \underline{s_g} \underline{c} \underline{f} \, B_{\mathrm{B}}(\underline{c}; \underline{d}) \wedge B^{\mathrm{q}} \to C_{\mathrm{B}}(\underline{s_E} \underline{c}; \underline{f})). \tag{9.2}$$

$$\tilde\forall \underline{\ell}' \, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}' \, \tilde\forall \underline{a}, \underline{f} \, \big(\tilde\forall \underline{b} \leq^{\mathrm{i}} \underline{r_g} (\underline{s_g} (\underline{r_C} \underline{a})) \underline{f} \, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to C_{\mathrm{B}}(\underline{s_E} (\underline{r_C} \underline{a}); \underline{f})\big). \tag{9.3}$$

Taking arbitrary monotone $\underline{d} \leq^{\mathrm{i}} \underline{s_g}(\underline{r_C}\underline{a})\underline{f}$ in (9.1) we get (9.4), from which we get (9.5); taking $\underline{c} = \underline{r_C}\underline{a}$ in (9.2) we get (9.6):

$$\begin{gathered} \tilde\forall \underline{\ell}' \, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}' \, \tilde\forall \underline{a} \, \tilde\forall \underline{d} \leq^{\mathrm{i}} \underline{s_g}(\underline{r_C}\underline{a})\underline{f} \, (\tilde\forall \underline{b} \leq^{\mathrm{i}} \underline{r_g} \underline{a} \underline{d} \, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \\ B_{\mathrm{B}}(\underline{r_C} \underline{a}; \underline{d})), \end{gathered} \tag{9.4}$$

$$\begin{gathered} \tilde\forall \underline{\ell}' \, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}' \, \tilde\forall \underline{a}, \underline{f} \, \big(\tilde\forall \underline{b} \leq^{\mathrm{i}} \underline{r_g} \underline{a} (\underline{s_g}(\underline{r_C}\underline{a})\underline{f}) \, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \\ \tilde\forall \underline{d} \leq^{\mathrm{i}} \underline{s_g}(\underline{r_C}\underline{a})\underline{f} \, B_{\mathrm{B}}(\underline{r_C}\underline{a}; \underline{d})\big), \end{gathered} \tag{9.5}$$

$$\begin{gathered} \tilde\forall \underline{\ell}' \, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}' \, \tilde\forall \underline{a}, \underline{f} \, (\tilde\forall \underline{d} \leq^{\mathrm{i}} \underline{s_g}(\underline{r_C}\underline{a})\underline{f} \, B_{\mathrm{B}}(\underline{r_C}\underline{a}; \underline{d}) \wedge B^{\mathrm{q}} \to \\ C_{\mathrm{B}}(\underline{s_E}(\underline{r_C}\underline{a}); \underline{f})). \end{gathered} \tag{9.6}$$

From (9.5) and (9.6) we get (9.3). If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved $A \to B$, so that part A^{q} in (9.3) implies the part B^{q} in (9.6). Analogously for A, $A \to B \,/\, B$ and $A \to B \,/\, C \vee A \to C \vee B$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ We have

$$\begin{gathered} (A \wedge B \to C)_{\mathrm{B}}(\underline{E}, \underline{g}, \underline{h}; \underline{a}, \underline{c}, \underline{f}) \equiv \\ \big(\tilde\forall \underline{b}, \underline{d} \leq^{\mathrm{i}} \underline{g}\underline{a}\underline{c}\underline{f}, \underline{h}\underline{a}\underline{c}\underline{f} \, (A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge B_{\mathrm{B}}(\underline{c}; \underline{d})) \wedge (A \wedge B)^{\mathrm{q}} \to \\ C_{\mathrm{B}}(\underline{E}\underline{a}\underline{c}; \underline{f})\big) \wedge (A \wedge B \to C)^{\mathrm{t}}, \end{gathered} \tag{9.7}$$

$$\begin{gathered} (A \to (B \to C))_{\mathrm{B}}(\underline{\mathrm{E}}, \underline{H}, \underline{g}; \underline{a}, \underline{c}, \underline{f}) \equiv \\ \big(\tilde\forall \underline{b} \leq^{\mathrm{i}} \underline{g}\underline{a}\underline{c}\underline{f} \, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \\ \big((\tilde\forall \underline{d} \leq^{\mathrm{i}} \underline{H}\underline{a}\underline{c}\underline{f} \, B_{\mathrm{B}}(\underline{c}; \underline{d}) \wedge B^{\mathrm{q}} \to C_{\mathrm{B}}(\underline{\mathrm{E}}\underline{a}\underline{c}; \underline{f})) \wedge (B \to C)^{\mathrm{t}}\big)\big) \wedge \\ (A \to (B \to C))^{\mathrm{t}}, \end{gathered} \tag{9.8}$$

$$\underline{t_{\mathrm{E}}} :\equiv \underline{s_E}, \quad \underline{t_H} :\equiv \underline{s_h}, \quad \underline{t_g} :\equiv \underline{s_g}.$$

If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{B}}(\underline{a}; \underline{b}) \to A$, so that the parts $(A \wedge B \to C)^{\mathrm{t}}$ in (9.7) and $A_{\mathrm{B}}(\underline{a}; \underline{b})$ in (9.8) together imply the part $(B \to C)^{\mathrm{t}}$ in (9.8). Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, A \to \forall x\, B}$ We have

$$(A \to B)_{\mathrm{B}}(\underline{C}, \underline{e}; \underline{a}, \underline{d}) \equiv$$
$$(\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{e}\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to B_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d})) \wedge (A \to B)^{\mathrm{t}},$$
$$(A \to \forall x\, B)_{\mathrm{B}}(\underline{\mathrm{C}}, \underline{e}; \underline{a}, f, \underline{d}) \equiv$$
$$(\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{e}\underline{a}f\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \forall x \leq^{\mathrm{i}} f\, B_{\mathrm{B}}(\underline{\mathrm{C}}\underline{a}f; \underline{d}) \wedge (\forall x\, B)^{\mathrm{t}}) \wedge$$
$$(A \to \forall x\, B)^{\mathrm{t}},$$
$$\underline{t}_{\underline{\mathrm{C}}}(\underline{\ell}) :\equiv \lambda \underline{a}, f \,.\, \underline{s}_{\underline{C}}(\underline{\ell}, f)\underline{a}, \quad \underline{t}_{\underline{e}}(\underline{\ell}) :\equiv \lambda \underline{a}, f, \underline{d} \,.\, \underline{s}_{\underline{e}}(\underline{\ell}, f)\underline{a}\underline{d}.$$

Let us see that the terms work. By induction hypothesis we have (9.9) and we want to prove (9.10):

$$\tilde{\forall}\underline{\ell}', f\, \forall \underline{\ell}, x \leq^{\mathrm{i}} \underline{\ell}', f\, \tilde{\forall}\underline{a}, \underline{d}\, \big((\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{s}_{\underline{e}}(\underline{\ell}', f)\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to B_{\mathrm{B}}(\underline{s}_{\underline{C}}(\underline{\ell}', f)\underline{a}; \underline{d})\big) \wedge (A \to B)^{\mathrm{t}}\big), \tag{9.9}$$

$$\tilde{\forall}\underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \tilde{\forall}\underline{a}, f, \underline{d}\, \big((\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{s}_{\underline{e}}(\underline{\ell}', f)\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \forall x \leq^{\mathrm{i}} f\, B_{\mathrm{B}}(\underline{s}_{\underline{C}}(\underline{\ell}', f)\underline{a}; \underline{d}) \wedge (\forall x\, B)^{\mathrm{t}}\big) \wedge (A \to \forall x\, B)^{\mathrm{t}}\big) \tag{9.10}$$

(if $x \notin \mathrm{FV}(A \to B)$, then in (9.9) where is $\tilde{\forall}\underline{\ell}', f\, \forall \underline{\ell}, x \leq^{\mathrm{i}} \underline{\ell}', f$ should be $\tilde{\forall}\underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'$). If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{B}}(\underline{a}; \underline{b}) \to A$, so that the parts $(A \to B)^{\mathrm{t}}$ (which was proved, so it can be upgraded to $(A \to \forall x\, B)^{\mathrm{t}}$) in (9.9) and $A_{\mathrm{B}}(\underline{a}; \underline{b})$ in (9.10) together imply the part $(\forall x\, B)^{\mathrm{t}}$ in (9.10). Analogously for $A \to B \,/\, \exists x\, A \to B$.

$\underline{\forall x \leq^{\mathrm{i}} t\, A \leftrightarrow \forall x\, (x \leq^{\mathrm{i}} t \to A)}$ To interpret $A \leftrightarrow B$ it suffices to interpret $A \to B$ and $B \to A$ separately.

$\underline{\to}$ We have

$$(\forall x \leq^{\mathrm{i}} t\, A \to \forall x\, (x \leq^{\mathrm{i}} t \to A))_{\mathrm{B}}(\underline{\mathrm{C}}, \underline{f}; \underline{a}, e, \underline{d}) \text{ “}\equiv\text{”}$$
$$\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{f}\underline{a}e\underline{d}\, \forall x \leq^{\mathrm{i}} t\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge (\forall x \leq^{\mathrm{i}} t\, A)^{\mathrm{q}} \to$$
$$\forall x \leq^{\mathrm{i}} e\, \big((x \leq^{\mathrm{i}} t \wedge (x \leq^{\mathrm{i}} t)^{\mathrm{q}} \to A_{\mathrm{B}}(\underline{\mathrm{C}}\underline{a}e; \underline{d})) \wedge (x \leq^{\mathrm{i}} t \to A)^{\mathrm{t}}\big) \wedge$$
$$(\forall x\, (x \leq^{\mathrm{i}} t \to A))^{\mathrm{t}},$$
$$\underline{t}_{\underline{\mathrm{C}}} :\equiv \lambda \underline{a}, e \,.\, \underline{a}, \quad \underline{t}_{\underline{f}} :\equiv \lambda \underline{a}, e, \underline{d} \,.\, \underline{d}.$$

If $\mathrm{t} = \mathrm{id}$, then we use $A_{\mathrm{B}}(\underline{a}; \underline{b}) \to A$, so that the part $\forall x \leq^{\mathrm{i}} t\, A_{\mathrm{B}}(\underline{a}; \underline{b})$ in the premise implies the parts $(x \leq^{\mathrm{i}} t \to A)^{\mathrm{t}}$ and $(\forall x\, (x \leq^{\mathrm{i}} t \to A))^{\mathrm{t}}$ in the conclusion.

$\underline{\leftarrow}$ We have

$$(\forall x\,(x \leq^{\mathrm{i}} t \to A) \to \forall x \leq^{\mathrm{i}} t\, A)_{\mathrm{B}}(\underline{C}, f, \underline{g}; \underline{A}, \underline{d}) \text{ “}\equiv\text{”}$$
$$\tilde{\forall} e, \underline{b} \leq^{\mathrm{i}} f\underline{A}\underline{d}, \underline{g}\underline{A}\underline{d}\, \forall x \leq^{\mathrm{i}} e$$
$$\big((x \leq^{\mathrm{i}} t \wedge (x \leq^{\mathrm{i}} t)^{\mathrm{q}} \to A_{\mathrm{B}}(\underline{A}e; \underline{b})) \wedge (x \leq^{\mathrm{i}} t \to A)^{\mathrm{t}}\big) \wedge$$
$$(\forall x\,(x \leq^{\mathrm{i}} t \to A))^{\mathrm{t}} \wedge (\forall x\,(x \leq^{\mathrm{i}} t \to A))^{\mathrm{q}} \to$$
$$\forall x \leq^{\mathrm{i}} t\, A_{\mathrm{B}}(\underline{C}\underline{A}; \underline{d}),$$

$$t_{\underline{C}}(\underline{\ell}) :\equiv \lambda \underline{A}\,.\, \underline{A} t^{\mathrm{m}}(\underline{\ell}), \quad t_f(\underline{\ell}) :\equiv \lambda \underline{A}, \underline{d}\,.\, t^{\mathrm{m}}(\underline{\ell}), \quad t_{\underline{g}} :\equiv \lambda \underline{A}, \underline{d}\,.\, \underline{d}.$$

Analogously for $\exists x \leq^{\mathrm{i}} t\, A \leftrightarrow \exists x\,(x \leq^{\mathrm{i}} t \wedge A)$.

<u>Axioms of $=_0$, S, Π, Σ, $\underline{\mathrm{R}}$ and $\leq^{\mathrm{i}}$, and rule of $\leq^{\mathrm{i}}$</u> Their formulas are bounded, so they are equivalent to their own interpretation.

<u>$A[0/x]$, $A \to A[\mathrm{S}x/x]\,/\,A$</u> We can assume $x \in \mathrm{FV}(A)$, otherwise $A[0/x] \equiv A$ and so the terms working for $A[0/x]$ also work for A. We have

$$A[0/x]_{\mathrm{B}}(\underline{a}; \underline{b}),$$

$$(A \to A[\mathrm{S}x/x])_{\mathrm{B}}(\underline{C}, \underline{e}; \underline{a}, \underline{d}) \text{ “}\equiv\text{”}$$
$$\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{e}\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to A[\mathrm{S}x/x]_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d}),$$

$$A_{\mathrm{B}}(\underline{a}; \underline{b}),$$

$$t_{\underline{a}}(\underline{\ell}, x) :\equiv \underline{\mathrm{R}} x r_{\underline{a}}(\underline{\ell})\, \lambda \underline{a}, x\,.\, \max(s_{\underline{C}}(\underline{\ell}, x)\underline{a}, \underline{a}).$$

By induction hypothesis we have (9.11) and (9.12), and we want to prove (9.13):

$$\tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \tilde{\forall} \underline{b}\, A[0/x]_{\mathrm{B}}(r_{\underline{a}}(\underline{\ell}'); \underline{b}), \tag{9.11}$$

$$\begin{aligned}\tilde{\forall} \underline{\ell}', x'\, \forall \underline{\ell}, x \leq^{\mathrm{i}} \underline{\ell}', x'\, \tilde{\forall} \underline{a}, \underline{d}\, \tilde{\forall} \underline{b} \leq^{\mathrm{i}} s_{\underline{e}}(\underline{\ell}', x')\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge A^{\mathrm{q}} \to \\ A[\mathrm{S}x/x]_{\mathrm{B}}(r_{\underline{C}}(\underline{\ell}, x')\underline{a}; \underline{d})),\end{aligned} \tag{9.12}$$

$$\tilde{\forall} \underline{\ell}', x'\, \forall \underline{\ell}, x \leq^{\mathrm{i}} \underline{\ell}', x'\, \tilde{\forall} \underline{b}\, A_{\mathrm{B}}(t_{\underline{a}}(\underline{\ell}', x'); \underline{b}). \tag{9.13}$$

First, we prove that $t_{\underline{a}}(\underline{\ell}, x)$ are monotone analogously to proof 4.11. Now, let us prove (9.13) by induction on x. We start by proving (9.13) with $x =_0 x'$. We take arbitrary monotone $\underline{\ell}'$ and arbitrary $\underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'$, and prove $\forall x\, \tilde{\forall} \underline{b}\, A_{\mathrm{B}}(t_{\underline{a}}(\underline{\ell}', x); \underline{b})$ by induction on x.

<u>Base case</u> The formula $\tilde{\forall} \underline{b}\, A_{\mathrm{B}}(t_{\underline{a}}(\underline{\ell}', x); \underline{b})[0/x]$, that is $\tilde{\forall} \underline{b}\, A[0/x]_{\mathrm{B}}(r_{\underline{a}}(\underline{\ell}'); \underline{b})$, is provable by (9.11).

<u>Induction step</u> By induction hypothesis we assume $\tilde{\forall} \underline{b}\, A_{\mathrm{B}}(t_{\underline{a}}(\underline{\ell}', x); \underline{b})$. Taking $x' = x$ and $\underline{a} = t_{\underline{a}}(\underline{\ell}', x)$ (which satisfies $\underline{a} \leq^{\mathrm{i}} \underline{a}$) in (9.12) we get $\tilde{\forall} \underline{d}\, A[\mathrm{S}x/x]_{\mathrm{B}}(s_{\underline{C}}(\underline{\ell}', x) t_{\underline{a}}(\underline{\ell}', x); \underline{d})$ (if q = id, then we use the assumption that we proved A, so as to have the part A^{q} in (9.12)). By monotonicity we get $\tilde{\forall} \underline{d}\, A[\mathrm{S}x/x]_{\mathrm{B}}\big(\max(s_{\underline{C}}(\underline{\ell}', x) t_{\underline{a}}(\underline{\ell}', x), t_{\underline{a}}(\underline{\ell}', x)); \underline{d}\big)$, that is $\tilde{\forall} \underline{b}\, A_{\mathrm{B}}(t_{\underline{a}}(\underline{\ell}', x); \underline{b})[\mathrm{S}x/x]$.

From (9.13) with $x =_0 x'$ we get (9.13) with $x \leq^{\mathrm{i}} x'$ by the monotonicity of $A_{\mathrm{B}}(\underline{a};\underline{b})$ and of $\underline{t_{\underline{a}}}(\underline{\ell}, x)$.

<u>BAC</u> We have

$$(\forall x\, \exists \underline{y}\, A)_{\mathrm{B}}(\underline{E}, \underline{A}; g, \underline{f}) \text{ “}\equiv\text{” }$$
$$\forall x \leq^{\mathrm{i}} g\, \exists \underline{y} \leq^{\mathrm{i}} \underline{E}g\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{f}\, A_{\mathrm{B}}(\underline{A}g; \underline{b}) \wedge A^{\mathrm{q}}) \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{t}},$$

$$(\tilde{\exists} \underline{v}\, \tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, A)_{\mathrm{B}}(\underline{j}, \underline{C}; k, \underline{l}) \text{ “}\equiv\text{” }$$
$$\tilde{\exists} \underline{v} \leq^{\mathrm{i}} \underline{j}\, \big(\forall i, \underline{h} \leq^{\mathrm{i}} k, \underline{l}\, \big(\tilde{\forall} u \leq^{\mathrm{i}} i\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, (\tilde{\forall} \underline{d} \leq^{\mathrm{i}} \underline{h}\, A_{\mathrm{B}}(\underline{C}i; \underline{d}) \wedge A^{\mathrm{q}}) \wedge$$
$$(\tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, A)^{\mathrm{t}}\big) \wedge (\tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, A)^{\mathrm{q}}\big),$$

$$\mathsf{BAC}_{\mathrm{B}}(\underline{J}, \underline{\mathrm{C}}, m, \underline{n}; \underline{E}, \underline{A}, k, \underline{l}) \text{ “}\equiv\text{” } \tilde{\forall} g, \underline{f} \leq^{\mathrm{i}} m\underline{E}\underline{A}k\underline{l}, \underline{n}\underline{E}\underline{A}k\underline{l}$$
$$\big(\forall x \leq^{\mathrm{i}} g\, \exists \underline{y} \leq^{\mathrm{i}} \underline{E}g\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{f}\, A_{\mathrm{B}}(\underline{A}g; \underline{b}) \wedge A^{\mathrm{q}}) \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{t}}\big) \wedge (\forall x\, \exists \underline{y}\, A)^{\mathrm{q}}$$
$$\to$$
$$\tilde{\exists} \underline{v} \leq^{\mathrm{i}} \underline{J}\underline{E}\underline{A}\, \big(\forall i, \underline{h} \leq^{\mathrm{i}} k, \underline{l}$$
$$\big(\tilde{\forall} u \leq^{\mathrm{i}} i\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, (\tilde{\forall} \underline{d} \leq^{\mathrm{i}} \underline{h}\, A_{\mathrm{B}}(\underline{\mathrm{C}}\underline{E}\underline{A}i; \underline{d}) \wedge A^{\mathrm{q}}) \wedge$$
$$(\tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, A)^{\mathrm{t}}\big) \wedge (\tilde{\forall} u\, \forall x \leq^{\mathrm{i}} u\, \exists \underline{y} \leq^{\mathrm{i}} \underline{v}u\, A)^{\mathrm{q}}\big),$$

$$t_{\underline{J}} :\equiv \lambda \underline{E}, \underline{A}\, .\, \underline{E}, \quad \underline{t_{\mathrm{C}}} :\equiv \lambda \underline{E}, \underline{A}, i\, .\, \underline{A}i,$$
$$t_m :\equiv \lambda \underline{E}, \underline{A}, \underline{l}, k\, .\, k, \quad \underline{t_{\underline{n}}} :\equiv \lambda \underline{E}, \underline{A}, \underline{l}, k\, .\, \underline{l}.$$

The terms only seem to work for $\mathrm{q} = \top$ and $\mathrm{t} = \top$.

<u>B-BCC</u> We have

$$(\tilde{\forall} \underline{v}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}})_{\mathrm{B}}(; \underline{a}) \text{ “}\equiv\text{” }$$
$$\tilde{\forall} \underline{v} \leq^{\mathrm{i}} \underline{a}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}} \wedge (\tilde{\forall} \underline{v}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}})^{\mathrm{t}},$$

$$(\exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y}\, A_{\mathrm{b}})_{\mathrm{B}}(; \underline{c}) \text{ “}\equiv\text{” }$$
$$\exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \big(\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{c}\, (\forall \underline{y} \leq^{\mathrm{i}} \underline{b}\, A_{\mathrm{b}} \wedge (\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{q}}\big),$$

$$(\mathsf{B\text{-}BCC})_{\mathrm{B}}(\underline{d}; \underline{c}) \text{ “}\equiv\text{” }$$
$$\tilde{\forall} \underline{a} \leq^{\mathrm{i}} \underline{d}\underline{c}\, (\tilde{\forall} \underline{v} \leq^{\mathrm{i}} \underline{a}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}} \wedge (\tilde{\forall} \underline{v}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge$$
$$(\tilde{\forall} \underline{v}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}})^{\mathrm{q}}$$
$$\to$$
$$\exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \big(\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{c}\, (\forall \underline{y} \leq^{\mathrm{i}} \underline{b}\, A_{\mathrm{b}} \wedge (\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{q}}\big),$$
$$\underline{t_{\underline{d}}} :\equiv \lambda \underline{c}\, .\, \underline{c}.$$

If $\mathrm{t} = \mathrm{id}$, then the part $(\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{t}}$ in the conclusion implies the part $\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{c}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{c}\, A_{\mathrm{b}}$ in the conclusion, so it suffices to prove $\exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, (\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{t}}$, which follows from the part $(\tilde{\forall} \underline{v}\, \exists \underline{x} \leq^{\mathrm{i}} \underline{u}\, \forall \underline{y} \leq^{\mathrm{i}} \underline{v}\, A_{\mathrm{b}})^{\mathrm{t}}$ in the premise by B-BCC. Analogously if $\mathrm{q} = \mathrm{id}$.

$\underline{\forall\text{-BIP}}$ We have

$$\begin{gathered}
(\forall \underline{x}\, A_{\mathrm{b}} \to \exists y\, B)_{\mathrm{B}}(f, \underline{a}, \underline{h}; \underline{g}) \text{ “}\equiv\text{”} \\
\big(\tilde{\forall} \underline{e} \leq^{\mathrm{i}} \underline{hg}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}} \to \\
\exists y \leq^{\mathrm{i}} f\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{g}\, B_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge B^{\mathrm{q}})\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y\, B)^{\mathrm{t}},
\end{gathered}$$

$$\begin{gathered}
(\tilde{\exists} z\, (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{i}} z\, B))_{\mathrm{B}}(l, \underline{c}, \underline{k}; \underline{m}) \text{ “}\equiv\text{”} \\
\tilde{\exists} z \leq^{\mathrm{i}} l\, \Big(\forall \underline{j} \leq^{\mathrm{i}} \underline{m}\, \big(\big(\tilde{\forall} \underline{i} \leq^{\mathrm{i}} \underline{kj}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{i}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}} \to \\
\exists y \leq^{\mathrm{i}} z\, (\tilde{\forall} \underline{d} \leq^{\mathrm{i}} \underline{j}\, B_{\mathrm{B}}(\underline{c}; \underline{d}) \wedge B^{\mathrm{q}})\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{i}} z\, B)^{\mathrm{t}}\big) \wedge \\
(z \leq^{\mathrm{i}} z \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{i}} z\, B))^{\mathrm{q}}\Big),
\end{gathered}$$

$$\begin{gathered}
(\forall\text{-BIP})_{\mathrm{B}}(L, \underline{C}, \underline{K}, \underline{n}; f, \underline{a}, \underline{h}, \underline{m}) \text{ “}\equiv\text{”} \\
\tilde{\forall} \underline{g} \leq^{\mathrm{i}} \underline{n} f \underline{a} \underline{h} \underline{m}\, \big(\big(\tilde{\forall} \underline{e} \leq^{\mathrm{i}} \underline{hg}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}} \to \\
\exists y \leq^{\mathrm{i}} f\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{g}\, B_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge B^{\mathrm{q}})\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y\, B)^{\mathrm{t}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y\, B)^{\mathrm{q}} \\
\downarrow \\
\tilde{\exists} z \leq^{\mathrm{i}} L f \underline{a} \underline{h}\, \Big(\forall \underline{j} \leq^{\mathrm{i}} \underline{m}\, \big(\big(\tilde{\forall} \underline{i} \leq^{\mathrm{i}} \underline{K} f \underline{a} \underline{h} \underline{j}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{i}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}} \to \\
\exists y \leq^{\mathrm{i}} z\, (\tilde{\forall} \underline{d} \leq^{\mathrm{i}} \underline{j}\, B_{\mathrm{B}}(\underline{C} f \underline{a} \underline{h}; \underline{d}) \wedge B^{\mathrm{q}})\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{i}} z\, B)^{\mathrm{t}}\big) \wedge \\
(z \leq^{\mathrm{i}} z \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{i}} z\, B))^{\mathrm{q}}\Big),
\end{gathered}$$

$$\begin{gathered}
t_L :\equiv \lambda f, \underline{a}, \underline{h} \,.\, f, \quad t_{\underline{C}} :\equiv \lambda f, \underline{a}, \underline{h} \,.\, \underline{a}, \\
t_{\underline{K}} :\equiv \lambda f, \underline{a}, \underline{h}, \underline{j} \,.\, \underline{hj}, \quad t_{\underline{n}} :\equiv \lambda f, \underline{a}, \underline{h}, \underline{m} \,.\, \underline{m}.
\end{gathered}$$

If $\mathrm{t} = \mathrm{id}$, then to prove $(\forall \underline{x}\, A_{\mathrm{b}} \to \exists y \leq^{\mathrm{i}} z\, B)^{\mathrm{t}}$ note that $\forall \underline{x}\, A_{\mathrm{b}}$ implies the part $\tilde{\forall} \underline{e} \leq^{\mathrm{i}} \underline{hg}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}}$ in the premise, which implies the part $\exists y \leq^{\mathrm{i}} f\, (\tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{g}\, B_{\mathrm{B}}(\underline{a}; \underline{b}) \wedge B^{\mathrm{q}})$ in the premise, which by truth implies $\exists y \leq^{\mathrm{i}} f\, B$, that is $\exists y \leq^{\mathrm{i}} z\, B$ with $z = f$. Analogously if $\mathrm{q} = \mathrm{id}$.

$\underline{\text{MAJ}}$ We have

$$\begin{gathered}
\mathsf{MAJ}_{\mathrm{B}}(A; b) \text{ “}\equiv\text{”} \forall x \leq^{\mathrm{i}} b\, \exists y \leq^{\mathrm{i}} Ab\, (x \leq^{\mathrm{i}} y \wedge (x \leq^{\mathrm{i}} y)^{\mathrm{q}}), \\
t_A :\equiv \lambda b \,.\, b.
\end{gathered}$$

$\underline{\text{B-BMP}}$ We have

$$\begin{gathered}
(\forall \underline{x}\, A_{\mathrm{b}} \to B_{\mathrm{b}})_{\mathrm{B}}(\underline{b};\,) \text{ “}\equiv\text{”} \\
\big(\tilde{\forall} \underline{a} \leq^{\mathrm{i}} \underline{b}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{a}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}} \to B_{\mathrm{b}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to B_{\mathrm{b}})^{\mathrm{t}},
\end{gathered}$$

$$\begin{gathered}
(\tilde{\exists} \underline{y}\, (\forall \underline{x} \leq^{\mathrm{i}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}}))_{\mathrm{B}}(\underline{c};\,) \text{ “}\equiv\text{”} \\
\tilde{\exists} \underline{y} \leq^{\mathrm{i}} \underline{c}\, ((\forall \underline{x} \leq^{\mathrm{i}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}}) \wedge (\forall \underline{x} \leq^{\mathrm{i}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}})^{\mathrm{q}}),
\end{gathered}$$

$$
\begin{gathered}
(\mathsf{B\text{-}BMP})_{\mathrm{B}}(\underline{C};\underline{b}) \text{ “}{\equiv}\text{”} \\
\big(\tilde{\forall} \underline{a} \le^{\mathrm{i}} \underline{b}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{a}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}} \to B_{\mathrm{b}}\big) \wedge (\forall \underline{x}\, A_{\mathrm{b}} \to B_{\mathrm{b}})^{\mathrm{t}} \wedge \\
(\forall \underline{x}\, A_{\mathrm{b}} \to B_{\mathrm{b}})^{\mathrm{q}} \\
\downarrow \\
\tilde{\exists} \underline{y} \le^{\mathrm{i}} \underline{C}\underline{b}\,((\forall \underline{x} \le^{\mathrm{i}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}}) \wedge (\forall \underline{x} \le^{\mathrm{i}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}})^{\mathrm{q}}),
\end{gathered}
$$

$$\underline{t}_{\underline{C}} :\equiv \lambda \underline{b} \,.\, \underline{b}.$$

These terms only seem to work for $\mathrm{q} = \top$ and $\mathrm{t} = \top$.

<u>B-BUD</u> We have

$$
\begin{gathered}
(\tilde{\forall} \underline{u}, \underline{v}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}))_{\mathrm{B}}(;\underline{a},\underline{b}) \text{ “}{\equiv}\text{”} \\
\tilde{\forall} \underline{u}, \underline{v} \le^{\mathrm{i}} \underline{a}, \underline{b}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}) \wedge (\tilde{\forall} \underline{u}, \underline{v}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}))^{\mathrm{t}},
\end{gathered}
$$

$$
\begin{gathered}
(\forall \underline{x}\, A_{\mathrm{b}} \vee \forall \underline{y}\, B_{\mathrm{b}})_{\mathrm{B}}(;\underline{e},\underline{f}) \text{ “}{\equiv}\text{”} \\
\big(\tilde{\forall} \underline{c} \le^{\mathrm{i}} \underline{e}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{c}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}}\big) \vee \\
\big(\tilde{\forall} \underline{d} \le^{\mathrm{i}} \underline{f}\,(\forall \underline{y} \le^{\mathrm{i}} \underline{d}\, B_{\mathrm{b}} \wedge (\forall \underline{y}\, B_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{y}\, B_{\mathrm{b}})^{\mathrm{q}}\big),
\end{gathered}
$$

$$
\begin{gathered}
(\mathsf{B\text{-}BUD})_{\mathrm{B}}(\underline{g},\underline{h};\underline{e},\underline{f}) \text{ “}{\equiv}\text{” } \tilde{\forall} \underline{a}, \underline{b} \le^{\mathrm{i}} \underline{g}\underline{e}\underline{f}, \underline{h}\underline{e}\underline{f} \\
\big(\tilde{\forall} \underline{u}, \underline{v} \le^{\mathrm{i}} \underline{a}, \underline{b}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}) \wedge (\tilde{\forall} \underline{u}, \underline{v}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}))^{\mathrm{t}}\big) \wedge \\
(\tilde{\forall} \underline{u}, \underline{v}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}))^{\mathrm{q}} \\
\downarrow \\
\big(\tilde{\forall} \underline{c} \le^{\mathrm{i}} \underline{e}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{c}\, A_{\mathrm{b}} \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{q}}\big) \vee \\
\big(\tilde{\forall} \underline{d} \le^{\mathrm{i}} \underline{f}\,(\forall \underline{y} \le^{\mathrm{i}} \underline{d}\, B_{\mathrm{b}} \wedge (\forall \underline{y}\, B_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{y}\, B_{\mathrm{b}})^{\mathrm{q}}\big),
\end{gathered}
$$

$$\underline{t}_{\underline{g}} :\equiv \lambda \underline{e}, \underline{f} \,.\, \underline{e}, \quad \underline{t}_{\underline{h}} :\equiv \lambda \underline{e}, \underline{f} \,.\, \underline{f}.$$

If $\mathrm{t} = \mathrm{id}$, then the parts $(\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}}$ and $(\forall \underline{y}\, B_{\mathrm{b}})^{\mathrm{t}}$ in the conclusion imply the parts $\tilde{\forall} \underline{c} \le^{\mathrm{i}} \underline{e}\, \forall \underline{x} \le^{\mathrm{i}} \underline{c}\, A_{\mathrm{b}}$ and $\tilde{\forall} \underline{d} \le^{\mathrm{i}} \underline{f}\, \forall \underline{y} \le^{\mathrm{i}} \underline{d}\, B_{\mathrm{b}}$ in the conclusion, so it suffices to prove $(\forall \underline{x}\, A_{\mathrm{b}})^{\mathrm{t}} \vee (\forall \underline{y}\, A_{\mathrm{b}})^{\mathrm{t}}$, which follows from the part $(\tilde{\forall} \underline{u}, \underline{v}\,(\forall \underline{x} \le^{\mathrm{i}} \underline{u}\, A_{\mathrm{b}} \vee \forall \underline{y} \le^{\mathrm{i}} \underline{v}\, B_{\mathrm{b}}))^{\mathrm{t}}$ in the premise by B-BUD. Analogously if $\mathrm{q} = \mathrm{id}$.

<u>Γ</u> We have

$$
\begin{gathered}
(\forall \underline{x}\, \exists \underline{y} \le^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, A_{\mathrm{b}})_{\mathrm{B}}(;\underline{c},\underline{b}) \text{ “}{\equiv}\text{”} \\
\forall \underline{x} \le^{\mathrm{i}} \underline{c}\, \exists \underline{y} \le^{\mathrm{i}} \underline{s}\, \big(\tilde{\forall} \underline{a} \le^{\mathrm{i}} \underline{b}\,(\forall \underline{z} \le^{\mathrm{i}} \underline{a}\, A_{\mathrm{b}} \wedge (\forall \underline{z}\, A_{\mathrm{b}})^{\mathrm{t}}) \wedge (\forall \underline{z}\, A_{\mathrm{b}})^{\mathrm{q}}\big).
\end{gathered}
$$

9.12 Remark.

1. The bounded functional interpretations with q-truth and t-truth do not seem to interpret BAC. To interpret it we (essentially and in particular) should present terms witnessing $\underline{v}$ in $\tilde{\forall} g \le^{\mathrm{i}} \ldots \forall x \le^{\mathrm{i}} g\, \exists \underline{y} \le^{\mathrm{i}} \underline{E} g\, A \to \tilde{\forall} u\, \forall x \le^{\mathrm{i}} u\, \exists \underline{y} \le^{\mathrm{i}} \underline{v} u\, A$ and this does not seem possible since the premise only gives us $\underline{E}$ working for bounded g but the conclusion asks for $\underline{v}$ working for unbounded u.

2. The bounded functional interpretations with q-truth and t-truth do not seem to interpret $\mathsf{B\text{-}BMP}$. To interpret it we (essentially and in particular) should present terms witnessing $\underline{y}$ in $(\forall \underline{x}\, A_{\mathrm{b}} \to B_{\mathrm{b}}) \to (\forall \underline{x} \leq^{\mathrm{i}} \underline{y}\, A_{\mathrm{b}} \to B_{\mathrm{b}})$ and this does not seem possible.

9.4 Characterisation

9.13 Theorem (characterisation)**.** Let us consider the theory $\mathsf{HA}^{\omega}_{\mathrm{i}} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD}$.

1. This theory proves $A \leftrightarrow A^{\mathrm{B}}$ for all formulas A of $\mathsf{HA}^{\omega}_{\mathrm{i}}$ [15, theorem 3].
2. This theory is the least theory, containing $\mathsf{HA}^{\omega}_{\mathrm{i}}$, satisfying the previous point.

9.14 Proof.

1. The proof is by induction on the structure of A.

$\underline{\vee}$ Using induction hypothesis in the first equivalence, and $\mathsf{B\text{-}BUD}$ in the third equivalence, we get

$$\begin{array}{r}
A \vee B \leftrightarrow \\
A^{\mathrm{B}} \vee B^{\mathrm{B}} \equiv \\
\tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \vee \tilde{\exists}\underline{c}\, \tilde{\forall}\underline{d}\, B_{\mathrm{B}}(\underline{c};\underline{d}) \leftrightarrow \\
\tilde{\exists}\underline{a}, \underline{c}\, (\tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \vee \tilde{\forall}\underline{d}\, B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\exists}\underline{a}, \underline{c}\, \tilde{\forall}\underline{e}, \underline{f}\, (\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \vee \tilde{\forall}\underline{d} \leq^{\mathrm{i}} \underline{f}\, B_{\mathrm{B}}(\underline{c};\underline{d})) \equiv \\
(A \vee B)^{\mathrm{B}}.
\end{array}$$

Analogously for A_{at} and $\wedge$.

$\underline{\to}$ Using induction hypothesis in the first equivalence, $\forall\text{-}\mathsf{BIP}$ in the third equivalence, monotonicity in the fourth equivalence, $\mathsf{B\text{-}BMP}$ in the sixth equivalence, and MAC (see point 2 of proposition 1.66) in the last two equivalences, we get

$$\begin{array}{r}
(A \to B) \leftrightarrow \\
(A^{\mathrm{B}} \to B^{\mathrm{B}}) \equiv \\
(\tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to \tilde{\exists}\underline{c}\, \tilde{\forall}\underline{d}\, B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\forall}\underline{a}\, (\tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to \tilde{\exists}\underline{c}\, \tilde{\forall}\underline{d}\, B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{c}\, (\tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to \tilde{\exists}\underline{c}' \leq^{\mathrm{i}} \underline{c}\, \tilde{\forall}\underline{d}\, B_{\mathrm{B}}(\underline{c}';\underline{d})) \leftrightarrow \\
\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{c}\, (\tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to \tilde{\forall}\underline{d}\, B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{c}\, \tilde{\forall}\underline{d}\, (\tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{c}\, \tilde{\forall}\underline{d}\, \tilde{\exists}\underline{e}\, (\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{e}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\forall}\underline{a}\, \tilde{\exists}\underline{c}\, \tilde{\exists}\underline{E}\, \tilde{\forall}\underline{d}\, (\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{E}\underline{d}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to B_{\mathrm{B}}(\underline{c};\underline{d})) \leftrightarrow \\
\tilde{\exists}\underline{C}, \underline{\mathrm{E}}\, \tilde{\forall}\underline{a}, \underline{d}\, (\tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{\mathrm{E}}\underline{a}\underline{d}\, A_{\mathrm{B}}(\underline{a};\underline{b}) \to B_{\mathrm{B}}(\underline{C}\underline{a};\underline{d})) \equiv \\
(A \to B)^{\mathrm{B}}.
\end{array}$$

$\underline{\forall\leq^{\mathrm{i}}}$ Using induction hypothesis in the first equivalence, BC (see point 1 of proposition 1.66) in the second equivalence, and monotonicity in the third equivalence, we get

$$\begin{aligned}
\forall x \leq^{\mathrm{i}} t\, A &\leftrightarrow \\
\forall x \leq^{\mathrm{i}} t\, A^{\mathrm{B}} &\equiv \\
\forall x \leq^{\mathrm{i}} t\, \tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}\underline{a}\, \forall x \leq^{\mathrm{i}} t\, \tilde{\exists}\underline{a}' \leq^{\mathrm{i}} \underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a}';\underline{b}) &\leftrightarrow \\
\tilde{\exists}\underline{a}\, \forall x \leq^{\mathrm{i}} t\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, \forall x \leq^{\mathrm{i}} t\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\equiv \\
(\forall x \leq^{\mathrm{i}} t\, A)^{\mathrm{B}}&.
\end{aligned}$$

$\underline{\forall}$ Using induction hypothesis in the first equivalence, BAC in the second equivalence, and monotonicity in the third equivalence, we get

$$\begin{aligned}
\forall x\, A &\leftrightarrow \\
\forall x\, A^{\mathrm{B}} &\equiv \\
\forall x\, \tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}\underline{A}\, \tilde{\forall}c\, \forall x \leq^{\mathrm{i}} c\, \tilde{\exists}\underline{a} \leq^{\mathrm{i}} \underline{Ac}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}\underline{A}\, \tilde{\forall}c\, \forall x \leq^{\mathrm{i}} c\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{A}c;\underline{b}) &\leftrightarrow \\
\tilde{\exists}\underline{A}\, \tilde{\forall}c,\underline{b}\, \forall x \leq^{\mathrm{i}} c\, A_{\mathrm{B}}(\underline{A}c;\underline{b}) &\equiv \\
(\forall x\, A)^{\mathrm{B}}&.
\end{aligned}$$

$\underline{\exists}$ Using induction hypothesis in the first equivalence, MAJ in the second equivalence, and $\mathsf{B\text{-}BCC}$ in the last equivalence, we get

$$\begin{aligned}
\exists x\, A &\leftrightarrow \\
\exists x\, A^{\mathrm{B}} &\equiv \\
\exists x\, \tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}c\, \exists x \leq^{\mathrm{i}} c\, \tilde{\exists}\underline{a}\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}c\, \tilde{\exists}\underline{a}\, \exists x \leq^{\mathrm{i}} c\, \tilde{\forall}\underline{b}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\leftrightarrow \\
\tilde{\exists}c,\underline{a}\, \tilde{\forall}\underline{d}\, \exists x \leq^{\mathrm{i}} c\, \tilde{\forall}\underline{b} \leq^{\mathrm{i}} \underline{d}\, A_{\mathrm{B}}(\underline{a};\underline{b}) &\equiv \\
(\exists x\, A)^{\mathrm{B}}&.
\end{aligned}$$

Analogously for $\exists\leq^{\mathrm{i}}$.

2. Analogous to point 2 of proof 3.15.

9.15 Remark. The characterisation theorem of B ensures that the soundness theorem of B is optimal, in the sense that the theory $\mathsf{HA}^{\omega}_{\mathrm{i}} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD} + \Gamma$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^{\omega}_{\mathrm{i}} + \Gamma \vdash A^{\mathrm{B}}$ (analogously to remark 3.16).

9.16. An (optimal) characterisation theorem of Bq and Bt is unknown.

9.5 Applications

9.17 Theorem (bounded existence property and bounded program extraction**).** Let $\mathsf{T} := \mathsf{HA}_{\mathrm{i}}^{\omega} \pm \mathsf{B\text{-}BCC} \pm \forall\text{-}\mathsf{BIP} \pm \mathsf{MAJ} \pm \mathsf{B\text{-}BUD}$.

1. Let $\mathrm{FV}(\exists \underline{x}\, A) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \exists \underline{x}\, A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ of T such that $\mathsf{T} \vdash \tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \exists \underline{x} \leq^{\mathrm{i}} \underline{t}(\underline{\ell}')\, A$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) = \mathrm{FV}(\exists \underline{x}\, A)$.

2. Let $\mathrm{FV}(\forall \underline{x}\, \exists \underline{y}\, A) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell}, \underline{x})$ of T such that $\mathsf{T} \vdash \tilde{\forall} \underline{\ell}', \underline{x}'\, \forall \underline{\ell}, \underline{x} \leq^{\mathrm{i}} \underline{\ell}', \underline{x}'\, \exists \underline{y} \leq^{\mathrm{i}} \underline{t}(\underline{\ell}', \underline{x}')\, A$ and $\mathrm{FV}(\underline{t}(\underline{\ell}, \underline{x})) = \mathrm{FV}(\exists \underline{y}\, A)$.

Analogously for $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD}$.

9.18 Proof. Analogous to proof 6.17.

9.19 Theorem (conservation and relative consistency**).**

1. Let $\tilde{\forall} \underline{x}'\, \forall \underline{x} \leq^{\mathrm{i}} \underline{x}'\, \exists \underline{y}\, A_{\mathrm{b}}$ be a sentence of $\mathsf{HA}_{\mathrm{i}}^{\omega}$. If $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD} \vdash \tilde{\forall} \underline{x}'\, \forall \underline{x} \leq^{\mathrm{i}} \underline{x}'\, \exists \underline{y}\, A_{\mathrm{b}}$, then $\mathsf{HA}_{\mathrm{i}}^{\omega} \vdash \tilde{\forall} \underline{x}'\, \forall \underline{x} \leq^{\mathrm{i}} \underline{x}'\, \exists \underline{y}\, A_{\mathrm{b}}$.

2. If $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD} \vdash \bot$, then $\mathsf{HA}_{\mathrm{i}}^{\omega} \vdash \bot$.

9.20 Proof. Analogous to proof 4.18.

9.6 Conclusion

9.21. We introduced the bounded functional interpretation as proof interpretation that aims at bounds from the start, instead of in a last step like the monotone functional interpretation. The main results about the bounded functional interpretation are the following.

Soundness theorem This theorem says that we can use the bounded functional interpretation to extract computational content from proofs in $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{BAC} + \mathsf{B\text{-}BCC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}BUD}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used the bounded functional interpretation to do applications on:

1. bounded existence property;
2. bounded program extraction;
3. conservation;
4. relative consistency.

Chapter 10

Shoenfield-like bounded functional interpretation

10.1 Introduction

10.1. In the same way that before we presented a unbounded interpretation S of PA^ω by composing D with Kr, now we present a bounded interpretation U of $\mathsf{PA}^\omega_{\mathrm{i}}$ by composing B with Kr. This is pictured in figure 10.1. This new composition $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$ is called Shoenfield-like bounded functional interpretation.

$$\begin{tikzcd} \mathsf{PA}^\omega \arrow[r, "\mathrm{Kr}"] \arrow[rr, bend right, "\mathrm{S=D\circ Kr}"'] & \mathsf{HA}^\omega \arrow[r, "\mathrm{D}"] & \mathsf{HA}^\omega \end{tikzcd} \qquad \begin{tikzcd} \mathsf{PA}^\omega_{\mathrm{i}} \arrow[r, "\mathrm{Kr}"] \arrow[rr, bend right, "\mathrm{U=B\circ Kr}"'] & \mathsf{HA}^\omega_{\mathrm{i}} \arrow[r, "\mathrm{U}"] & \mathsf{HA}^\omega_{\mathrm{i}} \end{tikzcd}$$

Figure 10.1: the compositions $\mathrm{S} = \mathrm{D} \circ \mathrm{Kr}$ and $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$.

10.2. Our main contribution to this topic is the factorisation $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$ (theorem 10.10).

10.2 Definition

10.3. In this chapter we consider $\mathsf{PA}^\omega_{\mathrm{i}}$ based on $\neg$, $\vee$, $\forall\leq^{\mathrm{i}}$ and $\forall$, so $A \wedge B :\equiv \neg(\neg A \vee \neg B)$, $A \to B :\equiv \neg A \vee B$, $\exists x \leq^{\mathrm{i}} t\, A :\equiv \neg\forall x \leq^{\mathrm{i}} t \,\neg A$ and $\exists x\, A :\equiv \neg\forall x\, \neg A$.

10.4 Definition. The *Shoenfield-like bounded functional interpretation* U [13, definition 1] assigns to each formula A of $\mathsf{PA}^\omega_{\mathrm{i}}$ the formula $A^{\mathrm{U}} :\equiv \tilde{\forall} \underline{a}\, \tilde{\exists} \underline{b}\, A_{\mathrm{U}}(\underline{a};\underline{b})$, where $A_{\mathrm{U}}(\underline{a};\underline{b})$ is defined by recursion on the structure of A by

$$\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{U}}(;) &:\equiv A_{\mathrm{at}},\\
(\neg A)_{\mathrm{U}}(\underline{B};\underline{c}) &:\equiv \tilde{\exists}\underline{a} \leq^{\mathrm{i}} \underline{c}\, \neg A_{\mathrm{U}}(\underline{a};\underline{B}\underline{a}),\\
(A \vee B)_{\mathrm{U}}(\underline{a},\underline{c};\underline{b},\underline{d}) &:\equiv A_{\mathrm{U}}(\underline{a};\underline{b}) \vee B_{\mathrm{U}}(\underline{c};\underline{d}),\\
(\forall x \leq^{\mathrm{i}} t\, A)_{\mathrm{U}}(\underline{a};\underline{b}) &:\equiv \forall x \leq^{\mathrm{i}} t\, A_{\mathrm{U}}(\underline{a};\underline{b}),\\
(\forall x\, A)_{\mathrm{U}}(c,\underline{a};\underline{b}) &:\equiv \forall x \leq^{\mathrm{i}} c\, A_{\mathrm{U}}(\underline{a};\underline{b}).
\end{aligned}$$

By $(A_{\mathrm{at}})_{\mathrm{U}}(;)$ we mean $(A_{\mathrm{at}})_{\mathrm{U}}(\underline{a};\underline{b})$ with the tuples $\underline{a}$ and $\underline{b}$ empty.

10.5. The letter U in the symbol for the Shoenfield-like bounded functional interpretation U seems to come from "uniformity" since the paper [13] where this interpretation is presented puts the emphasis on uniformities.

10.6 Remark. The formulas $A_{\mathrm{U}}(\underline{a};\underline{b})$ are bounded.

10.7 Remark. The Shoenfield-like bounded functional interpretation U acts as the identity on bounded formulas of $\mathsf{PA}^{\omega}_{\mathrm{i}}$ in the sense of: $(A_{\mathrm{b}})_{\mathrm{U}}(;) \equiv A_{\mathrm{b}}$ for all bounded formulas A_{b} of $\mathsf{PA}^{\omega}_{\mathrm{i}}$.

10.3 Factorisation

10.8 Lemma (monotonicity)**.** We have $\mathsf{HA}^{\omega}_{\mathrm{i}} \vdash \forall \underline{a}\, \forall \underline{b}'\, \forall \underline{b} \leq^{\mathrm{i}} \underline{b}' \,(A_{\mathrm{U}}(\underline{a};\underline{b}) \to A_{\mathrm{U}}(\underline{a};\underline{b}'))$ [13, lemma 1].

10.9 Proof. Analogous to proof 4.9.

10.10 Theorem (factorisation $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$)**.** For all formulas A of $\mathsf{PA}^{\omega}_{\mathrm{i}}$ we have:

1. $\mathsf{HA}^{\omega}_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{B}, \underline{a}\, (A_{\mathrm{U}}(\underline{a};\underline{B}\underline{a}) \leftrightarrow (A^{\mathrm{Kr}})_{\mathrm{B}}(\underline{B};\underline{a}))$ [20, theorem 4.1];

2. $\mathsf{HA}^{\omega}_{\mathrm{i}} + \mathsf{B\text{-}LEM} + \mathsf{B\text{-}MAC} \vdash A^{\mathrm{U}} \leftrightarrow (A^{\mathrm{Kr}})^{\mathrm{B}}$ [20, theorem 4.1].

10.11 Proof.

1. (a) First we prove $(*)$ $\mathsf{HA}^{\omega}_{\mathrm{i}}{+}\mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{a}, \underline{b}\, (A_{\mathrm{U}}(\underline{a};\underline{b}) \leftrightarrow \neg \tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, (A_{\mathrm{Kr}})_{\mathrm{B}}(\underline{a};\underline{b}'))$ by induction on the structure of A. (The essential difference between this proof and proof 7.8 is in the quantification $\tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}$ used here.)

   $\underline{\vee}$ Let us assume $\underline{a}, \underline{c}, \underline{b}, \underline{d} \leq^{\mathrm{i}} \underline{a}, \underline{c}, \underline{b}, \underline{d}$. Using the induction hypothesis in the first equivalence, $\mathsf{B\text{-}LEM}$ in the second equivalence, and $\underline{b}, \underline{d} \leq^{\mathrm{i}} \underline{b}, \underline{d}$ in the third equivalence (because to prove $\forall x \leq^{\mathrm{i}} y\, (C \wedge D) \to \forall x \leq^{\mathrm{i}} y\, C \wedge D$, with $x \notin \mathrm{FV}(D)$, we use $y \leq^{\mathrm{i}} y$), we get

$$\begin{aligned}
(A \vee B)_{\mathrm{U}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &\equiv \\
A_{\mathrm{U}}(\underline{a};\underline{b}) \vee B_{\mathrm{U}}(\underline{c};\underline{d}) &\leftrightarrow \\
\neg \tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, (A_{\mathrm{Kr}})_{\mathrm{B}}(\underline{a};\underline{b}') \vee \neg \tilde{\forall} \underline{d}' \leq^{\mathrm{i}} \underline{d}\, (B_{\mathrm{Kr}})_{\mathrm{B}}(\underline{c};\underline{d}') &\leftrightarrow \\
\neg (\tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, (A_{\mathrm{Kr}})_{\mathrm{B}}(\underline{a};\underline{b}') \wedge \tilde{\forall} \underline{d}' \leq^{\mathrm{i}} \underline{d}\, (B_{\mathrm{Kr}})_{\mathrm{B}}(\underline{c};\underline{d}')) &\leftrightarrow \\
\neg \tilde{\forall} \underline{b}, \underline{d}' \leq^{\mathrm{i}} \underline{b}, \underline{d}\, ((A_{\mathrm{Kr}})_{\mathrm{B}}(\underline{a};\underline{b}') \vee (B_{\mathrm{Kr}})_{\mathrm{B}}(\underline{c};\underline{d}')) &\equiv \\
\neg \tilde{\forall} \underline{b}', \underline{d}' \leq^{\mathrm{i}} \underline{b}, \underline{d}\, ((A \vee B)_{\mathrm{Kr}})_{\mathrm{B}}(\underline{a}, \underline{c}; \underline{b}', \underline{d}')&.
\end{aligned}$$

   Analogously for A_{at} and $\neg$.

$\underline{\forall}$ Let us assume $\underline{c}, \underline{a}, \underline{b} \leq^{\mathrm{i}} c, \underline{a}, \underline{b}$. Using the induction hypothesis in the first equivalence we get

$$\begin{aligned}
(\forall x\, A)_{\mathrm{U}}(c, \underline{a}; \underline{b}) &\equiv \\
\forall x \leq^{\mathrm{i}} c\, A_{\mathrm{U}}(\underline{a}; \underline{b}) &\leftrightarrow \\
\forall x \leq^{\mathrm{i}} c\, \neg \tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, (A_{\mathrm{Kr}})_{\mathrm{U}}(\underline{a}; \underline{b}') &\leftrightarrow \\
\neg \exists x \leq^{\mathrm{i}} c\, \tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, (A_{\mathrm{Kr}})_{\mathrm{U}}(\underline{a}; \underline{b}') &\leftrightarrow \\
\neg \tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, \exists x \leq^{\mathrm{i}} c\, \tilde{\forall} \underline{b}'' \leq^{\mathrm{i}} \underline{b}'\, (A_{\mathrm{Kr}})_{\mathrm{U}}(\underline{a}; \underline{b}'') &\equiv \\
\neg \tilde{\forall} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, ((\forall x\, A)_{\mathrm{Kr}})_{\mathrm{B}}(c, \underline{a}; \underline{b}')&.
\end{aligned}$$

Analogously for $\forall \underline{\leq}^{\mathrm{i}}$.

(b) Now we prove $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{B}, \underline{a}\, (A_{\mathrm{U}}(\underline{a}; \underline{B}\underline{a}) \leftrightarrow (A^{\mathrm{Kr}})_{\mathrm{B}}(\underline{B}; \underline{a}))$. Let us assume $\underline{B}, \underline{a} \leq^{\mathrm{i}} \underline{B}, \underline{a}$. Using $(*)$ in the equivalence we get

$$\begin{aligned}
A_{\mathrm{U}}(\underline{a}; \underline{B}\underline{a}) &\leftrightarrow \\
\neg \tilde{\forall} \underline{b} \leq^{\mathrm{i}} \underline{B}\underline{a}\, (A_{\mathrm{Kr}})_{\mathrm{B}}(\underline{a}; \underline{b}) &\equiv \\
(\neg(A_{\mathrm{Kr}}))_{\mathrm{B}}(\underline{B}; \underline{a}) &\equiv \\
(A^{\mathrm{Kr}})_{\mathrm{B}}(\underline{B}; \underline{a})&.
\end{aligned}$$

2. Using $\mathsf{B\text{-}MAC}$ in the first equivalence (having in mind the monotonicity of U), and point 1 in the second equivalence, we get

$$\begin{aligned}
A^{\mathrm{U}} &\equiv \\
\tilde{\forall} \underline{a}\, \tilde{\exists} \underline{b}\, A_{\mathrm{U}}(\underline{a}; \underline{b}) &\leftrightarrow \\
\tilde{\exists} \underline{B}\, \tilde{\forall} \underline{a}\, A_{\mathrm{U}}(\underline{a}; \underline{B}\underline{a}) &\leftrightarrow \\
\tilde{\exists} \underline{B}\, \tilde{\forall} \underline{a}\, (A^{\mathrm{Kr}})_{\mathrm{B}}(\underline{B}; \underline{a}) &\equiv \\
(A^{\mathrm{Kr}})^{\mathrm{B}}&.
\end{aligned}$$

10.4 Soundness

10.12 Theorem (soundness)**.** Let A be a formula of $\mathsf{PA}_{\mathrm{i}}^{\omega}$ with $\mathrm{FV}(A) = \{\underline{\ell}\}$, and let Γ be a set of formulas of $\mathsf{PA}_{\mathrm{i}}^{\omega}$ of the form $\forall \underline{x}\, \exists \underline{y} \leq^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, A_{\mathrm{b}}$ where $\underline{s}$ are terms of $\mathsf{PA}_{\mathrm{i}}^{\omega}$. If $\mathsf{PA}_{\mathrm{i}}^{\omega} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ} + \Gamma \vdash A$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ such that $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{B\text{-}LEM} + \Gamma \vdash \tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \tilde{\forall} \underline{a}\, A_{\mathrm{U}}(\underline{a}; \underline{t}(\underline{\ell}'))$ and $\mathrm{FV}(\underline{t}) \subseteq \{\underline{\ell}, \underline{a}\}$ [13, theorem 1].

10.13 Proof. Say $\underline{y} \equiv y_1, \ldots, y_n$ and $\underline{s} \equiv s_1, \ldots, s_n$. Recall $\forall \underline{x}\, \exists \underline{y} \leq^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, A_{\mathrm{b}} \equiv \forall \underline{x}\, \neg \forall y_1 \leq^{\mathrm{i}} s_1 \neg \cdots \neg \forall y_n \leq^{\mathrm{i}} s_n \neg \forall \underline{z}\, A_{\mathrm{b}}$ (in $\mathsf{PA}_{\mathrm{i}}^{\omega}$ based on $\neg$, $\vee$, $\forall \underline{\leq}^{\mathrm{i}}$ and $\forall$). We can prove $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{B\text{-}LEM} \vdash A_{\mathrm{b}}^{\mathrm{Kr}} \leftrightarrow A_{\mathrm{b}}$ by induction on the structure of A_{b}, so $(\forall \underline{x}\, \exists \underline{y} \leq^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, A_{\mathrm{b}})^{\mathrm{Kr}} \equiv \neg \exists \underline{x}\, \neg \exists y_1 \leq^{\mathrm{i}} s_1 \neg \cdots \neg \exists y_n \leq^{\mathrm{i}} s_n \neg \exists \underline{z}\, (A_{\mathrm{b}})_{\mathrm{Kr}}$ is equivalent in $\mathsf{HA}_{\mathrm{i}}^{\omega}$ to $\forall \underline{x}\, \neg\neg \exists \underline{y} \leq^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, \neg (A_{\mathrm{b}})_{\mathrm{Kr}} \equiv \forall \underline{x}\, \neg\neg \exists \underline{y} \leq^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, A_{\mathrm{b}}^{\mathrm{Kr}}$, which is implied in $\mathsf{HA}_{\mathrm{i}}^{\omega} + \mathsf{B\text{-}LEM}$ by $\forall \underline{x}\, \exists \underline{y} \leq^{\mathrm{i}} \underline{s}\, \forall \underline{z}\, A_{\mathrm{b}}$. So $(*)$ $\mathsf{HA}^{\omega} + \mathsf{B\text{-}LEM} + \Gamma^{\mathrm{Kr}} \subseteq \mathsf{HA}^{\omega} + \mathsf{B\text{-}LEM} + \Gamma$.

If $\mathsf{PA}^\omega_\mathrm{i} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ} + \Gamma \vdash A$, then $\mathsf{HA}^\omega_\mathrm{i} + \mathsf{B\text{-}BAC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \Gamma^{\mathrm{Kr}} \vdash A^{\mathrm{Kr}}$ by the soundness theorem of Kr, therefore $\mathsf{HA}^\omega_\mathrm{i} + \mathsf{B\text{-}BAC} + \forall\text{-}\mathsf{BIP} + \mathsf{MAJ} + \mathsf{B\text{-}BMP} + \mathsf{B\text{-}LEM} + \Gamma \vdash A^{\mathrm{Kr}}$ by $(*)$, so by the soundness theorem of B (taking Γ as actually being $\Gamma \cup \{\mathsf{B\text{-}LEM}\}$, where $\{\mathsf{B\text{-}LEM}\}$ denotes the set of instances of $\mathsf{B\text{-}LEM}$, which are bounded formulas) we can extract monotone terms $\underline{t}'(\underline{\ell})$ of $\mathsf{HA}^\omega_\mathrm{i}$ such that $\mathsf{HA}^\omega_\mathrm{i} + \mathsf{B\text{-}LEM} + \Gamma \vdash \tilde{\forall}\underline{\ell}' \, \forall \underline{\ell} \leq^\mathrm{i} \underline{\ell}' \, \tilde{\forall}\underline{a} \, (A^{\mathrm{Kr}})_\mathrm{B}(\underline{t}'(\underline{\ell}'); \underline{a})$ and $\mathrm{FV}(\underline{t}'(\underline{\ell})) \subseteq \mathrm{FV}(A^{\mathrm{Kr}}) = \mathrm{FV}(A)$. By point 1 of the factorisation $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$ we get $\mathsf{HA}^\omega_\mathrm{i} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall}\underline{\ell}' \, \forall \underline{\ell} \leq^\mathrm{i} \underline{\ell}' \, \tilde{\forall}\underline{a} \, A_\mathrm{U}(\underline{a}; \underline{t}'(\underline{\ell}')\underline{a})$. Take $\underline{t}(\underline{\ell}) :\equiv \underline{t}'(\underline{\ell})\underline{a}$.

10.5 Characterisation

10.14 Theorem (characterisation)**.** Let us consider the theory $\mathsf{PA}^\omega_\mathrm{i} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ}$.

1. This theory proves $A \leftrightarrow A^\mathrm{U}$ for all formulas A of $\mathsf{PA}^\omega_\mathrm{i}$ [13, theorem 3].

2. This theory is the least theory, containing $\mathsf{HA}^\omega_\mathrm{i} + \mathsf{B\text{-}LEM}$, satisfying the previous point.

10.15 Proof.

1. The proof is by induction on the structure of A.

$\underline{\neg}$ Using induction hypothesis in the first equivalence, $\mathsf{B\text{-}MAC}$ (which is provable in $\mathsf{PA}^\omega_\mathrm{i} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ}$ by point 2 of proposition 1.66 together with $\mathsf{PA}^\omega_\mathrm{i} + \mathsf{MAJ} \vdash \forall\text{-}\mathsf{BIP}$) in the third equivalence (having in mind the monotonicity of U), and MAJ in the last equivalence, we get

$$\begin{aligned} &\neg A \leftrightarrow \\ &\neg A^\mathrm{U} \equiv \\ &\neg \tilde{\forall}\underline{a} \, \tilde{\exists}\underline{b} \, A_\mathrm{U}(\underline{a}; \underline{b}) \leftrightarrow \\ &\neg \tilde{\exists}\underline{B} \, \tilde{\forall}\underline{a} \, A_\mathrm{U}(\underline{a}; \underline{B}\underline{a}) \leftrightarrow \\ &\neg \tilde{\exists}\underline{B} \, \tilde{\forall}\underline{c} \, \tilde{\forall}\underline{a} \leq^\mathrm{i} \underline{c} \, A_\mathrm{U}(\underline{a}; \underline{B}\underline{a}) \leftrightarrow \\ &\tilde{\forall}\underline{B} \, \tilde{\exists}\underline{c} \, \tilde{\exists}\underline{a} \leq^\mathrm{i} \underline{c} \, \neg A_\mathrm{U}(\underline{a}; \underline{B}\underline{a}) \equiv \\ &(\neg A)^\mathrm{U}. \end{aligned}$$

Analogously for A_{at} and $\vee$.

$\underline{\forall}$ Using induction hypothesis in the first equivalence, MAJ in the second equivalence, the contrapositive of $\mathsf{B\text{-}BCC}$ in the third equivalence, and the monotonicity of U in the last equivalence, we get

$$\begin{aligned}
\forall x\, A &\leftrightarrow \\
\forall x\, A^{\mathrm{U}} &\equiv \\
\forall x\, \tilde{\forall} \underline{a}\, \tilde{\exists} \underline{b}\, A_{\mathrm{U}}(\underline{a}; \underline{b}) &\leftrightarrow \\
\tilde{\forall} c, \underline{a}\, \forall x \leq^{\mathrm{i}} c\, \tilde{\exists} \underline{b}\, A_{\mathrm{U}}(\underline{a}; \underline{b}) &\leftrightarrow \\
\tilde{\forall} c, \underline{a}\, \tilde{\exists} \underline{b}\, \forall x \leq^{\mathrm{i}} c\, \tilde{\exists} \underline{b}' \leq^{\mathrm{i}} \underline{b}\, A_{\mathrm{U}}(\underline{a}; \underline{b}') &\leftrightarrow \\
\tilde{\forall} c, \underline{a}\, \tilde{\exists} \underline{b}\, \forall x \leq^{\mathrm{i}} c\, A_{\mathrm{U}}(\underline{a}; \underline{b}) &\equiv \\
(\forall x\, A)^{\mathrm{U}}&.
\end{aligned}$$

Analogously for $\forall \leq^{\mathrm{i}}$.

2. Analogous to point 2 of proof 3.15.

10.16 Remark. The characterisation theorem of U ensures that the soundness theorem of U is optimal, in the sense that the theory $\mathsf{PA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ} + \Gamma$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \;\Rightarrow\; \mathsf{HA}^\omega_{\mathrm{i}} + \Gamma \vdash A^{\mathrm{U}}$ (analogously to remark 3.16) [13, section 5].

10.6 Applications

10.17 Theorem (bounded existence property for bounded formulas and bounded program extraction for bounded formulas)**.** Let $\mathsf{T} := \mathsf{PA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ}$.

1. Let $\mathrm{FV}(\exists \underline{x}\, A_{\mathrm{b}}) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \exists \underline{x}\, A_{\mathrm{b}}$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell})$ of T such that $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \exists \underline{x} \leq^{\mathrm{i}} \underline{t}(\underline{\ell}')\, A_{\mathrm{b}}$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) = \mathrm{FV}(\exists \underline{x}\, A_{\mathrm{b}})$ [13, corollary 1].

2. Let $\mathrm{FV}(\forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{b}}) = \{\underline{\ell}\}$. If $\mathsf{T} \vdash \forall \underline{x}\, \exists \underline{y}\, A_{\mathrm{b}}$, then we can extract from such a proof monotone terms $\underline{t}(\underline{\ell}, \underline{x})$ of T such that $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{\ell}', \underline{x}'\, \forall \underline{\ell}, \underline{x} \leq^{\mathrm{i}} \underline{\ell}', \underline{x}'\, \exists \underline{y} \leq^{\mathrm{i}} \underline{t}(\underline{\ell}', \underline{x}')\, A_{\mathrm{b}}$ and $\mathrm{FV}(\underline{t}(\underline{\ell}, \underline{x})) = \mathrm{FV}(\exists \underline{y}\, A_{\mathrm{b}})$ [13, corollary 1].

10.18 Proof.

1. Say $\underline{x} \equiv x_1, \ldots, x_n$. Recall $\exists \underline{x}\, A_{\mathrm{b}} \equiv \neg \forall x_1 \neg \cdots \neg \forall x_n \neg A_{\mathrm{b}}$ (in $\mathsf{PA}^\omega_{\mathrm{i}}$ based on $\neg$, $\vee$, $\forall \leq^{\mathrm{i}}$ and $\forall$). We have $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash (\exists \underline{x}\, A_{\mathrm{b}})_{\mathrm{U}}(; \underline{a}) \leftrightarrow \exists \underline{x} \leq^{\mathrm{i}} \underline{a}\, A_{\mathrm{b}}$. Assuming the premise of the theorem, by the soundness theorem of U we can extract monotone terms $\underline{t}(\underline{\ell})$ of $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM}$ such that $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{\ell}'\, \forall \underline{\ell} \leq^{\mathrm{i}} \underline{\ell}'\, \exists \underline{x} \leq^{\mathrm{i}} \underline{t}(\underline{\ell}')\, A_{\mathrm{b}}$ and $\mathrm{FV}(\underline{t}(\underline{\ell})) \subseteq \mathrm{FV}(\exists \underline{x}\, A_{\mathrm{b}})$.

2. Follows from the previous point.

10.19 Theorem (conservation and relative consistency)**.**

1. Let $\tilde{\forall} \underline{x}'\, \forall \underline{x} \leq^{\mathrm{i}} \underline{x}'\, \exists \underline{y}\, A_{\mathrm{b}}$ be a sentence of $\mathsf{PA}^\omega_{\mathrm{i}}$. If $\mathsf{PA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ} \vdash \tilde{\forall} \underline{x}'\, \forall \underline{x} \leq^{\mathrm{i}} \underline{x}'\, \exists \underline{y}\, A_{\mathrm{b}}$, then $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash \tilde{\forall} \underline{x}'\, \forall \underline{x} \leq^{\mathrm{i}} \underline{x}'\, \exists \underline{y}\, A_{\mathrm{b}}$.

2. If $\mathsf{PA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ} \vdash \bot$, then $\mathsf{HA}^\omega_{\mathrm{i}} + \mathsf{B\text{-}LEM} \vdash \bot$ [13, section 4].

10.20 Proof. Analogous to proof 4.18.

10.7 Conclusion

10.21. We introduced the Shoenfield-like bounded functional interpretation U and motivated it by the composition $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$. The main results about the Shoenfield-like bounded functional interpretation are the following.

Factorisation We proved $\mathrm{U} = \mathrm{B} \circ \mathrm{Kr}$.

Soundness theorem This theorem says that we can use the Shoenfield-like bounded functional interpretation to extract computational content from proofs in $\mathsf{PA}^{\omega}_{\mathrm{i}} + \mathsf{B\text{-}BAC} + \mathsf{B\text{-}BCC} + \mathsf{MAJ}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used the Shoenfield-like bounded functional interpretation to do applications on:

1. bounded existence property for bounded formulas;
2. bounded program extraction for bounded formulas;
3. conservation;
4. relative consistency.

Chapter 11

Slash

11.1 Introduction

11.1. The slash is a proof interpretation different from the other ones: instead of mapping a formula to an interpreted formula, the slash maps a formula to a condition in metalevel. Essentially, the slash interprets the internal symbols

$$A_{\mathrm{at}}, \qquad \wedge, \qquad \vee, \qquad \rightarrow, \qquad \forall, \qquad \exists$$

into, respectively, the metalevel symbols

$$\vdash A_{\mathrm{at}}, \quad \text{and}, \quad \text{or}, \quad \text{implies}, \quad \text{for all closed terms}, \quad \text{exists a closed term}.$$

We are going to see three slashes.

Slash without truth $|$ This slash is (essentially) Tarski's definition of truth treated in a proof interpretation style.

Slash with q-truth $|_{\mathrm{q}}$ This slash is (roughly speaking) the previous one with some information about provability hardwired in it.

Slash with t-truth $|_{\mathrm{t}}$ This slash is similar to the previous one but with more information about provability hardwired, to the point that "Tarski's truth" implies provability.

11.2. Our main contributions to this topic are the following.

1. The slash without truth $|$ and its soundness theorem [22, section 4] (definition 11.3 and theorem 11.11).

2. The use of $\omega\mathsf{R}$ and extensionality in the soundness and characterisation theorems of $|$, $|_{\mathrm{q}}$ and $|_{\mathrm{t}}$ (theorems 11.11 and 11.15).

3. The use of $|$ to prove the disjunction and existence properties of classical theories with $\omega\mathsf{R}$ (theorem 11.19).

11.2 Definition

11.3 Definition. Let Γ be a set of formulas of HA^ω.

1. The *slash* $|$ [22, section 4] (on $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$) assigns to each formula A of HA^ω the condition $\Gamma \,|\, A$ defined by recursion on the structure of A by

$$\begin{aligned}
\Gamma \,|\, A_{\mathrm{at}} &:\equiv \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A_{\mathrm{at}}, \\
\Gamma \,|\, (A \wedge B) &:\equiv \Gamma \,|\, A \text{ and } \Gamma \,|\, B, \\
\Gamma \,|\, (A \vee B) &:\equiv \Gamma \,|\, A \text{ or } \Gamma \,|\, B, \\
\Gamma \,|\, (A \to B) &:\equiv \Gamma \,|\, A \text{ implies } \Gamma \,|\, B, \\
\Gamma \,|\, \forall x^\rho\, A(x) &:\equiv \text{for all closed terms } t^\rho\ \Gamma \,|\, A(t), \\
\Gamma \,|\, \exists x^\rho\, A(x) &:\equiv \text{exists a closed term } t^\rho\ \Gamma \,|\, A(t).
\end{aligned}$$

2. The *slash with q-truth* $|_{\mathrm{q}}$ [44, section 2.2] [75, section 3.1.19] (on $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$) is defined analogously to $\Gamma \,|\, A$ except for

$$\begin{aligned}
\Gamma \,|_{\mathrm{q}}\, (A \vee B) &:\equiv (\Gamma \,|_{\mathrm{q}}\, A \text{ and } \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A) \text{ or} \\
&\quad\ \ (\Gamma \,|_{\mathrm{q}}\, B \text{ and } \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash B), \\
\Gamma \,|_{\mathrm{q}}\, (A \to B) &:\equiv \Gamma \,|_{\mathrm{q}}\, A \text{ and } \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A \text{ implies } \Gamma \,|_{\mathrm{q}}\, B, \\
\Gamma \,|_{\mathrm{q}}\, \exists x\, A(x) &:\equiv \text{exists a closed term } t\ (\Gamma \,|_{\mathrm{q}}\, A(t) \text{ and } \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A(t)).
\end{aligned}$$

3. The *slash with t-truth* $|_{\mathrm{t}}$ [1, section 4] (on $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$) is defined analogously to $\Gamma \,|\, A$ except for

$$\begin{aligned}
\Gamma \,|_{\mathrm{t}}\, (A \to B) &:\equiv (\Gamma \,|_{\mathrm{t}}\, A \text{ implies } \Gamma \,|_{\mathrm{t}}\, B) \text{ and } \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A \to B, \\
\Gamma \,|_{\mathrm{t}}\, \forall x\, A(x) &:\equiv \text{for all closed terms } t\ \Gamma \,|_{\mathrm{t}}\, A(t) \text{ and } \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash \forall x\, A(x).
\end{aligned}$$

Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$.

11.4. Let us note that, contrarily to what is done for mrt, in $|_{\mathrm{t}}$ we added "and $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash \forall x\, A$" in the clause of $\forall$; this will be discussed later in chapter 13.

11.5. When we write something along the lines of "if $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$, then $\Gamma \,|_{\mathrm{t}}\, A$", we implicitly assume that the slash in question is based on $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$, that is the theory that appears in the clauses

$$\begin{aligned}
\Gamma \,|\, A_{\mathrm{at}} &:\equiv \mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A_{\mathrm{at}}, \\
\Gamma \,|_{\mathrm{t}}\, (A \to B) &:\equiv (\Gamma \,|_{\mathrm{t}}\, A \text{ implies } \Gamma \,|_{\mathrm{t}}\, B) \text{ and } \mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A \to B, \\
\Gamma \,|_{\mathrm{t}}\, \forall x\, A(x) &:\equiv \text{for all closed terms } t\ \Gamma \,|_{\mathrm{t}}\, A(t) \text{ and } \mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash \forall x\, A(x).
\end{aligned}$$

is $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$.

In the particular case of $|$, with respect to sentences, the theory $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$ in $\Gamma \,|\, A_{\mathrm{at}} :\equiv \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A_{\mathrm{at}}$, if consistent, does not matter, because HA^ω is complete with respect to atomic sentences (by point 2 of theorem 1.53).

11.6. We can think of $|$ as Tarski's definition of truth with:

1. Tarski's condition for atomic formulas "A_{at} is true" replaced by "A_{at} is provable";
2. in a model whose objects are exactly the closed terms, so that Tarski's condition for universal quantifications "for all objects x we have $A(x)$" becomes "for all closed terms t we have $A(t)$", and analogously for existential quantifications.

This relation between the slash and Tarski's definition of truth can be given a rigorous meaning: for all sentences A of HA^ω we have the equivalence $\emptyset \mid A \Leftrightarrow \mathbb{T}^\omega \models A$ (where $\mathbb{T}^\omega$ is the term model of HA^ω from definition 12.4).

11.7 Remark.

1. The slash with q-truth $|_{\mathrm{q}}$ has truth in the sense of: $\Gamma\,|_{\mathrm{q}}\, A$ implies $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$ for all disjunctive and existential formulas A of HA^ω [22, remark 4.2].
2. The slash with t-truth $|_{\mathrm{t}}$ has truth in the sense of: $\Gamma\,|_{\mathrm{t}}\, A$ implies $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$ for all formulas A of HA^ω [78, section 5.7 in chapter 3].

The slash with t-truth $|_{\mathrm{t}}$ is a $(*_1)$ strengthening of $|_{\mathrm{q}}$ which $(*_2)$ has truth for all formulas. This can be given a rigorous meaning: $(*_3)$ $\Gamma\,|_{\mathrm{t}}\, A \Leftrightarrow (\Gamma\,|_{\mathrm{q}}\, A$ and $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A)$, for all formulas A of $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$ [78, exercise 3.5.3 in chapter 3]. From $(*_3)$ we get: $\Gamma\,|_{\mathrm{t}}\, A \Rightarrow \Gamma\,|_{\mathrm{q}}\, A$, that is $(*_1)$; $\Gamma\,|_{\mathrm{t}}\, A \Rightarrow \mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$, that is $(*_2)$. Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$.

11.8 Proposition. The slash $|$ acts as the identity on quantifier-free sentences of $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$ in the sense of: $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A_{\mathrm{qf}} \Leftrightarrow \Gamma\,|\, A_{\mathrm{qf}}$, for all quantifier-free sentences A_{qf} of $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$. Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $|_{\mathrm{q}}$ and $|_{\mathrm{t}}$.

11.9 Proof. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth, with q-truth and with t-truth, by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12. We do the proof for $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$; the cases of the other theories are analogous. Let us abbreviate $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$ by $\Gamma \vdash A$. The proof is by induction on the structure of A_{qf}. We only see the case of $\to$; the cases of A_{at}, $\wedge$ and $\vee$ are analogous. Using point 2 of theorem 1.53 in right-to-left implication of the first equivalence (in the following way: if $\Gamma \vdash \neg A$ or $\Gamma \vdash B$, then $\Gamma \vdash A \to B$; if $\Gamma \vdash A$ and $\Gamma \vdash \neg B$, then $\Gamma \vdash B \wedge \neg B$, that is $\Gamma \vdash \bot$, so $\Gamma \vdash A \to B$), and induction hypothesis in the last equivalence, we get

$$\begin{aligned} \Gamma \vdash A \to B \;&\Leftrightarrow \\ (\Gamma \vdash A \text{ implies } \Gamma \vdash B) \text{ and } \Gamma \vdash (A \to B)^{\mathrm{t}} \;&\Leftrightarrow \\ (\Gamma \vdash A \text{ and } \Gamma \vdash A^{\mathrm{q}} \text{ implies } \Gamma \vdash B) \text{ and } \Gamma \vdash (A \to B)^{\mathrm{t}} \;&\Leftrightarrow \\ (\Gamma\,|\,A \text{ and } \Gamma \vdash A^{\mathrm{q}} \text{ implies } \Gamma\,|\,B) \text{ and } \Gamma \vdash (A \to B)^{\mathrm{t}} \;&\equiv \\ \Gamma\,|\,(A \to B).& \end{aligned}$$

11.10 Definition. Let A be a formula of $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$ and Γ be a set of formulas of $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$.

1. We denote a universal closure of A by $\bar{A}$.
2. We define $\bar{\Gamma} := \{\bar{B} : B \in \Gamma\}$.
3. We define $\Gamma \,|\, \bar{\Gamma}$ as meaning "$\Gamma \,|\, \bar{B}$ for all $B \in \Gamma$".

Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$.

11.3 Soundness

11.11 Theorem (soundness)**.** Let Γ be a set of formulas such that $\Gamma \,|\, \bar{\Gamma}$.

1. If $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$, then $\Gamma \,|\, \bar{A}$ [22, section 4]. Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \Gamma \pm \omega\mathsf{R}$.
2. If $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$, then $\Gamma \,|_{\mathrm{q}}\, \bar{A}$. Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$ [75, section 3.1.20].
3. If $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$, then $\Gamma \,|_{\mathrm{t}}\, \bar{A}$ [78, theorem 5.9 in chapter 3]. Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$.

11.12 Proof. Let us make some remarks. We make them for $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$ and $|$, but they also apply to the other theories and slashes.

1. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth, with q-truth and with t-truth, by means of $\mathrm{q}, \mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12.
2. Let us shorten $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$ by $\Gamma \vdash A$, $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A^{\mathrm{q}}$ by $\Gamma \vdash_{\mathrm{q}} A$ and $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A^{\mathrm{t}}$ by $\Gamma \vdash_{\mathrm{t}} A$. Also, let us shorten "$\Gamma \,|\, A$ and $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A^{\mathrm{q}}$" by $\Gamma \Vdash_{\mathrm{q}} A$, and "$\Gamma \,|\, A$ and $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A^{\mathrm{t}}$" by $\Gamma \Vdash_{\mathrm{t}} A$.
3. Say $\mathrm{FV}(A(\underline{\ell})) = \{\underline{\ell}\}$. When interpreting $A(\underline{\ell})$, we first take a universal closure $\bar{A} \equiv \forall \underline{\ell}\, A(\underline{\ell})$ of $A(\underline{\ell})$ and then we slash $\bar{A}$; this slash is equivalent to $(*)$ "for all closed terms $\underline{t}$ $\Gamma \,|\, A(\underline{t})$ and $\Gamma \vdash_{\mathrm{t}} \bar{A}$". We will do this implicitly by directly writing $(*)$. When we do it, we write "$\equiv$" instead of $\equiv$.
4. If $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash \bot$, then for all formulas A of $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$ we have $\Gamma \,|\, A$ [44, section 2.3].

   Let us prove the remark. The condition $\Gamma \,|\, A$ is a combination by means of "and", "or", "implies", "for all closed terms" and "exists a closed term" of "atomic" conditions of the form $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash A$. From the assumption we get that all theses "atomic" conditions are true, so $\Gamma \,|\, A$ is also true.
5. If $\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R} \vdash s =_0 t$, then $\Gamma \,|\, A(s) \;\Leftrightarrow\; \Gamma \,|\, A(t)$. The same holds for $\mathsf{E\text{-}HA}^\omega + \Gamma \pm \omega\mathsf{R}$ with $=_\rho$ instead of $=_0$. The proof is analogous to the proof of remark 4.

Let us prove the theorem by induction on the derivation of A.

$\underline{\bot \to A(\underline{\ell})}$ We have

$$\begin{gathered}\Gamma \,|\, \overline{\bot \to A(\underline{\ell})} \text{ “}\equiv\text{” for all closed terms } \underline{t} \\ \big((\Gamma \vdash \bot \text{ and } \Gamma \vdash_{\mathrm{q}} \bot \text{ implies } \Gamma \,|\, A(\underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} \bot \to A(\underline{t})\big) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{\bot \to A(\underline{\ell})}.\end{gathered}$$

Here we use remark 4.

$\underline{A(t(x,\underline{\ell}),\underline{\ell}) \to \exists x\, A(x,\underline{\ell})}$ We have

$$\begin{gathered}\Gamma \,|\, \overline{A(t(x,\underline{\ell}),\underline{\ell}) \to \exists x\, A(x,\underline{\ell})} \text{ “}\equiv\text{” for all closed terms } s', \underline{s} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(t(s',\underline{s}),\underline{s}) \text{ implies exists a closed term } r\ \Gamma \Vdash_{\mathrm{q}} A(r,\underline{s})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} A(t(r,\underline{s}),\underline{s}) \to \exists x\, A(x,\underline{s})\big) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{A(t(x,\underline{\ell}),\underline{\ell}) \to \exists x\, A(x,\underline{\ell})}.\end{gathered}$$

(actually, if $x \notin \mathrm{FV}\big(A(t(x,\underline{\ell}),\underline{\ell}) \to \exists x\, A(x,\underline{\ell})\big)$, then there is no quantification of s'). Analogously for $A \to A \wedge A$, $A \vee A \to A$, $A \wedge B \to A$, $A \to A \vee B$, $A \wedge B \to B \wedge A$, $A \vee B \to B \vee A$ and $\forall x\, A(x) \to A(t)$.

$\underline{A(\underline{\ell}) \to B(\underline{\ell}),\ B(\underline{\ell}) \to C(\underline{\ell}) \,/\, A(\underline{\ell}) \to C(\underline{\ell})}$ We have

$$\begin{gathered}\Gamma \,|\, \overline{A(\underline{\ell}) \to B(\underline{\ell})} \equiv \text{ for all closed terms } \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(\underline{t}) \text{ implies } \Gamma \,|\, B(\underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} A(\underline{t}) \to B(\underline{t})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \to B(\underline{\ell})},\end{gathered} \tag{11.1}$$

$$\begin{gathered}\Gamma \,|\, \overline{B(\underline{\ell}) \to C(\underline{\ell})} \equiv \text{ for all closed terms } \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} B(\underline{t}) \text{ implies } \Gamma \,|\, C(\underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} B(\underline{t}) \to C(\underline{t})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{B(\underline{\ell}) \to C(\underline{\ell})},\end{gathered} \tag{11.2}$$

$$\begin{gathered}\Gamma \,|\, \overline{A(\underline{\ell}) \to C(\underline{\ell})} \equiv \text{ for all closed terms } \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(\underline{t}) \text{ implies } \Gamma \,|\, C(\underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} A(\underline{t}) \to C(\underline{t})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \to C(\underline{\ell})}\end{gathered} \tag{11.3}$$

(actually, in (11.1) there are no quantifications of terms in $\underline{t}$ corresponding to variables of $\underline{\ell}$ that are in $\mathrm{FV}(C(\underline{\ell})) \setminus \big(\mathrm{FV}(A(\underline{\ell})) \cup \mathrm{FV}(B(\underline{\ell}))\big)$, and analogously for (11.2) and (11.3)). If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved $A(\underline{\ell}) \to B(\underline{\ell})$, so that the part $\Gamma \vdash_{\mathrm{q}} A(\underline{t})$ in (11.3) implies the part $\Gamma \vdash_{\mathrm{q}} B(\underline{t})$ in (11.2). Analogously for $A,\ A \to B \,/\, B$ and $A \to B \,/\, C \vee A \to C \vee B$.

$\underline{A(\underline{\ell}) \wedge B(\underline{\ell}) \to C(\underline{\ell}) \,/\, A(\underline{\ell}) \to (B(\underline{\ell}) \to C(\underline{\ell}))}$ We have

$$\begin{gathered}\Gamma \,|\, \overline{A(\underline{\ell}) \wedge B(\underline{\ell}) \to C(\underline{\ell})} \equiv \text{for all closed terms } \underline{t} \\ \big((\Gamma \,|\, A(\underline{t}) \text{ and } \Gamma \,|\, B(\underline{t}) \text{ and } \Gamma \vdash_{\mathrm{q}} A(\underline{t}) \wedge B(\underline{t}) \text{ implies } \Gamma \,|\, C(\underline{t})) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} A(\underline{t}) \wedge B(\underline{t}) \to C(\underline{t})\big) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \wedge B(\underline{\ell}) \to C(\underline{\ell})},\end{gathered} \tag{11.4}$$

$$\begin{gathered}\Gamma \,|\, \overline{A(\underline{\ell}) \to (B(\underline{\ell}) \to C(\underline{\ell}))} \equiv \text{for all closed terms } \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(\underline{t}) \text{ implies } (\Gamma \Vdash_{\mathrm{q}} B(\underline{t}) \text{ implies } \Gamma \,|\, C(\underline{t})) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} B(\underline{t}) \to C(\underline{t})\big) \text{ and } \Gamma \vdash_{\mathrm{t}} A(\underline{t}) \to (B(\underline{t}) \to C(\underline{t}))\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \to (B(\underline{\ell}) \to C(\underline{\ell}))}.\end{gathered} \tag{11.5}$$

If $\mathrm{t} = \mathrm{id}$, then we use that $\Gamma \,|\, A(\underline{t})$ implies $\Gamma \vdash A(\underline{t})$, so that the parts $\Gamma \,|\, A(\underline{t})$ in (11.5) and $\Gamma \vdash_{\mathrm{t}} A(\underline{t}) \wedge B(\underline{t}) \to C(\underline{t})$ in (11.4) together imply the part $\Gamma \vdash_{\mathrm{t}} B(\underline{t}) \to C(\underline{t})$ in (11.5). Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A(\underline{\ell}) \to B(x, \underline{\ell}) \,/\, A(\underline{\ell}) \to \forall x\, B(x, \underline{\ell})}$ We have

$$\begin{gathered}\Gamma \,|\, \overline{A(\underline{\ell}) \to B(x, \underline{\ell})} \equiv \text{for all closed terms } t', \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(\underline{t}) \text{ implies } \Gamma \,|\, B(t', \underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} A(\underline{t}) \to B(t', \underline{t})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \to B(x, \underline{\ell})},\end{gathered} \tag{11.6}$$

$$\begin{gathered}\Gamma \,|\, \overline{A(\underline{\ell}) \to \forall x\, B(x, \underline{\ell})} \equiv \text{for all closed terms } \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(\underline{t}) \text{ implies for all closed terms } s\ \Gamma \,|\, B(s, \underline{t}) \\ \text{and } \Gamma \vdash_{\mathrm{t}} \forall x\, B(x, \underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} A(\underline{t}) \to \forall x\, B(x, \underline{t})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \to \forall x\, B(x, \underline{\ell})}\end{gathered} \tag{11.7}$$

(actually, if $x \notin \mathrm{FV}(A(\underline{\ell}) \to B(x, \underline{\ell}))$, then there is no quantification of t' in (11.6) and the quantification of s in (11.7) is dummy). If $\mathrm{t} = \mathrm{id}$, then we use that $\Gamma \,|\, A(\underline{t})$ implies $\Gamma \vdash A(\underline{t})$, so that the parts $\Gamma \,|\, A(\underline{t})$ in (11.7) and $\Gamma \vdash_{\mathrm{t}} \overline{A(\underline{\ell}) \to B(x, \underline{\ell})}$ in (11.6) together imply the part $\Gamma \vdash_{\mathrm{t}} \forall x\, B(x, \underline{t})$ in (11.7). Analogously for $A \to B \,/\, \exists x\, A \to B$.

Axioms of $=_0$, S, Π, Σ and $\underline{\mathrm{R}}$ Let $A_{\mathrm{qf}}(\underline{\ell})$ be one of these axioms. We have

$$\Gamma \,|\, \overline{A_{\mathrm{qf}}(\underline{\ell})} \text{ ``}\equiv\text{'' for all closed terms } \underline{t}\ \Gamma \,|\, A_{\mathrm{qf}}(\underline{t}) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{A_{\mathrm{qf}}(\underline{\ell})}$$

so $\Gamma \,|\, \overline{A_{\mathrm{qf}}(\underline{\ell})}$ is equivalent to "for all closed terms $\underline{t}$ $\Gamma \vdash A_{\mathrm{qf}}(\underline{t})$ and $\Gamma \vdash_{\mathrm{t}} \overline{A_{\mathrm{qf}}(\underline{\ell})}$" by proposition 11.8, thus $\Gamma \,|\, \overline{A_{\mathrm{qf}}(\underline{\ell})}$ follows from the axiom $A_{\mathrm{qf}}(\underline{\ell})$ itself.

$\underline{A(0, \underline{\ell}),\ A(x, \underline{\ell}) \to A(\mathrm{S}x, \underline{\ell}) \,/\, A(x, \underline{\ell})}$ We can assume $x \in \mathrm{FV}(A)$, otherwise $A[0/x] \equiv$

A and so $\Gamma \,|\, \overline{A[0/x]} \equiv \Gamma \,|\, \bar{A}$. We have

$$\Gamma \,|\, \overline{A(0,\underline{\ell})} \equiv \text{for all closed terms } \underline{t}\ \Gamma \,|\, A(0,\underline{t}) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{A(0,\underline{\ell})}, \tag{11.8}$$

$$\begin{gathered} \Gamma \,|\, \overline{A(x,\underline{\ell}) \to A(\mathrm{S}x,\underline{\ell})} \text{ “}\equiv\text{” for all closed terms } t', \underline{t} \\ \big((\Gamma \Vdash_{\mathrm{q}} A(t',\underline{t}) \text{ implies } \Gamma \,|\, A(\mathrm{S}t',\underline{t})) \text{ and } \Gamma \vdash_{\mathrm{t}} A(t',\underline{t}) \to A(\mathrm{S}t',\underline{t})\big) \text{ and} \\ \Gamma \vdash_{\mathrm{t}} \overline{A(x,\underline{\ell}) \to A(\mathrm{S}x,\underline{\ell})}, \end{gathered} \tag{11.9}$$

$$\Gamma \,|\, \overline{A(x,\underline{\ell})} \equiv \text{for all closed terms } t', \underline{t}\ \Gamma \,|\, A(t',\underline{t}) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{A(x,\underline{\ell})}. \tag{11.10}$$

By point 3 of theorem 1.30 and remark 5, to prove (11.10) it suffices to prove it when t' is a numeral $\bar{n}$. We do this by induction on n using (11.8) and (11.9). If $\mathrm{q} = \mathrm{id}$, then we use the assumption that we proved A, so as to have the part $\Gamma \vdash_{\mathrm{q}} A(t',\underline{t})$ in (11.9).

$\underline{A_{\mathrm{at}} \to s = t \,/\, A_{\mathrm{at}} \to r(s) =_0 r(t)}$ Using the assumption that we proved the premise of the rule, then we proved the conclusion of the rule, so $\Gamma \,|\, \overline{A_{\mathrm{at}} \to r(s) =_0 r(t)}$ arguing as in the case of the axioms of $=_0$, S, Π, Σ and $\underline{\mathrm{R}}$.

$\underline{z \approx z}$ We have

$$\Gamma \,|\, \overline{z \approx z} \equiv \text{for all closed terms } t\ \Gamma \,|\, (t \approx t) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{z \approx z}.$$

At this point of the proof, we already proved the following: if $\mathsf{HA}^\omega \vdash A$, then $\Gamma \,|\, \bar{A}$ (where the slash is on $\mathsf{E}\text{-}\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$). So, since $\mathsf{HA}^\omega \vdash t \approx t$ by point 6 of proposition 1.26, then $\Gamma \,|\, (t \approx t)$ (where the slash is on $\mathsf{E}\text{-}\mathsf{HA}^\omega + \Gamma \pm \omega\mathsf{R}$).

$\underline{\omega\mathsf{R}}$ We have

$$\Gamma \,|\, \overline{A(t,\underline{\ell})} \text{ “}\equiv\text{” for all closed terms } \underline{s}\ \Gamma \,|\, A(t,\underline{s}) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{A(t,\underline{\ell})},$$

$$\begin{gathered} \Gamma \,|\, \overline{\forall x\, A(x,\underline{\ell})} \text{ “}\equiv\text{” for all closed terms } \underline{s} \\ (\text{for all closed terms } t\ \Gamma \,|\, A(t,\underline{s}) \text{ and } \Gamma \vdash_{\mathrm{t}} \forall x\, A(x,\underline{s})) \text{ and } \Gamma \vdash_{\mathrm{t}} \overline{\forall x\, A(x,\underline{\ell})}. \end{gathered}$$

$\underline{A(\underline{\ell}) \vee \neg A(\underline{\ell})}$ Let $\mathrm{q} = \top$ and $\mathrm{t} = \top$. We have

$$\begin{gathered} \Gamma \,|\, (A(\underline{\ell}) \vee \neg A(\underline{\ell})) \text{ “}\equiv\text{”} \\ \text{for all closed terms } \underline{t}\ \big(\Gamma \,|\, A(\underline{t}) \text{ or } (\Gamma \,|\, A(\underline{t}) \text{ implies } \Gamma \vdash \bot)\big). \end{gathered}$$

$\underline{\Gamma}$ By hypothesis we have $\Gamma \,|\, \bar{\Gamma}$.

11.13. We need the terms t in

$$\Gamma \,|\, \forall x\, A(x) :\equiv \text{for all closed terms } t\ \Gamma \,|\, A(t)$$

to be closed because when verifying the induction rule in the proof of the soundness theorem we need to reduce t to a numeral. Once settled that the terms need to be closed, we are forced to take universal closures of all formulas, otherwise we could not show

$$\Gamma \,|\, (\forall x\, A(x) \to A(x)) \equiv \text{for all closed terms } t \ \Gamma \,|\, A(t) \text{ implies } \Gamma \,|\, A(x)$$

to be true by taking t to be the non-closed term x. But taking a universal closure, the slash will replace the free variable x by closed terms s, and then we can take q to be the closed term s:

$$\begin{gathered}\Gamma \,|\, \overline{\forall x\, A(x) \to A(x)} \equiv \text{for all closed terms } s \\ (\text{for all closed terms } t \ \Gamma \,|\, A(t) \text{ implies } \Gamma \,|\, A(s))\end{gathered}$$

(where $\mathrm{FV}(A(x)) = \{x\}$).

11.14. The slashes $|_{\mathrm{q}}$ and $|_{\mathrm{t}}$ on PA^ω do not interpret PA^ω because their truth prevents them from interpreting the law of excluded middle. Indeed, in the case of $|_{\mathrm{t}}$, if for all sentences A of PA^ω the slash

$$\emptyset \,|_{\mathrm{t}}\, (A \vee \neg A) \equiv \emptyset \,|_{\mathrm{t}}\, A \text{ or } \emptyset \,|_{\mathrm{t}}\, \neg A$$

were true, then (by truth) for all sentences A of PA^ω we would have

$$\mathsf{PA}^\omega \vdash A \text{ or } \mathsf{PA}^\omega \vdash \neg A,$$

so PA^ω would be complete, which is false. Analogously for $\mathsf{WE\text{-}PA}^\omega$, $\mathsf{E\text{-}PA}^\omega$ and $|_{\mathrm{q}}$.

11.4 Characterisation

11.15 Theorem (characterisation)**.** Let Γ be a set of formulas of $\mathsf{HA}^\omega + \omega\mathsf{R}$. For all formulas A of $\mathsf{HA}^\omega + \Gamma + \omega\mathsf{R}$, we have the equivalence $\mathsf{HA}^\omega + \Gamma + \omega\mathsf{R} \vdash \bar{A} \Leftrightarrow \Gamma \,|\, \bar{A}$. Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{E\text{-}PA}^\omega + \Gamma + \omega\mathsf{R}$, $|_{\mathrm{q}}$ and $|_{\mathrm{t}}$ [78, sections 5.7 and 5.9 in chapter 3].

11.16 Proof. Let us do the proof only for $\mathsf{HA}^\omega + \Gamma + \omega\mathsf{R}$; the cases of the other theories are analogous. Since $\bar{A}$ is a sentence, it suffices to prove the theorem for sentences A. We adopt here the remarks made in the beginning of proof 11.12. We assume $\mathsf{HA}^\omega + \Gamma + \omega\mathsf{R} \nvdash \bot$, otherwise the claim of the theorem follows from remark 4 in proof 11.12. Let us denote "not $\Gamma \,|\, A$" by $\Gamma \nmid A$. We prove

$$\begin{aligned}\Gamma \,|\, A &\Leftrightarrow \Gamma \vdash A,\\ \Gamma \nmid A &\Leftrightarrow \Gamma \vdash \neg A,\end{aligned}$$

by simultaneous induction on the structure of the sentence A [68, suggested for the different context of proof 12.13]. Note that the equivalences imply $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.

$\underline{A_{\mathrm{at}}}$ We have $\Gamma \,|\, A_{\mathrm{at}} \equiv \Gamma \vdash A_{\mathrm{at}}$.

Using point 2 of theorem 1.53 in the equivalence, we get

$$\begin{aligned}\Gamma \nmid A_{\mathrm{at}} &\equiv\\ \Gamma \nvdash A_{\mathrm{at}} &\Leftrightarrow\\ \Gamma \vdash \neg A_{\mathrm{at}}&.\end{aligned}$$

$\underline{\to}$ From the induction hypothesis we get ($*_1$) $\Gamma \vdash A$ or $\Gamma \vdash \neg A$, and ($*_2$) $\Gamma \vdash B$ or $\Gamma \vdash \neg B$.

Using induction hypothesis in the first equivalence, and ($*_1$) in the third equivalence, we get

$$\begin{aligned}
\Gamma \,|\, (A \to B) \;&\equiv\\
(\Gamma \,|\, A \text{ and } \Gamma \vdash_{\mathrm{q}} A \text{ implies } \Gamma \,|\, B) \text{ and } \Gamma \vdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
(\Gamma \vdash A \text{ and } \Gamma \vdash_{\mathrm{q}} A \text{ implies } \Gamma \vdash B) \text{ and } \Gamma \vdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
(\Gamma \vdash A \text{ implies } \Gamma \vdash B) \text{ and } \Gamma \vdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
\Gamma \vdash A \to B \text{ and } \Gamma \vdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
\Gamma \vdash A \to B.&
\end{aligned}$$

Using induction hypothesis in the second equivalence, and ($*_1$) and ($*_2$) in the last two equivalences, we get

$$\begin{aligned}
\Gamma \nmid (A \to B) \;&\Leftrightarrow\\
(\Gamma \,|\, A \text{ and } \Gamma \vdash_{\mathrm{q}} A \text{ and } \Gamma \nmid B) \text{ or } \Gamma \nvdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
(\Gamma \vdash A \text{ and } \Gamma \vdash_{\mathrm{q}} A \text{ and } \Gamma \vdash \neg B) \text{ or } \Gamma \nvdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
(\Gamma \vdash A \text{ and } \Gamma \vdash \neg B) \text{ or } \Gamma \nvdash_{\mathrm{t}} A \to B \;&\Leftrightarrow\\
\Gamma \vdash A \text{ and } \Gamma \vdash \neg B \;&\Leftrightarrow\\
\Gamma \vdash \neg(A \to B).&
\end{aligned}$$

Analogously for $\wedge$ and $\vee$.

$\underline{\forall}$ As before, we have $\Gamma \vdash A(t)$ or $\Gamma \vdash \neg A(t)$ for all closed terms t.

We have that ($*_1$) $\Gamma \vdash \neg\forall x\, A(x)$ implies that there exists a closed term t such that $\Gamma \vdash \neg A(t)$: if the conclusion is false, then for all closed terms t we have $\Gamma \vdash A(t)$, so $\Gamma \vdash \forall x\, A(x)$ by $\omega\mathsf{R}$, thus the premise is false.

We have that ($*_2$) $\Gamma \nvdash \forall x\, A(x)$ implies $\Gamma \vdash \neg\forall x\, A(x)$: if for all closed terms t we would have $\Gamma \vdash A(t)$, then by $\omega\mathsf{R}$ we would have $\Gamma \vdash \forall x\, A(x)$ contradicting the premise, therefore for some closed term t we have $\Gamma \vdash \neg A(t)$, thus $\Gamma \vdash \neg\forall x\, A(x)$.

Using induction hypothesis in the first equivalence, and $\omega\mathsf{R}$ in the second equivalence, we get

$$\begin{aligned}
\Gamma \,|\, \forall x\, A(x) \;&\equiv\\
\text{for all closed terms } t\ \Gamma \,|\, A(t) \text{ and } \Gamma \vdash_{\mathrm{t}} \forall x\, A(x) \;&\Leftrightarrow\\
\text{for all closed terms } t\ \Gamma \vdash A(t) \text{ and } \Gamma \vdash_{\mathrm{t}} \forall x\, A(x) \;&\Leftrightarrow\\
\Gamma \vdash \forall x\, A(x) \text{ and } \Gamma \vdash_{\mathrm{t}} \forall x\, A(x) \;&\Leftrightarrow\\
\Gamma \vdash \forall x\, A(x).&
\end{aligned}$$

Using induction hypothesis in the second equivalence, $(*_1)$ in the third equivalence, and $(*_2)$ in the last equivalence, we get

$$\begin{aligned} \Gamma \nmid \forall x\, A(x) \; &\Leftrightarrow \\ \text{exists a closed term } t\ \Gamma \nmid A(t) \text{ or } \Gamma \nvdash_{\mathrm{t}} \forall x\, A(x) \; &\Leftrightarrow \\ \text{exists a closed term } t\ \Gamma \vdash \neg A(t) \text{ or } \Gamma \nvdash_{\mathrm{t}} \forall x\, A(x) \; &\Leftrightarrow \\ \Gamma \vdash \neg\forall x\, A(x) \text{ or } \Gamma \nvdash_{\mathrm{t}} \forall x\, A(x) \; &\Leftrightarrow \\ \Gamma \vdash \neg\forall x\, A(x). \end{aligned}$$

Analogously for $\exists$.

11.17 Remark. The characterisation theorem of $|$ ensures that the soundness theorem of $|$ is optimal, in the sense that the theory $\mathsf{HA}^\omega + \Gamma + \omega\mathsf{R}$ there considered is the strongest theory T such that $\mathsf{T} \vdash A \;\Rightarrow\; \Gamma \,|\, A$ where the slash $\Gamma \,|\, A$ is on $\mathsf{HA}^\omega + \Gamma + \omega\mathsf{R}$ (analogously to remark 3.16). Analogously for $\mathsf{WE\text{-}HA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{PA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \Gamma + \omega\mathsf{R}$, $\mathsf{E\text{-}PA}^\omega + \Gamma + \omega\mathsf{R}$, $|_{\mathrm{q}}$ and $|_{\mathrm{t}}$.

11.18. The characterisation theorem of $|_{\mathrm{t}}$ tells us that $|_{\mathrm{t}}$ (on $\mathsf{PA}^\omega + \omega\mathsf{R}$) interprets PA^ω, but we saw in paragraph 11.14 that $|_{\mathrm{t}}$ (on PA^ω) does not interpret PA^ω. This is not a contradiction (because the former slash is on $\mathsf{PA}^\omega + \omega\mathsf{R}$ and the latter slash is on PA^ω), but may look confusing. So it may help to say that the reason why $|_{\mathrm{t}}$ (on $\mathsf{PA}^\omega + \omega\mathsf{R}$) interprets $\mathsf{PA}^\omega + \omega\mathsf{R}$ is because $\mathsf{PA}^\omega + \omega\mathsf{R}$ is complete (as we will prove in theorem 12.8), and the reason why $|_{\mathrm{t}}$ (on PA^ω) does not interpret PA^ω is because PA^ω is incomplete (as we saw in paragraph 11.14). Analogously for $|_{\mathrm{q}}$.

11.5 Applications

11.19 Theorem (disjunction property and existence property)**.** Let $\mathsf{T} := \mathsf{HA}^\omega \pm \omega\mathsf{R}$.

1. Let A and B be sentences of T. If $\mathsf{T} \vdash A \vee B$, then $\mathsf{T} \vdash A$ or $\mathsf{T} \vdash B$ [75, section 3.1.20].

2. Let $\exists \underline{x}\, A(\underline{x})$ be a sentence of T. If $\mathsf{T} \vdash \exists \underline{x}\, A(\underline{x})$, then there exist closed terms $\underline{t}$ of T such that $\mathsf{T} \vdash A(\underline{t})$ [75, section 3.1.20].

Analogously for $\mathsf{WE\text{-}HA}^\omega \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}HA}^\omega \pm \omega\mathsf{R}$ [75, section 3.1.20], $\mathsf{PA}^\omega + \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \omega\mathsf{R}$.

11.20 Proof. We do two slightly different proofs: one for the intuitionistic theories and another one for the classical theories.

<u>$\mathsf{HA}^\omega \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}HA}^\omega \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}HA}^\omega \pm \omega\mathsf{R}$</u>

1. Assuming the premise of the theorem, by the soundness theorem of $|_{\mathrm{t}}$ we have $\emptyset \,|_{\mathrm{t}}\, (A \vee B) \equiv$ "$\emptyset \,|_{\mathrm{t}}\, A$ or $\emptyset \,|_{\mathrm{t}}\, B$". By truth we get the conclusion of the theorem.
2. Assuming the premise of the theorem, by the soundness theorem of $|_{\mathrm{t}}$ we have $\emptyset \,|_{\mathrm{t}}\, \exists \underline{x}\, A(\underline{x}) \equiv$ "exists a closed term $\underline{t}$ $\emptyset \,|_{\mathrm{t}}\, A(\underline{t})$". By truth we get the conclusion of the theorem.

$\mathsf{PA}^\omega + \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \omega\mathsf{R}$ Analogous to the previous proof but using $|$ and its characterisation theorem instead of $|_{\mathrm{t}}$ and its truth.

11.21. In chapter 12 we are going to see that $\mathsf{HA}^\omega + \omega\mathsf{R}$ even has a property stronger than the disjunction and existence properties: it is a complete theory.

11.6 Conclusion

11.22. We saw three slashes $|$, $|_{\mathrm{q}}$ and $|_{\mathrm{t}}$ which interpret the internal symbols

$$A_{\mathrm{at}}, \qquad \wedge, \qquad \vee, \qquad \to, \qquad \forall, \qquad \exists$$

as the metalevel symbols

$$\Gamma \vdash A_{\mathrm{at}}, \quad \text{and}, \quad \text{or}, \quad \text{implies}, \quad \text{for all closed terms}, \quad \text{exists a closed term},$$

resembling Tarski's definition of truth (with some provability hardwired). The main results about the slash are the following.

Soundness theorem This theorem says that we can use the slash to guarantee the existence of computational content from proofs in $\mathsf{E\text{-}PA}^\omega + \omega\mathsf{R}$.

Characterisation theorem This theorem guarantees that the soundness theorem is optimal.

Applications We used the slash to do applications on:

1. disjunction property;
2. existence property.

Part III

Theoretical contributions

Chapter 12

Completeness and ω-rule

12.1 Introduction

12.1. In this chapter we introduce the term model $\mathbb{T}^\omega$ of $\mathsf{PA}^\omega + \omega\mathsf{R}$ and prove the completeness of $\mathsf{PA}^\omega + \omega\mathsf{R}$:

Syntactic completeness for all sentences A we have $\mathsf{PA}^\omega + \omega\mathsf{R} \vdash A$ or $\mathsf{PA}^\omega + \omega\mathsf{R} \vdash \neg A$;

Semantic completeness for all sentences A we have $\mathsf{PA}^\omega + \omega\mathsf{R} \vdash A \;\Leftrightarrow\; \mathbb{T}^\omega \models A$.

The chapter is admittedly light since the results and proofs are quite simple. Despite this, the results have some interest from a historical perspective as they relate to Hilbert's program.

12.2 Hilbert's program and ω-rule

12.2. In the early 20th century there were attempts to ground all mathematics in secure foundations (for example, naive set theory), but unsuccessfully due to paradoxes (such as Russell's paradox). David Hilbert proposed a foundation in his famous program [81], namely to ground mathematics in a system consisting of the following.

Language A well defined and precise language in which all statements should be written.

Axioms and rules A well defined and precise list of axioms and rules according to which the statements in the language should be manipulated.

Moreover, the system should have the following properties.

Completeness The system should prove all true statements in the language.

Consistency The system should not fall into contradiction, and this fact should be proved using only "finitary methods". (The exact meaning of "finitary methods" is open to interpretation, but they should be methods obviously true, easy to survey and working with finite objects.)

Conservation If a statement about "real objects" (concrete, finite and easy to survey objects such as natural numbers) is proved using "ideal objects" (abstract, infinite and difficult to survey objects such as infinite sets), then in principle we can eliminate the use of "ideal objects", obtaining a proof in the system that only uses "real objects".

Decidability There should be an algorithm that correctly decides the truth value of any statement in the language.

Gödel's incompleteness theorems showed that Hilbert's program is unattainable for the following reasons.

1. Gödel's first incompleteness theorem implies that Hilbert's system, being decidable and consistent, cannot be complete.

2. Gödel's second incompleteness theorem implies that Hilbert's "finitary methods", which (independently of what they are) were believed to be formalisable in Peano arithmetic PA, do not prove the consistency of a system grounding all mathematics.

One day after Gödel's announcement of his first incompleteness theorem, Hilbert gives a talk where he claims that there is no *ignorabimus* (impossibility to know the truth) in mathematics, in contradiction to Gödel's theorem [9, pages 69 and 71]. In the same talk, Hilbert proposes to add the ω-rule $\omega\mathsf{R}'$

$$\frac{A(\bar{0}) \quad A(\bar{1}) \quad A(\bar{2}) \quad \dots}{\forall x\, A(x)}$$

to PA in order to get a complete theory: $\mathbb{N} \models A \;\Leftrightarrow\; \mathsf{PA}{+}\omega\mathsf{R}' \vdash A$ [35, pages 491–492]. (Hilbert is not very clear; under one possible interpretation he seems to argue completeness only for Π^0_1 sentences A.) The ω-rule turns proofs into infinite objects without a provability predicate, putting them out of the range of Gödel's incompleteness theorems.

12.3 Term model

12.3. The rule $\omega\mathsf{R}$ gives us the left-to-right implication in the equivalence

$$\text{for all closed terms } t^\rho\ A(t) \quad \Leftrightarrow \quad \forall x^\rho\, A(x).$$

This equivalence suggests that for $\omega\mathsf{R}$ to be true in a model of HA^ω, the objects of type ρ of the model should be exactly the closed terms of type ρ. There is such a model: the term model.

12.4 Definition. The *term model* $\mathbb{T}^\omega$ [75, definition 2.5.1] is a model of HA^ω defined as follows.

1. The universe of $\mathbb{T}^\omega$ is the set of all closed terms of HA^ω.

2. The constants 0, S, Π, Σ and R_i are interpreted as themselves.

3. Term application $s^{\rho\sigma}, t^{\sigma} \rightsquigarrow (st)^{\rho}$ is interpreted as itself.

4. Equality $=_0$ is interpreted as: $s =_0 t \;\Leftrightarrow\; s^{\mathrm{n}} \equiv t^{\mathrm{n}}$.

12.5 Proposition. The term model $\mathbb{T}^{\omega}$ is a model of $\mathsf{HA}^{\omega} \pm \omega\mathsf{R}$ [75, definition 2.5.1], $\mathsf{WE\text{-}HA}^{\omega} \pm \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^{\omega} \pm \omega\mathsf{R}$, $\mathsf{PA}^{\omega} \pm \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^{\omega} \pm \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^{\omega} \pm \omega\mathsf{R}$.

12.6 Proof. It suffices to prove the proposition for $\mathsf{E\text{-}PA}^{\omega} + \omega\mathsf{R}$, since the other theories are subtheories of $\mathsf{E\text{-}PA}^{\omega} + \omega\mathsf{R}$. The logical axioms and rules (given in table 1.1 plus LEM) hold true in any model, so we only have to verify the arithmetical axioms and rules (given in table 1.3), $\forall z\,(z \approx z)$ (see point 3 of proposition 1.26) and $\omega\mathsf{R}$. First, let us make some remarks.

1. When interpreting in $\mathbb{T}^{\omega}$ a formula, we implicitly interpret an universal closure of the formula.

2. We have $\mathbb{T}^{\omega} \models x =_0 y \to (A[x/z] \leftrightarrow A[y/z])$.

   Let us prove this claim by induction on the structure of A. We only see the base case $A_{\mathrm{at}}(z, \underline{\ell}) \equiv s(z, \underline{\ell}) =_0 t(z, \underline{\ell})$, where $\mathrm{FV}(A_{\mathrm{at}}(z, \underline{\ell})) \subseteq \{z, \underline{\ell}\}$, since the case $A \equiv \perp$ and the induction step are easy. The interpretation of $x =_0 y \to (s(x, \underline{\ell}) =_0 t(x, \underline{\ell}) \leftrightarrow s(y, \underline{\ell}) =_0 t(y, \underline{\ell}))$ is "for all closed terms $\underline{p}$, q and r, if $q^{\mathrm{n}} \equiv r^{\mathrm{n}}$ then: $s(q, \underline{p})^{\mathrm{n}} \equiv t(q, \underline{p})^{\mathrm{n}} \;\Leftrightarrow\; s(r, \underline{p})^{\mathrm{n}} \equiv t(r, \underline{p})^{\mathrm{n}}$". It is true because if $q^{\mathrm{n}} \equiv r^{\mathrm{n}}$, then $s(q, \underline{p}), s(r, \underline{p}) \succeq s(q^{\mathrm{n}}, \underline{p}) \equiv s(r^{\mathrm{n}}, \underline{p})$ and $t(q, \underline{p}), t(r, \underline{p}) \succeq t(q^{\mathrm{n}}, \underline{p}) \equiv t(r^{\mathrm{n}}, \underline{p})$, so $s(q, \underline{p})^{\mathrm{n}} \equiv s(r, \underline{p})^{\mathrm{n}}$ and $t(q, \underline{p})^{\mathrm{n}} \equiv t(r, \underline{p})^{\mathrm{n}}$.

$\underline{x =_0 x}$ Its interpretation is the true "for all closed terms t^0 we have $t^{\mathrm{n}} \equiv t^{\mathrm{n}}$".

$\underline{x =_0 y \wedge A_{\mathrm{at}}[x/z] \to A_{\mathrm{at}}[y/z]}$ Follows from remark 2.

$\underline{\mathrm{S}x \neq_0 0}$ Its interpretation is "for all closed terms t^0 we have $(\mathrm{S}t)^{\mathrm{n}} \not\equiv 0^{\mathrm{n}}$". It is true because $(\mathrm{S}t)^{\mathrm{n}} \equiv \mathrm{S}t^{\mathrm{n}} \not\equiv 0$.

$\underline{\mathrm{S}x =_0 \mathrm{S}y \to x =_0 y}$ Its interpretation is "for all closed terms s^0 and t^0, if $(\mathrm{S}s)^{\mathrm{n}} \equiv (\mathrm{S}t)^{\mathrm{n}}$ then $s^{\mathrm{n}} \equiv t^{\mathrm{n}}$". It is true because $(\mathrm{S}s)^{\mathrm{n}} \equiv \mathrm{S}s^{\mathrm{n}}$ and $(\mathrm{S}t)^{\mathrm{n}} \equiv \mathrm{S}t^{\mathrm{n}}$.

$\underline{A_{\mathrm{at}}[\underline{\mathrm{R}}(\mathrm{S}x)\underline{y}\underline{z}/\underline{w}] \leftrightarrow A_{\mathrm{at}}[\underline{z}(\underline{\mathrm{R}}x\underline{y}\underline{z})x/\underline{w}]}$ Let us say $A_{\mathrm{at}}(\underline{w}, \underline{\ell}) \equiv s(\underline{w}, \underline{\ell}) =_0 t(\underline{w}, \underline{\ell})$, where $\mathrm{FV}(A_{\mathrm{at}}(\underline{w}, \underline{\ell})) \subseteq \{\underline{w}, \underline{\ell}\}$ (the case $A_{\mathrm{at}} \equiv \perp$ is trivial). The interpretation of the axiom is "for all closed terms $\underline{o}$, p, $\underline{q}$ and $\underline{r}$ we have: $s(\underline{\mathrm{R}}(\mathrm{S}p)\underline{q}\underline{r}, \underline{o})^{\mathrm{n}} \equiv t(\underline{\mathrm{R}}(\mathrm{S}p)\underline{q}\underline{r}, \underline{o})^{\mathrm{n}} \;\Leftrightarrow\; s(\underline{r}(\underline{\mathrm{R}}p\underline{q}\underline{r})p, \underline{o})^{\mathrm{n}} \equiv t(\underline{r}(\underline{\mathrm{R}}p\underline{q}\underline{r})p, \underline{o})^{\mathrm{n}}$". This interpretation is true since $s(\underline{\mathrm{R}}(\mathrm{S}p)\underline{q}\underline{r}, \underline{o})^{\mathrm{n}} \equiv s(\underline{r}(\underline{\mathrm{R}}p\underline{q}\underline{r})p, \underline{o})^{\mathrm{n}}$ and $t(\underline{\mathrm{R}}(\mathrm{S}p)\underline{q}\underline{r}, \underline{o})^{\mathrm{n}} \equiv t(\underline{r}(\underline{\mathrm{R}}p\underline{q}\underline{r})p, \underline{o})^{\mathrm{n}}$ because $\mathrm{R}_i(\mathrm{S}p)\underline{q}\underline{r} \succeq r_i(\mathrm{R}_i p\underline{q}\underline{r})p$. Analogously for the other axioms of $\underline{\mathrm{R}}$ and the axioms of Π and Σ.

$\underline{A(x),\ A(x) \to A(\mathrm{S}x) \,/\, A(x)}$ Its interpretation is

$$
\begin{array}{l}
\quad\text{if } \mathbb{T}^{\omega} \models A(0) \text{ and} \\
\qquad\text{for all closed terms } s \text{ we have } (\mathbb{T}^{\omega} \models A(s) \;\Rightarrow\; \mathbb{T}^{\omega} \models A(\mathrm{S}s)) \\
\text{then for all closed terms } t \text{ we have } \mathbb{T}^{\omega} \models A(t).
\end{array}
$$

It suffices to prove the conclusion for numerals: by point 3 of theorem 1.30 we have $t =_0 \bar{n}$ for some $n \in \mathbb{N}$, thus $\mathbb{T}^\omega \models A(t)$ is equivalent to $\mathbb{T}^\omega \models A(\bar{n})$ by remark 2. From the premise we get $\mathbb{T}^\omega \models A(\bar{n})$ for all $n \in \mathbb{N}$ by induction on n.

$\underline{z \approx z}$ Its interpretation is "for all closed terms t we have $\mathbb{T}^\omega \models t \approx t$". At this point of the proof we already proved that $\mathbb{T}^\omega$ is a model of HA^ω. So $\mathbb{T}^\omega \models t \approx t$ for all closed terms t by point 6 of proposition 1.26.

$\underline{\omega\mathsf{R}}$ The interpretations of the premises and conclusion of the rule coincide.

12.4 Completeness

12.7. We are going to prove the completeness of the intuitionistic theories $\mathsf{HA}^\omega + \omega\mathsf{R}$, $\mathsf{WE\text{-}HA}^\omega + \omega\mathsf{R}$ and $\mathsf{E\text{-}HA}^\omega + \omega\mathsf{R}$, and the classical theories $\mathsf{PA}^\omega + \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \omega\mathsf{R}$. We give five different proofs. Some proofs work for both the classical and intuitionistic theories, other ones only work for the classical theories. Some proofs use as tool the slash, others use as tool provability, and others use as tool truth (in $\mathbb{T}^\omega$). We summarise this in table 12.1.

	theories		tools		
	intuitionistic	classical	slash	provability	truth
proof 12.9		✓	✓		
proof 12.10	✓	✓	✓		
proof 12.11		✓		✓	
proof 12.12	✓	✓		✓	
proof 12.13	✓	✓			✓

Table 12.1: proofs of completeness, the theories for which they work, and the tools that they use.

12.8 Theorem.

1. The theory $\mathsf{HA}^\omega + \omega\mathsf{R}$ is syntactically complete: for all sentences A of $\mathsf{HA}^\omega + \omega\mathsf{R}$ we have $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$ or $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$.

2. The theory $\mathsf{HA}^\omega + \omega\mathsf{R}$ is semantically complete with respect to $\mathbb{T}^\omega$: for all sentences A of $\mathsf{HA}^\omega + \omega\mathsf{R}$ we have $\mathbb{T}^\omega \models A \;\Leftrightarrow\; \mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$.

Analogously for $\mathsf{WE\text{-}HA}^\omega + \omega\mathsf{R}$, $\mathsf{E\text{-}HA}^\omega + \omega\mathsf{R}$, $\mathsf{PA}^\omega + \omega\mathsf{R}$, $\mathsf{WE\text{-}PA}^\omega + \omega\mathsf{R}$ and $\mathsf{E\text{-}PA}^\omega + \omega\mathsf{R}$.

12.9 Proof. This proof only works for the classical theories. We only do the proof for $\mathsf{PA}^\omega + \omega\mathsf{R}$; the cases of the other classical theories are analogous.

1. We have $\mathsf{PA}^\omega + \omega\mathsf{R} \vdash A \vee \neg A$, so $\emptyset \mid (A \vee \neg A)$, that is "$\emptyset \mid A$ or $\emptyset \mid \neg A$", by the soundness theorem of $\mid$. Then $\mathsf{PA}^\omega + \omega\mathsf{R} \vdash A$ or $\mathsf{PA}^\omega + \omega\mathsf{R} \vdash \neg A$ by the characterisation theorem of $\mid$.

2. This point follows from the previous one since $\mathbb{T}^\omega$ is a model of $\mathsf{PA}^\omega + \omega\mathsf{R}$.

12.10 Proof. We only do the proof for $\mathsf{HA}^\omega + \omega\mathsf{R}$; the cases of the other theories are analogous.

1. We have $\emptyset \mid A$ or $\emptyset \nmid A$, where $\emptyset \nmid A$ implies $\emptyset \mid \neg A$. So by the characterisation theorem of $\mid$ we get $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$ or $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$.

2. This point follows from the previous one since $\mathbb{T}^\omega$ is a model of $\mathsf{HA}^\omega + \omega\mathsf{R}$.

12.11 Proof. This proof only works for the classical theories. We only do the proof for $\mathsf{PA}^\omega + \omega\mathsf{R}$; the cases of the other classical theories are analogous. We prove first the last point of the theorem.

2. The right-to-left implication follows from $\mathbb{T}^\omega$ being a model of $\mathsf{PA}^\omega + \omega\mathsf{R}$. Let us see the left-to-right implication. In $\mathbb{T}^\omega$ and in $\mathsf{PA}^\omega + \omega\mathsf{R}$ every formula A is equivalent to a formula $A^p \equiv Q_1 x_1 \ldots Q_n x_n \, A_{\mathrm{qf}}(x_1, \ldots, x_n)$ in prenex normal form, where $Q_1, \ldots, Q_n \in \{\forall, \exists\}$. So it suffices to prove $\mathbb{T}^\omega \models A^p \;\Rightarrow\; \mathsf{PA}^\omega + \omega\mathsf{R} \vdash A^p$ by induction on the structure of sentences in prenex normal form.

   $\underline{A_{\mathrm{qf}}}$ This case follows from point 2 of theorem 1.53 and from $\mathbb{T}^\omega$ being a model of $\mathsf{PA}^\omega + \omega\mathsf{R}$.

   $\underline{\forall}$ Using the induction hypothesis in the second implication (note that $A(t)$ is a sentence because $\forall x\, A(x)$ is a sentence) and $\omega\mathsf{R}$ in the third implication, we get

   $$\begin{aligned} \mathbb{T}^\omega \models \forall x\, A(x) \;&\Rightarrow \\ \text{for all closed terms } t \;\; \mathbb{T}^\omega \models A(t) \;&\Rightarrow \\ \text{for all closed terms } t \;\; \mathsf{PA}^\omega + \omega\mathsf{R} \vdash A(t) \;&\Rightarrow \\ \mathsf{PA}^\omega + \omega\mathsf{R} \vdash \forall x\, A(x). & \end{aligned}$$

   Analogously for $\exists$.

1. This point follows from the other point.

12.12 Proof. We only do the proof for $\mathsf{HA}^\omega + \omega\mathsf{R}$; the cases of the other theories are analogous. Let us abbreviate $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$ by $\vdash A$. We prove $\vdash A$ or $\vdash \neg A$ by induction on the structure of A.

1. $\underline{A_{\mathrm{at}}}$ This case follows from point 2 of theorem 1.53.

   $\underline{\rightarrow}$ Using the induction hypothesis in the second implication, we get

   $$\begin{aligned} \nvdash A \to B \;&\Rightarrow \\ \nvdash \neg A \text{ and } \nvdash B \;&\Rightarrow \\ \vdash A \text{ and } \vdash \neg B \;&\Rightarrow \\ \vdash \neg(A \to B). & \end{aligned}$$

   Analogously for $\wedge$ and $\vee$.

$\underline{\forall}$ Using $\omega\mathsf{R}$ (that gives the equivalence between $\vdash \forall x\, A(x)$ and "for all closed terms $t\ \vdash A(t)$") in the first implication, and induction hypothesis in the second implication, we get

$$\begin{aligned} \nvdash \forall x\, A(x) \ &\Rightarrow \\ \text{there exists a closed term } t\ \nvdash A(t) \ &\Rightarrow \\ \text{there exists a closed term } t\ \vdash \neg A(t) \ &\Rightarrow \\ \vdash \exists x\, \neg A(x) \ &\Rightarrow \\ \vdash \neg\forall x\, A(x). & \end{aligned}$$

Analogously for $\exists$.

2. This point follows from the previous one since $\mathbb{T}^\omega$ is a model of $\mathsf{HA}^\omega + \omega\mathsf{R}$.

12.13 Proof. We only do the proof for $\mathsf{HA}^\omega + \omega\mathsf{R}$; the cases of the other theories are analogous. Let us abbreviate $\mathbb{T}^\omega \models A$ by $\models A$, and $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$ by $\vdash A$. We prove first the last point of the theorem.

2. We prove the equivalences

$$\mathbb{T}^\omega \models A \ \Leftrightarrow\ \mathsf{HA}^\omega + \omega\mathsf{R} \vdash A,$$
$$\mathbb{T}^\omega \models \neg A \ \Leftrightarrow\ \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$$

by simultaneous induction on the structure of A [68]. The right-to-left implications follow from $\mathbb{T}^\omega$ being a model of $\mathsf{HA}^\omega + \omega\mathsf{R}$, so we only have to prove the left-to-right implications.

$\underline{A_{\mathrm{at}}}$ This case follows from point 2 of theorem 1.53.

$\underline{\to}$ Using the induction hypothesis in the second implication of both columns, we get

$$\begin{aligned} \models A \to B \ &\Rightarrow & \models \neg(A \to B) \ &\Rightarrow \\ \models \neg A \text{ or } \models B \ &\Rightarrow & \models A \text{ and } \models \neg B \ &\Rightarrow \\ \vdash \neg A \text{ or } \vdash B \ &\Rightarrow & \vdash A \text{ and } \vdash \neg B \ &\Rightarrow \\ \vdash A \to B, & & \vdash \neg(A \to B). & \end{aligned}$$

Analogously for $\vee$ and $\wedge$.

$\underline{\exists}$ Using the induction hypothesis in the second implication of both columns, and $\omega\mathsf{R}$ in the third implication of the right column, we get

$$\begin{aligned} \models \exists x\, A(x) \ &\Rightarrow & \models \neg\exists x\, A(x) \ &\Rightarrow \\ \text{exists a closed term } t \models A(t) \ &\Rightarrow & \text{for all closed terms } t \models \neg A(t) \ &\Rightarrow \\ \text{exists a closed term } t \vdash A(t) \ &\Rightarrow & \text{for all closed terms } t \vdash \neg A(t) \ &\Rightarrow \\ \vdash \exists x\, A(x), & & \vdash \forall x\, \neg A(x) \ &\Rightarrow \\ & & \vdash \neg\exists x\, A(x). & \end{aligned}$$

Analogously for $\forall$.

1. This point follows from the other point.

12.14. In proof 12.13 we used the idea of proving the equivalences $(*_1)$ $\mathbb{T}^\omega \models A \Leftrightarrow \mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$ and $(*_2)$ $\mathbb{T}^\omega \models \neg A \Leftrightarrow \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$ by simultaneous induction on the structure of A [68]. Since we are only interested in $(*_1)$, the more natural thing to do would be to only prove $(*_1)$. But this seems to fail in the case of negation (a particular case of implication): assuming that $(*_1)$ holds for A, we have to show $\mathbb{T}^\omega \models \neg A \Leftrightarrow \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$, that is $\mathbb{T}^\omega \not\models A \Leftrightarrow \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$; but the induction hypothesis only tells us that $\mathbb{T}^\omega \not\models A \Leftrightarrow \mathsf{HA}^\omega + \omega\mathsf{R} \nvdash A$, and we do not know how to show $(*_3)$ $\mathsf{HA}^\omega + \omega\mathsf{R} \nvdash A \Rightarrow \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$. In fact, $(*_3)$ is the essence of completeness, exactly what we want to prove. But by having simultaneously $(*_1)$ and $(*_2)$ we do get $(*_3)$.

12.15. The following corollary tells us that $\omega\mathsf{R}$ encapsulates full classical logic.

12.16 Corollary. We have $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash \mathsf{LEM}$. Analogously for $\mathsf{WE\text{-}HA}^\omega + \omega\mathsf{R}$ and $\mathsf{E\text{-}HA}^\omega + \omega\mathsf{R}$.

12.17 Proof. We only prove $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash \mathsf{LEM}$; the cases of the other theories follow. Let A be an arbitrary formula of $\mathsf{HA}^\omega + \omega\mathsf{R}$ and $\bar{A}$ be a universal closure of A. Using point 2 of theorem 12.8 in the second equivalence, we get

$$\begin{aligned} \mathsf{HA}^\omega + \omega\mathsf{R} \vdash A \vee \neg A \;&\Leftrightarrow \\ \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \overline{A \vee \neg A} \;&\Leftrightarrow \\ \mathbb{T}^\omega \models \overline{A \vee \neg A}&, \end{aligned}$$

where the last line is true [52].

Alternatively, we prove $\mathsf{HA}^\omega + \omega\mathsf{R} = \mathsf{PA}^\omega + \omega\mathsf{R}$:

$$\begin{aligned} \mathsf{HA}^\omega + \omega\mathsf{R} \vdash A \;&\Leftrightarrow \\ \mathsf{HA}^\omega + \omega\mathsf{R} \vdash \bar{A} \;&\Leftrightarrow \\ \mathbb{T}^\omega \models \bar{A} \;&\Leftrightarrow \\ \mathsf{PA}^\omega + \omega\mathsf{R} \vdash \bar{A} \;&\Leftrightarrow \\ \mathsf{PA}^\omega + \omega\mathsf{R} \vdash A&. \end{aligned}$$

12.5 Conclusion

12.18. We saw that $\mathsf{HA}^\omega + \omega\mathsf{R}$ (and its variants with extensionality and classical logic) is complete:

Syntactic completeness for all sentences A we have $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash A$ or $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash \neg A$;

Semantic completeness for all sentences A we have $\mathsf{HA}^\omega + \omega\mathsf{R} \vdash A \Leftrightarrow \mathbb{T}^\omega \models A$.

Chapter 13

Proof interpretations with truth

13.1 Introduction

13.1. Let us recall that a proof interpretation I is a mapping $A \mapsto A_{\mathrm{I}}(\underline{a})$ with the properties

Soundness $\vdash A \;\Rightarrow\; \vdash A_{\mathrm{I}}(\underline{t})$ for suitable terms $\underline{t}$;

Truth $A \in \Gamma \;\Rightarrow\; \vdash A_{\mathrm{I}}(\underline{a}) \to A$ for a suitable class Γ of formulas.

We can use I to prove closure under rules for formulas in Γ. For example, using soundness in the first implications and truth in the second implications below, we prove the disjunction property, existence property and program extraction for $A, B \in \Gamma$:

$$\begin{array}{rclclcl}
\vdash A \vee B & \Rightarrow & \vdash A_{\mathrm{I}} \vee_t B_{\mathrm{I}} & \Rightarrow & \vdash A \vee_t B & \Rightarrow & \vdash A \text{ or } \vdash B, \\
\vdash \exists x\, A(x) & \Rightarrow & \vdash A_{\mathrm{I}}(t) & \Rightarrow & \vdash A(t), & & \\
\vdash \forall x\, \exists y\, A(x,y) & \Rightarrow & \vdash \forall x\, A_{\mathrm{I}}(x, t(x)) & \Rightarrow & \vdash \forall x\, A(x, t(x)). & &
\end{array}$$

But this only works for formulas in Γ. So, naturally, we wish Γ to be as large as possible, ideally we even want Γ to be the class of all formulas. To enlarge Γ we hardwire truth in I, that is we change I getting It by adding copies of the formulas under interpretation in some clauses of the definition of I:

$$\begin{array}{rlrl}
(A_{\mathrm{at}})_{\mathrm{I}} & :\equiv \ldots, & (A_{\mathrm{at}})_{\mathrm{It}} & :\equiv \ldots, \\
(A \wedge B)_{\mathrm{I}} & :\equiv \ldots, & (A \wedge B)_{\mathrm{It}} & :\equiv \ldots, \\
(A \vee B)_{\mathrm{I}} & :\equiv \ldots, & (A \vee B)_{\mathrm{It}} & :\equiv \ldots, \\
(A \to B)_{\mathrm{I}} & :\equiv \ldots, & (A \to B)_{\mathrm{It}} & :\equiv \ldots \wedge (A \to B), \\
(\forall x\, A)_{\mathrm{I}} & :\equiv \ldots, & (\forall x\, A)_{\mathrm{It}} & :\equiv \ldots \wedge \forall x\, A, \\
(\exists x\, A)_{\mathrm{I}} & :\equiv \ldots, & (\exists x\, A)_{\mathrm{It}} & :\equiv \ldots .
\end{array}$$

The questions are: in which clauses? and is It sound? We are going to answer these questions with three heuristics.

Heuristic 1 In intuitionistic linear logic ILL^{ω}, to hardwire truth we only have to add a copy in the clause of the bang !. Using Girard's embeddings, we move from IL^ω to ILL^ω, hardwire truth in ILL^ω, and then return to IL^ω. Girard's embeddings propagate the copy added in ILL^ω to the clauses of A_{at}, $\to$ and $\forall$, as illustrated in figure 13.1.

$$\mathsf{IL}^\omega \xrightarrow{\text{embedding}} \begin{array}{c}\mathsf{ILL}^\omega \text{ with}\\ \text{copy in !}\end{array} \xrightarrow{\text{embedding}} \begin{array}{c}\mathsf{IL}^\omega \text{ with copies}\\ \text{in } A_{\mathrm{at}}, \to, \forall\end{array}$$

Figure 13.1: Girard's embeddings propagating the copy in ! to A_{at}, $\to$ and $\forall$.

Heuristic 2 We can usually hardwire q-truth in I, getting Iq, just by imitating the way in which we hardwire q-truth in mr getting mrq. Then we try to upgrade q-truth to t-truth by defining $A_{\mathrm{It}} :\equiv A_{\mathrm{Iq}} \wedge A$.

Heuristic 3 We add copies in all clauses, and then we see that if I is "well-behaved" in a certain sense, then It is sound.

13.2 Heuristic 1

13.2. Intuitionistic linear logic is based on two conjunctions $\otimes$ and $\&$, a disjunction $\oplus$, an implication $\multimap$, quantifications $\forall$ and $\exists$, and a symbol ! called bang. The quantifiers behave as in the usual logic, but the remaining symbols do not, so we motivate the remaining symbols.

Conjunction In intuitionistic logic, the treatment of conjunction can be formalised by two sets of rules, $\{\wedge\mathrm{L}, \wedge\mathrm{R}\}$ and $\{\wedge\mathrm{L}', \wedge\mathrm{R}'\}$ (where $i \in \{1, 2\}$):

$$\frac{\Gamma, A_i \vdash B}{\Gamma, A_1 \wedge A_2 \vdash B}\,\wedge\mathrm{L} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge\mathrm{R}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge\mathrm{L}' \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge\mathrm{R}'$$

In the presence of contraction con and weakening wkn rules,

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathrm{con} \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathrm{wkn}$$

these two treatments are equivalent: one set can be deduced from the other by (where $i, j \in \{1, 2\}$ and $i \neq j$)

$$\dfrac{\dfrac{\Gamma, A_i \vdash B}{\Gamma, A_i, A_j \vdash B}\,\mathrm{wkn}}{\Gamma, A_1 \wedge A_2 \vdash B}\,\wedge\mathrm{L}' \qquad \dfrac{\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma, \Gamma \vdash A \wedge B}\,\wedge\mathrm{R}'}{\Gamma \vdash A \wedge B}\,\mathrm{con}$$

$$\dfrac{\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B, A \wedge B \vdash C}\,\wedge\mathrm{L}}{\Gamma, A \wedge B \vdash C}\,\mathrm{con} \qquad \dfrac{\dfrac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\,\mathrm{wkn} \quad \dfrac{\Delta \vdash B}{\Gamma, \Delta \vdash B}\,\mathrm{wkn}}{\Gamma, \Delta \vdash A \wedge B}\,\wedge\mathrm{R}$$

But in a contraction-and-weakening-free context, the two treatments lead to two different conjunctions.

$\underline{\{\wedge\mathrm{L}, \wedge\mathrm{R}\}}$ Leads to a context-sensitive conjunction & because $\wedge$R requires A and B to be proved from the same context Γ. This conjunction supports contraction in the sense that from $\Gamma \vdash A$ and $\Gamma \vdash B$ we get $\Gamma \vdash A \mathbin{\&} B$, not only $\Gamma, \Gamma \vdash A \mathbin{\&} B$. It also supports weakening in the sense that from $A \vdash C$ we get $A \mathbin{\&} B \vdash C$.

$\underline{\{\wedge\mathrm{L}', \wedge\mathrm{R}'\}}$ Leads to a context-insensitive conjunction $\otimes$ because in $\wedge$R$'$ there is no requirement on the contexts Γ of A and Δ of B. This conjunction does not support contraction in the sense that from $\Gamma \vdash A$ and $\Gamma \vdash B$ we get $\Gamma, \Gamma \vdash A \otimes B$, not $\Gamma \vdash A \otimes B$. It also does not support weakening in the sense that from $A \vdash C$ we do not get $A \otimes B \vdash C$ [77, section 1.5].

Disjunction Let us for a moment change to a sequent calculus with tuples not only on the left of $\vdash$, but also on the right. In classical logic, the treatment of disjunction can be formalised by two sets of rules, $\{\vee\mathrm{L}, \vee\mathrm{R}\}$ and $\{\vee\mathrm{L}', \vee\mathrm{R}'\}$ (where $i \in \{1, 2\}$):

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,\vee\mathrm{L} \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}\,\vee\mathrm{R}$$

$$\frac{\Gamma, A \vdash \Pi \quad \Delta, B \vdash \Sigma}{\Gamma, \Delta, A \vee B \vdash \Pi, \Sigma}\,\vee\mathrm{L}' \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,\vee\mathrm{R}'$$

In the presence of contraction con and weakening wkn rules,

$$\frac{\Gamma, \Delta, \Delta \vdash \Pi, \Sigma, \Sigma}{\Gamma, \Delta \vdash \Pi, \Sigma}\,\mathrm{con} \qquad \frac{\Gamma \vdash \Pi}{\Gamma, \Delta \vdash \Pi, \Sigma}\,\mathrm{wkn}$$

these two treatments are equivalent: one set can be deduced from the other by (where $i, j \in \{1, 2\}$ and $i \neq j$)

$$\frac{\dfrac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, \Gamma, A \vee B \vdash \Delta, \Delta}\,\vee\mathrm{L}'}{\Gamma, A \vee B \vdash \Delta}\,\mathrm{con} \qquad \frac{\dfrac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_i, A_j, \Delta}\,\mathrm{wkn}}{\Gamma \vdash A_1 \vee A_2, \Delta}\,\vee\mathrm{R}'$$

$$\frac{\dfrac{\Gamma, A \vdash \Pi}{\Gamma, \Delta, A \vdash \Pi, \Sigma}\,\mathrm{wkn} \quad \dfrac{\Delta, B \vdash \Sigma}{\Gamma, \Delta, B \vdash \Pi, \Sigma}\,\mathrm{wkn}}{\Gamma, \Delta, A \vee B \vdash \Pi, \Sigma}\,\vee\mathrm{L} \qquad \frac{\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, A \vee B, \Delta}\,\vee\mathrm{R}}{\Gamma \vdash A \vee B, \Delta}\,\mathrm{con}$$

But in a contraction-and-weakening-free context, the two treatments lead to two different disjunctions.

$\underline{\{\vee\mathrm{L}, \vee\mathrm{R}\}}$ Leads to a context-sensitive disjunction $\oplus$ because $\vee$L requires Δ to be proved from the contexts Γ, A and Γ, B with a common Γ. This disjunction supports contraction in the sense that from $\Gamma, A \vdash \Delta$ and $\Gamma, B \vdash \Delta$ we get $\Gamma, A \oplus B \vdash \Delta$, not only $\Gamma, \Gamma, A \oplus B \vdash \Delta$. It also supports weakening in the sense that from $\Gamma \vdash A$ we get $\Gamma \vdash A \oplus B$.

$\underline{\{\vee\mathrm{L}', \vee\mathrm{R}'\}}$ Leads to a context-insensitive disjunction $\parr$ because in $\vee$L$'$ there is no requirement on the contexts Γ, A of Π and Δ, B of Σ. This disjunction does not support contraction in the sense that from $\Gamma, A \vdash \Pi$ and $\Gamma, B \vdash \Sigma$ we get $\Gamma, \Gamma, A \parr B \vdash \Pi, \Sigma$, not $\Gamma, A \parr B \vdash \Pi, \Sigma$. It also does not support weakening in the sense that from $\Gamma \vdash A$ we do not get $\Gamma \vdash A \parr B$.

It is well-known that intuitionistic logic can be obtained from classical logic by restricting the sequent calculus for classical logic to only one formula on the right side of $\vdash$. Copying this, intuitionistic linear logic is defined from classical linear logic making the same restriction. But since the rule $\vee$R$'$ makes no sense under this restriction, the disjunction $\parr$ is left out in intuitionistic linear logic. (It could happen that just copying into linear logic a restriction that works for the usual logic would result in a linear logic that is not intuitionistic in some reasonably sense. However, Girard's embeddings from intuitionistic logic into intuitionistic linear logic, and vice versa, give some justification to regard intuitionistic linear logic as really intuitionistic.)

Implication Linear implication $\multimap$ is intended to satisfy the following equivalence: $(*)$ $A \vdash B$ if and only if $\vdash A \multimap B$. In our contraction-free context, $A \vdash B$ is not the same that $A, A \vdash B$, so $(*)$ translates to say that $\multimap$ is sensitive to how many times we use the premise of $\multimap$. Similarly in our weakening-free context, $\vdash B$ does not imply $A \vdash B$, so $(*)$ translates to say that $\multimap$ does not allow dummy premises. This leads us to interpret $A \multimap B$ as meaning that from A we get B using A exactly once.

Bang We saw that the two conjunctions & and $\otimes$ and the two disjunctions $\oplus$ and $\parr$ arise due to a contraction-and-weakening-free context. Nevertheless, we may wish to apply contraction or weakening to a formula A. This is allowed provided that we mark the formula A with a symbol !, getting $!A$, to signal that contraction and weakening are allowed on $!A$ and to keep track of where contraction and weakening are used. For example, the sequent $!A \vdash B \mathbin{\&} C$ means that from A we prove $B \mathbin{\&} C$ provided that we are allowed to use contraction or weakening on A. Technically, this enforcing of marking the formulas where contraction or weakening is applied is achieved by restricting the rules con and wkn to marked formulas:

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \text{con} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \text{wkn}$$

Informally, we may think of $!A$ as being $A, A, A, \ldots, A$ (n times), for any value of n that we may want (including $n = 0$). For example, from $A \vdash B$ and $A \vdash C$ we get $!A \vdash B \otimes C$, where we may think of $!A$ as being A, A (so $n = 2$). And from $\vdash B$ we get $!A \vdash B$, where we may think of $!A$ as being an empty list of As (so $n = 0$).

13.3 Definition.

1. Let us define *intuitionistic linear logic* ILL^ω (with primitive λ-abstraction and with booleans) [16, sections 1.1 and 1.2] [22, section 3.1].

   (a) The language of ILL^ω is the following.

      i. The language of ILL^ω is a typed language based on two ground types, 0 and b (booleans), and has the following symbols.

         A. The logical constants *zero* 0, *true* $\top$, *times* $\otimes$, *with* &, *plus* $\oplus$, *linear implication* $\multimap$, *bang* !, $\forall$ and $\exists$.

B. Countable many variables $x_1^\rho, x_2^\rho, x_3^\rho, \ldots$ for each type ρ.

C. For each arity $n \geq 0$, at most countable many (possibly none) n-ary function symbols $f_1, f_2, f_3, \ldots$.

D. For each arity $n \geq 0$, at most countable many (possibly none) n-ary predicate symbols $P_1, P_2, P_3, \ldots$.

E. The constant c.

F. The constants *true* t and *false* f.

G. The constant λ-*abstraction* $\lambda \cdot\ .\ \cdot$.

H. The constant *definition by cases* $\cdot \oplus .\ \cdot$.

I. The binary relation *equality* $=$ (between booleans).

ii. Terms are defined as follows (their types indicated in superscripts).

A. Variables x^ρ, and constants c^0, t^{b} and f^{b} are terms.

B. If $\underline{x}^{\underline{\varrho}}$ is a tuple of variables and r^σ, s^{b}, and t^σ are terms, then $(\lambda \underline{x} \,.\, t)^{\sigma \underline{\varrho}^{\mathrm{t}}}$ and $(r \oplus_s t)^\sigma$ are terms.

C. If $s^{\rho\sigma}$ and t^σ are terms, then $(st)^\rho$ is a term.

iii. Formulas are defined as follows.

A. Predicate symbols, 0 and $\top$ are atomic formulas.

B. The expressions $s = t$ are atomic formula (where s^{b} and t^{b} are terms).

C. Formulas are built from atomic formulas by means of $\otimes$, $\&$, $\oplus$, $\multimap$, $!$, $\forall$ and $\exists$.

(b) We define the following in ILL^ω.

i. The formula $1 :\equiv !\top$.

ii. The term $\mathcal{O}^\rho :\equiv \lambda x_1^{\rho_1}, \ldots, x_1^{\rho_n} \,.\, c$, where $\rho = 0\rho_n \cdots \rho_1$ (possibly with no ρ_is).

iii. The *linear equivalence* $A \multimapboth B :\equiv (A \multimap B) \,\&\, (B \multimap A)$, where A and B are formulas of ILL^ω.

iv. If $\Gamma \equiv A_1, \ldots, A_n$, then $!\Gamma :\equiv !A_1, \ldots, !A_n$, where $A_1, \ldots, A_n$ are formulas of ILL^ω.

v. The tuple of terms $\underline{r} \oplus_s \underline{t} :\equiv r_1 \oplus_s t_1, \ldots, r_n \oplus_s t_n$, where $\underline{s} \equiv s_1, \ldots, s_n$ and $\underline{t} \equiv t_1, \ldots, t_n$ are tuples of terms of ILL^ω.

vi. The formula $A \oplus_t B :\equiv (!(t = \mathrm{t}) \multimap A) \,\&\, (!(t = \mathrm{f}) \multimap B)$, where t^{b} is a term of ILL^ω and A and B are formulas of ILL^ω.

(c) We adopt the following convention to save on parentheses: $!$, $\forall$ and $\exists$ bind stronger than $\otimes$, $\&$ and $\oplus$, which in turn bind stronger than $\multimap$ and $\multimapboth$.

(d) The axioms and rules are expressed in a sequent calculus where on the left of $\vdash$ we have (finite and possibly empty) multisets (that is multiplicity matters but order does not) and on the right of $\vdash$ we have exactly one formula. The logical axioms and rules of ILL^ω are given in table 13.1. The axioms for λ-abstraction, term application, $\cdot \oplus .\ \cdot$ and $=$ are given in table 13.2.

$A_{\mathrm{at}} \vdash A_{\mathrm{at}} \quad \mathrm{id}$	$\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \mathrm{cut}$
$\Gamma, 0 \vdash A \quad 0\mathrm{L}$	$\Gamma \vdash \top \quad \top\mathrm{R}$
$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\mathrm{L}$	$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\mathrm{R}$
$\dfrac{\Gamma, A_i \vdash B}{\Gamma, A_1 \mathbin{\&} A_2 \vdash B}\ \&\mathrm{L}$	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B}\ \&\mathrm{R}$
$\dfrac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus\mathrm{L}$	$\dfrac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2}\ \oplus\mathrm{R}$ with $i \in \{1, 2\}$
$\dfrac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ \multimap\mathrm{L}$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\mathrm{R}$
$\dfrac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x\, A \vdash B}\ \forall\mathrm{L}$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash \forall x\, A}\ \forall\mathrm{R}$ with $x \notin \mathrm{FV}(\Gamma)$
$\dfrac{\Gamma, A \vdash B}{\Gamma, \exists x\, A \vdash B}\ \exists\mathrm{L}$ with $x \notin \mathrm{FV}(B)$	$\dfrac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x\, A}\ \exists\mathrm{R}$
$\dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ !\mathrm{L}$	$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ !\mathrm{R}$
$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \mathrm{con}$	$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \mathrm{wkn}$

Table 13.1: logical axioms and rules of ILL^{ω}.

$\vdash A_{\mathrm{at}}[(\lambda \underline{x}\,.\, \underline{t})\underline{s}/\underline{y}] \multimapboth A_{\mathrm{at}}[\underline{t}[\underline{s}/\underline{x}]/\underline{y}]$	$\vdash A_{\mathrm{at}}[\lambda \underline{x}\,.\,(\underline{t}\underline{x})/\underline{y}] \multimapboth A_{\mathrm{at}}[\underline{t}/\underline{y}]$
$\vdash A_{\mathrm{at}}[\underline{s} \oplus_{\mathrm{t}} \underline{t}/\underline{x}] \multimapboth A_{\mathrm{at}}[\underline{s}/\underline{x}]$	$\vdash A_{\mathrm{at}}[\underline{s} \oplus_{\mathrm{f}} \underline{t}/\underline{x}] \multimapboth A_{\mathrm{at}}[\underline{t}/\underline{x}]$
$\vdash x = x$	$!(x = y) \vdash y = x$
$!(x = y), !(y = z) \vdash x = z$	$!(x = y), A_{\mathrm{at}}[x/z] \vdash A_{\mathrm{at}}[y/z]$
$\Gamma, !(\mathrm{t} = \mathrm{f}) \vdash 0$	$\vdash\ !(x = \mathrm{t}) \oplus\, !(x = \mathrm{f})$

Table 13.2: axioms of ILL^{ω} for λ-abstraction, term application, $\cdot \oplus_{\cdot} \cdot$ and $=$.

2. Let us define *intuitionistic logic* IL^ω (with primitive λ-abstraction and with booleans) [22, section 3.1]. It is defined like ILL^ω, except for the following differences.

   (a) Instead of having the logical constants 0, $\top$, $\otimes$, $\&$, $\oplus$, $\multimap$, $!$, $\forall$ and $\exists$, it has the logical constants $\bot$, $\wedge$, $\vee$, $\to$, $\forall$ and $\exists$.

   (b) Its constant $\cdot \oplus . \cdot$ is denoted by $\cdot \vee . \cdot$.

   (c) We replace $A \oplus_t B$ by $A \vee_t B :\equiv (t = \mathrm{t} \to A) \wedge (t = \mathrm{f} \to B)$, where t^{b} is a term of IL^ω and A and B are formulas of IL^ω.

   (d) Instead of the axioms given in tables 13.1 and 13.2, it has the axioms given in tables 1.1 and 13.3.

$A_{\mathrm{at}}[(\lambda \underline{x} . \underline{t})\underline{s}/\underline{y}] \leftrightarrow A_{\mathrm{at}}[\underline{t}[\underline{s}/\underline{x}]/\underline{y}]$	$A_{\mathrm{at}}[\lambda \underline{x} . (\underline{t}\underline{x})/\underline{y}] \leftrightarrow A_{\mathrm{at}}[\underline{t}/\underline{y}]$
$A_{\mathrm{at}}[\underline{s} \vee_{\mathrm{t}} \underline{t}/\underline{x}] \leftrightarrow A_{\mathrm{at}}[\underline{s}/\underline{x}]$	$A_{\mathrm{at}}[\underline{s} \vee_{\mathrm{f}} \underline{t}/\underline{x}] \leftrightarrow A_{\mathrm{at}}[\underline{t}/\underline{x}]$
$x = x$	$x = y \to y = x$
$x = y \wedge y = z \to x = z$	$x = y \wedge A_{\mathrm{at}}[x/z] \to A_{\mathrm{at}}[y/z]$
$\neg(\mathrm{t} = \mathrm{f})$	$x = \mathrm{t} \vee x = \mathrm{f}$

Table 13.3: axioms of IL^ω for λ-abstraction, term application, $\cdot \vee . \cdot$ and $=$.

13.4. Let us explain the role of some symbols and axioms of ILL^ω.

$\underline{0, \top \text{ and } 1}$ The formula 0 is the identity of $\oplus$, in the sense of $\vdash A \oplus 0 \multimap\!\!\!\circ\, A$. Analogously, $\top$ and 1 are the identities of $\&$ and $\otimes$, respectively.

$\underline{c}$ We assume that there exists a constant c^0 to ensure that every type ρ is inhabited by a closed term $\mathcal{O}^\rho$. This is necessary when we have to produce a dummy closed term of an arbitrary type.

$\underline{\cdot \oplus . \cdot}$ The term $r \oplus_s t$ is intended to be a definition by cases: it reduce to r when the boolean s is true, and to t when s is false. Analogously, the formula $A \oplus_t B$ reduces to A when t is true, and to B when t is false.

$\underline{0 \text{ and } \mathrm{b}}$ The ground type 0 is the one of interest (in arithmetic it would stand for the type of the natural numbers). The boolean ground type b is introduced only to allow the definitions by cases $r \oplus_s t$ and $A \oplus_t B$. (In arithmetic we would not need booleans as we could take $r \oplus_s t :\equiv \mathrm{R}s^0 r \lambda x, y . t$ with $x, y \notin \mathrm{FV}(t)$, and $A \oplus_t B :\equiv (t^0 =_0 0 \to A) \wedge (t^0 \neq_0 0 \to B)$.)

$\underline{\Gamma, !(\mathrm{t} = \mathrm{f}) \vdash 0 \text{ and } \vdash !(x = \mathrm{t}) \oplus !(x = \mathrm{f})}$ The axiom $\Gamma, !(\mathrm{t} = \mathrm{f}) \vdash 0$ states $\mathrm{t} \neq \mathrm{f}$, and the axiom $\vdash !(x = \mathrm{t}) \oplus !(x = \mathrm{f})$ states that any boolean is true or false. So together they state that any boolean as exactly one of the truth values true and false.

13.5. In the next lemma we collect some basic derivations in ILL^ω that we will need later on to work smoothly with ILL^ω. Admittedly, the proofs are tedious, so the reader may want to skip them.

13.6 Lemma. The following is provable in ILL^ω.

1. $A \vdash A$.
2. $\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$ and $\dfrac{\Gamma, A \otimes B \vdash C}{\Gamma, A, B \vdash C}$.
3. $\dfrac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \,\&\, A_2}$ and $\dfrac{\Gamma \vdash A_1 \,\&\, A_2}{\Gamma \vdash A_i}$ (where $i \in \{1, 2\}$).
4. $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$ and $\dfrac{\Gamma \vdash A \multimap B}{\Gamma, A \vdash B}$.
5. $\dfrac{\Gamma \vdash \forall x\, A}{\Gamma \vdash A[t/x]}$.
6. $\dfrac{\Gamma \vdash A}{\Gamma[t/x] \vdash A[t/x]}$.
7. The axioms of λ-abstraction and equality generalise to arbitrary formulas.
8. $!(A \multimapboth B) \vdash C(A) \multimapboth C(B)$ where $C(S)$ is a formula of ILL^ω, S a subformula of $C(S)$ and $\mathrm{BV}(C) \cap (\mathrm{FV}(A) \cup \mathrm{FV}(B)) = \emptyset$.
9. $\dfrac{A_1, \ldots, A_n \vdash B \quad \Gamma_i \vdash A_i \multimapboth A_i',\ i = 1, \ldots, n \quad \Delta \vdash B \multimapboth B'}{\Gamma_1, \ldots, \Gamma_n, \Delta, A_1', \ldots, A_n' \vdash B'}$.
10. $A \multimapboth A \oplus_{\mathrm{t}} B$ and $B \multimapboth A \oplus_{\mathrm{f}} B$ [16, lemma 1(iv)].
11. $\dfrac{\Gamma(\underline{s}) \vdash A \quad \Gamma(\underline{t}) \vdash B}{\Gamma(\underline{s} \oplus_x \underline{t}) \vdash A \oplus_x B}$ [22, table 2].
12. $\dfrac{\Gamma(\underline{q}), A \vdash C(\underline{s}) \quad \Gamma(\underline{r}), B \vdash C(\underline{t})}{\Gamma(\underline{q} \oplus_x \underline{r}), A \oplus_x B \vdash C(\underline{s} \oplus_x \underline{t})}$ [22, table 2].

13.7 Proof.

1. The proof is by induction on the structure of A. For example, let us see the case of $\multimap$. By induction hypothesis we assume $A \vdash A$ and $B \vdash B$, and we want to prove $A \multimap B \vdash A \multimap B$. From $A \vdash A$ and $B \vdash B$ we get $A, A \multimap B \vdash B$ by $\multimap$L, and so we conclude $A \multimap B \vdash A \multimap B$ by $\multimap$R.
2. The first rule is $\otimes$L, so let us prove the second rule. By point 1 we have $A \vdash A$ and $B \vdash B$, so $A, B \vdash A \otimes B$ by $\otimes$R. From here and the premise $\Gamma, A \otimes B \vdash C$ we conclude $\Gamma, A, B \vdash C$ by cut.
3. The first rule is &R, so let us prove the second rule. By point 1 we have $A_i \vdash A_i$, so $A_1 \,\&\, A_2 \vdash A_i$ by &L. From here and the premise $\Gamma \vdash A_1 \,\&\, A_2$ we conclude $\Gamma \vdash A_i$ by cut.

4. The first rule is $\multimap$R, so let us prove the second rule. By point 1 we have $A \vdash A$ and $B \vdash B$, so $A, A \multimap B \vdash B$ by $\multimap$L. From here and the premise $\Gamma \vdash A \multimap B$ we conclude $\Gamma, A \vdash B$ by cut.

5. By point 1 we have $A[t/x] \vdash A[t/x]$, so $\forall x\, A \vdash A[t/x]$ by $\forall$L. From here and the premise $\Gamma \vdash \forall x\, A$ we conclude $\Gamma \vdash A[t/x]$ by cut.

6. Say $\Gamma \equiv B_1, \ldots, B_n$. By point 4, it is equivalent to prove $\vdash B_1 \multimap \cdots \multimap B_n \multimap A \,/\, \vdash B_1[t/x] \multimap \cdots \multimap B_n[t/x] \multimap A[t/x]$ (associating $\multimap$ to the right). By $\forall$R we introduce $\forall x$ on the right side of $\vdash$ in the premise and then we replace x by t using point 5, getting the conclusion.

7. It suffices to prove that if for all tuples of variables $\underline{x}$ and tuples of terms $\underline{s}$ and $\underline{t}$ and for all atomic formulas A_{at} we have $!\Gamma \vdash A_{\mathrm{at}}[\underline{s}/\underline{x}] \multimapboth A_{\mathrm{at}}[\underline{t}/\underline{x}]$, then for all tuples of variables $\underline{x}$ and tuples of terms $\underline{s}$ and $\underline{t}$ and for all formulas A such that $\mathrm{BV}(A) \cap \mathrm{FV}(\Gamma) = \emptyset$ we have $!\Gamma \vdash A[\underline{s}/\underline{x}] \multimapboth A[\underline{x}/\underline{t}]$. It is convenient to note that $!\Gamma \vdash A[\underline{s}/\underline{x}] \multimapboth A[\underline{t}/\underline{x}]$ is equivalent to the conjunction of $(*_1)$ $!\Gamma, A[\underline{s}/\underline{x}] \vdash A[\underline{t}/\underline{x}]$ and $(*_2)$ $!\Gamma, A[\underline{t}/\underline{x}] \vdash A[\underline{s}/\underline{x}]$ by points 3 and 4, and it suffices to prove $(*_1)$ since the proof of $(*_2)$ is analogous. The proof is by induction on the structure of A. Let us see the cases of $\multimap$ and $\forall$.

   $\underline{\multimap}$ By induction hypothesis we have $!\Gamma, A[\underline{t}/\underline{x}] \vdash A[\underline{s}/\underline{x}]$ and $!\Gamma, B[\underline{s}/\underline{x}] \vdash B[\underline{t}/\underline{x}]$, so $!\Gamma, A[\underline{t}/\underline{x}], A[\underline{s}/\underline{x}] \multimap B[\underline{s}/\underline{x}] \vdash B[\underline{t}/\underline{x}]$ by $\multimap$L and con. From here we conclude $!\Gamma, A[\underline{s}/\underline{x}] \multimap B[\underline{s}/\underline{x}] \vdash A[\underline{t}/\underline{x}] \multimap B[\underline{t}/\underline{x}]$ by $\multimap$R.

   $\underline{\forall}$ If y is not one of the variables in $\underline{x} = x_1, \ldots, x_n$, then from the induction hypothesis $!\Gamma, A[\underline{s}/\underline{x}] \vdash A[\underline{t}/\underline{x}]$ we get $!\Gamma, (\forall y\, A)[\underline{s}/\underline{x}] \equiv \forall y\, A[\underline{s}/\underline{x}] \vdash \forall y\, A[\underline{t}/\underline{x}] \equiv (\forall y\, A)[\underline{t}/\underline{x}]$ by $\forall$L and $\forall$R, where we can apply $\forall$R because $y \notin \mathrm{FV}(\Gamma)$ since by hypothesis $\mathrm{BV}(\forall y\, A) \cap \mathrm{FV}(\Gamma) = \emptyset$. If y is x_i, then from the induction hypothesis $!\Gamma, A[\underline{s}'/\underline{x}'] \vdash A[\underline{t}'/\underline{x}']$ applied to the tuples $\underline{x}' = x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, $\underline{s}' :\equiv s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$ and $\underline{t}' :\equiv t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ we get $!\Gamma, (\forall y\, A)[\underline{s}/\underline{x}] \equiv \forall y\, A[\underline{s}'/\underline{x}'] \vdash \forall y\, A[\underline{t}'/\underline{x}'] \equiv (\forall y\, A)[\underline{t}/\underline{x}]$ by $\forall$L and $\forall$R.

8. The proof is by induction on the structure of C. Let us see the case of $\multimap$. As in the previous point we only see one of the implications since the other one is analogous. By induction hypothesis we have $!(A \multimapboth B), C(B) \vdash C(A)$ and $!(A \multimapboth B), D(A) \vdash D(B)$, so $!(A \multimapboth B), C(B), C(A) \multimap D(A) \vdash D(B)$ by $\multimap$L and con, thus we conclude $!(A \multimapboth B), C(A) \multimap D(A) \vdash C(B) \multimap D(B)$ by $\multimap$R.

9. First we prove the result with $B \equiv B'$ and with Δ empty by induction on n.

   Base case From the premise $\Gamma_1 \vdash A_1 \multimapboth A_1'$ we get $\Gamma_1, A_1' \vdash A_1$ by points 3 and 4. From here and the premise $A_1 \vdash B$ we conclude $\Gamma_1, A_1' \vdash B$ by cut.

   Induction step Assume the premises $A_1, \ldots, A_{n+1} \vdash B$ and $\Gamma_i \vdash A_i \multimapboth A_i'$, $i = 1, \ldots, n+1$. So $A_1, \ldots, A_n \vdash A_{n+1} \multimap B$ by $\multimap$R, and $\Gamma_i \vdash A_i \multimapboth A_i'$, $i = 1, \ldots, n$. By induction hypothesis we get $\Gamma_1, \ldots, \Gamma_n, A_1', \ldots, A_n' \vdash$

$A_{n+1} \multimap B$, so $(*_1)$ $\Gamma_1, \ldots, \Gamma_n, A'_1, \ldots, A'_n, A_{n+1} \vdash B$ by point 4. From $\Gamma_{n+1} \vdash A_{n+1} \multimapboth A'_{n+1}$ we get $(*_2)$ $\Gamma_{n+1}, A'_{n+1} \vdash A_{n+1}$. From $(*_1)$ and $(*_2)$ we conclude $\Gamma_1, \ldots, \Gamma_{n+1}, A'_1, \ldots, A'_{n+1} \vdash B$ by cut.

Now we prove the result for arbitrary B, B' and Δ. From the premise $\Delta \vdash B \multimapboth B'$ we get $\Delta, B \vdash B'$, and from here and the already proved $\Gamma_1, \ldots, \Gamma_n, A'_1, \ldots, A'_n \vdash B$ we conclude $\Gamma_1, \ldots, \Gamma_n, \Delta, A'_1, \ldots, A'_n \vdash B'$ by cut.

10. We prove only $A \multimapboth A \oplus_{\mathrm{t}} B$ since $B \multimapboth A \oplus_{\mathrm{f}} B$ is analogous.

    $\underline{\multimap}$ We have $A, !(\mathrm{t} = \mathrm{t}) \vdash A$ by point 1 and wkn, so $(*_1)$ $A \vdash !(\mathrm{t} = \mathrm{t}) \multimap A$ by $\multimap$R. We have $A, 0 \vdash B$ by 0L, so replacing 0 by the linearly equivalent $!(\mathrm{t} = \mathrm{f})$ we get $A, !(\mathrm{t} = \mathrm{f}) \vdash B$ by point 9, thus $(*_2)$ $A \vdash !(\mathrm{t} = \mathrm{f}) \multimap B$ by $\multimap$R. From $(*_1)$ and $(*_2)$ we conclude $A \vdash A \oplus_{\mathrm{t}} B$.

    $\underline{\multimapinv}$ We have $!(\mathrm{t} = \mathrm{t}) \vdash !(\mathrm{t} = \mathrm{t})$ and $A \vdash A$ by point 1, so $!(\mathrm{t} = \mathrm{t}), !(\mathrm{t} = \mathrm{t}) \multimap A \vdash A$ by $\multimap$L. We cut out $!(\mathrm{t} = \mathrm{t})$ (that is provable from the axiom $\vdash x = x$ and !R) and get $!(\mathrm{t} = \mathrm{t}) \multimap A \vdash A$. Then by &L we conclude $A \oplus_{\mathrm{t}} B \vdash A$.

11. From $\Gamma(\underline{s}) \vdash A$ we get $\Gamma(\underline{s} \oplus_{\mathrm{t}} \underline{t}) \vdash A$ (by points 7 and 8), so $(*_1)$ $!(x = \mathrm{t}), \Gamma(\underline{s} \oplus_x \underline{t}) \vdash !(x = \mathrm{t}) \multimap A$ (by wkn, $\multimap$R and point 7). Since $\Gamma(\underline{s} \oplus_x \underline{t}), !(x = \mathrm{t}), !(x = \mathrm{f}) \vdash 0$ (by replacing t by x in the axiom $\Gamma(\underline{s} \oplus_x \underline{t}), !(\mathrm{t} = \mathrm{f}) \vdash 0$ assuming $!(x = \mathrm{t})$), then $(*_2)$ $!(x = \mathrm{t}), \Gamma(\underline{s} \oplus_x \underline{t}) \vdash !(x = \mathrm{f}) \multimap B$ (by the axiom $0 \vdash B$, cut and $\multimap$R). From $(*_1)$ and $(*_2)$ we get $(*_3)$ $!(x = \mathrm{t}), \Gamma(\underline{s} \oplus_x \underline{t}) \vdash A \oplus_x B$ by &R. Analogously we prove $(*_4)$ $!(x = \mathrm{f}), \Gamma(\underline{s} \oplus_x \underline{t}) \vdash A \oplus_x B$. From $(*_3)$ and $(*_4)$ we get $!(x = \mathrm{t}) \oplus !(x = \mathrm{f}), \Gamma(\underline{s} \oplus_x \underline{t}) \vdash A \oplus_x B$ by $\oplus$L. Then we cut out the axiom $!(x = \mathrm{t}) \oplus !(x = \mathrm{f})$ from the context, getting our conclusion.

12. From $\Gamma(\underline{q}), A \vdash C(\underline{s})$ we get $\Gamma(\underline{q} \oplus_{\mathrm{t}} \underline{r}), A \oplus_{\mathrm{t}} B \vdash C(\underline{s} \oplus_{\mathrm{t}} \underline{t})$. Replacing t by x assuming $!(x = \mathrm{t})$ and get $(*_1)$ $!(x = \mathrm{t}), \Gamma(\underline{q} \oplus_x \underline{r}), A \oplus_x B \vdash C(\underline{s} \oplus_x \underline{t})$. Analogously we prove $(*_2)$ $!(x = \mathrm{f}), \Gamma(\underline{q} \oplus_x \underline{r}), A \oplus_x B \vdash C(\underline{s} \oplus_x \underline{t})$. From $(*_1)$ and $(*_2)$ we get $!(x = \mathrm{t}) \oplus !(x = \mathrm{f}), \Gamma(\underline{q} \oplus_x \underline{r}), A \oplus_x B \vdash C(\underline{s} \oplus_x \underline{t})$ by $\oplus$L. Then we cut out the axiom $!(x = \mathrm{t}) \oplus !(x = \mathrm{f})$ from the context, getting our conclusion.

13.8. Now we are going to define Girard's embeddings: two embeddings q and t of IL^ω into ILL^ω, and one embedding i of ILL^ω into IL^ω. This is pictured in figure 13.2. As curiosities, we can mention $\mathrm{i} \circ \mathrm{q} = \mathrm{id} = \mathrm{i} \circ \mathrm{t}$ (that is $(A^{\mathrm{q}})^{\mathrm{i}} \equiv A \equiv (A^{\mathrm{t}})^{\mathrm{i}}$ [26,

$$\begin{tikzcd} \mathsf{IL}^\omega \arrow[r, bend left, "\mathrm{q}"] \arrow[r, bend right, "\mathrm{t}"'] & \mathsf{ILL}^\omega \arrow[r, "\mathrm{i}"] & \mathsf{IL}^\omega \end{tikzcd}$$

Figure 13.2: Girard's embeddings q, t and i.

page 81]) and $\mathrm{t} = !\mathrm{q}$ (that is $\mathsf{ILL}^\omega \vdash A^{\mathrm{t}} \multimapboth !A^{\mathrm{q}}$ [16, proposition 1] [22, lemma 3.2]).

13.9 Definition.

1. *Girard's q-embedding* q [26, section 5.1] assigns to each formula A of IL^ω the formula A^{q} of ILL^ω defined by recursion on the structure of A by (where $A_{\mathrm{at}} \not\equiv \bot$)

$$\begin{aligned} A_{\mathrm{at}}^{\mathrm{q}} &:\equiv A_{\mathrm{at}}, \\ \bot^{\mathrm{q}} &:\equiv 0, \\ (A \wedge B)^{\mathrm{q}} &:\equiv A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}, \\ (A \vee B)^{\mathrm{q}} &:\equiv {!A^{\mathrm{q}}} \oplus {!B^{\mathrm{q}}}, \\ (A \to B)^{\mathrm{q}} &:\equiv {!A^{\mathrm{q}}} \multimap B^{\mathrm{q}}, \\ (\forall x\, A)^{\mathrm{q}} &:\equiv \forall x\, A^{\mathrm{q}}, \\ (\exists x\, A)^{\mathrm{q}} &:\equiv \exists x\, {!A^{\mathrm{q}}}. \end{aligned}$$

2. *Girard's t-embedding* t [26, page 81] assigns to each formula A of IL^ω the formula A^{t} of ILL^ω defined by recursion on the structure of A by (where $A_{\mathrm{at}} \not\equiv \bot$)

$$\begin{aligned} A_{\mathrm{at}}^{\mathrm{t}} &:\equiv {!A_{\mathrm{at}}}, \\ \bot^{\mathrm{t}} &:\equiv 0, \\ (A \wedge B)^{\mathrm{t}} &:\equiv A^{\mathrm{t}} \otimes B^{\mathrm{t}}, \\ (A \vee B)^{\mathrm{t}} &:\equiv A^{\mathrm{t}} \oplus B^{\mathrm{t}}, \\ (A \to B)^{\mathrm{t}} &:\equiv {!(A^{\mathrm{t}} \multimap B^{\mathrm{t}})}, \\ (\forall x\, A)^{\mathrm{t}} &:\equiv {!\forall x\, A^{\mathrm{t}}}, \\ (\exists x\, A)^{\mathrm{t}} &:\equiv \exists x\, A^{\mathrm{t}}. \end{aligned}$$

3. *Girard's i-embedding* i [26, page 81] assigns to each formula A of ILL^ω the formula A^{i} of IL^ω defined by induction on the structure of A by (where $A_{\mathrm{at}} \not\equiv 0, \top$)

$$\begin{aligned} A_{\mathrm{at}}^{\mathrm{i}} &:\equiv A_{\mathrm{at}}, \\ 0^{\mathrm{i}} &:\equiv \bot, \\ \top^{\mathrm{i}} &:\equiv \neg\bot, \\ (A \otimes B)^{\mathrm{i}} &:\equiv A^{\mathrm{i}} \wedge B^{\mathrm{i}}, \\ (A \,\&\, B)^{\mathrm{i}} &:\equiv A^{\mathrm{i}} \wedge B^{\mathrm{i}}, \\ (A \oplus B)^{\mathrm{i}} &:\equiv A^{\mathrm{i}} \vee B^{\mathrm{i}}, \\ (A \multimap B)^{\mathrm{i}} &:\equiv A^{\mathrm{i}} \to B^{\mathrm{i}}, \\ (\forall x\, A)^{\mathrm{i}} &:\equiv \forall x\, A^{\mathrm{i}}, \\ (\exists x\, A)^{\mathrm{i}} &:\equiv \exists x\, A^{\mathrm{i}}, \\ (!A)^{\mathrm{i}} &:\equiv A^{\mathrm{i}}. \end{aligned}$$

13.10 Theorem (soundness)**.**

1. If $\mathsf{IL}^\omega \vdash A$, then $\vdash A^{\mathrm{q}}$ in ILL^ω [26, section 5.1].

2. If $\mathsf{IL}^\omega \vdash A$, then $\vdash A^{\mathrm{t}}$ in ILL^ω [26, page 81].

3. If $A_1, \ldots, A_n \vdash B$ in ILL^ω, then $\mathsf{IL}^\omega \vdash A_1^{\mathrm{i}} \wedge \cdots \wedge A_n^{\mathrm{i}} \to B^{\mathrm{i}}$ [26, page 81].

13.11 Proof. First, let us make some remarks.

1. We will frequently use points 1 and 4 of lemma 13.6 without mentioning it.

   To prove $\vdash !A \multimap B$ it suffices to prove $A \vdash B$, so we will systematically only prove $A \vdash B$.

2. For all formulas A of ILL^ω we have $\mathsf{ILL}^\omega \vdash A^{\mathrm{t}} \multimapboth !A^{\mathrm{t}}$.

   Let us prove this claim. It suffices to prove $A^{\mathrm{t}} \vdash !A^{\mathrm{t}}$ in ILL^ω since $!B \vdash B$ holds always. The proof is by induction on the structure of A.

   $\underline{A_{\mathrm{at}}}$ The case of $A_{\mathrm{at}} \not\equiv \perp$ is trivial since its interpretation is a banged formula $!A$ and $!A \vdash !!A$. If $A_{\mathrm{at}} \equiv \perp$, then $\perp^{\mathrm{t}} \equiv 0 \vdash !0 \equiv !\perp^{\mathrm{t}}$ by 0L. Analogously for $\to$ and $\forall$.

   $\underline{\wedge}$ From $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ and $B^{\mathrm{t}} \vdash B^{\mathrm{t}}$ we get $A^{\mathrm{t}}, B^{\mathrm{t}} \vdash A^{\mathrm{t}} \otimes B^{\mathrm{t}}$ by $\otimes$R. So $A^{\mathrm{t}}, B^{\mathrm{t}} \vdash !(A^{\mathrm{t}} \otimes B^{\mathrm{t}})$ by !R using that A^{t} and B^{t} are equivalent to banged formulas (by the induction hypothesis). So $(A \wedge B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \otimes B^{\mathrm{t}} \vdash !(A^{\mathrm{t}} \otimes B^{\mathrm{t}}) \equiv !(A \wedge B)^{\mathrm{t}}$ by $\otimes$L.

   $\underline{\vee}$ From $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ we get $A^{\mathrm{t}} \vdash A^{\mathrm{t}} \oplus B^{\mathrm{t}}$ by $\oplus$R. So $(*_1)$ $A^{\mathrm{t}} \vdash !(A^{\mathrm{t}} \oplus B^{\mathrm{t}})$ by !R using that A^{t} is equivalent to a banged formula (by the induction hypothesis). Analogously, $(*_2)$ $B^{\mathrm{t}} \vdash !(A^{\mathrm{t}} \oplus B^{\mathrm{t}})$. From $(*_1)$ and $(*_2)$ we conclude $(A \vee B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \oplus B^{\mathrm{t}} \vdash !(A^{\mathrm{t}} \oplus B^{\mathrm{t}}) \equiv !(A \vee B)^{\mathrm{t}}$ by $\oplus$L.

   $\underline{\exists}$ From $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ we get $A^{\mathrm{t}} \vdash \exists x\, A^{\mathrm{t}}$ by $\exists$R. So $A^{\mathrm{t}} \vdash !\exists x\, A^{\mathrm{t}}$ using that A^{t} is equivalent to a banged formula (by the induction hypothesis). Thus $(\exists x\, A)^{\mathrm{t}} \equiv \exists x\, A^{\mathrm{t}} \vdash !\exists x\, A^{\mathrm{t}} \equiv !(\exists x\, A)^{\mathrm{t}}$ by $\exists$L.

Let us prove the theorem by induction on the derivation of A.

1. $\underline{A \vee A \to A}$ Its interpretation is $!(!A^{\mathrm{q}} \oplus !A^{\mathrm{q}}) \multimap A^{\mathrm{q}}$. We have $A^{\mathrm{q}} \vdash A^{\mathrm{q}}$, so $!A^{\mathrm{q}} \vdash A$ by !L, thus $!A^{\mathrm{q}} \oplus !A^{\mathrm{q}} \vdash A$ by $\oplus$L. Analogously for $A \to A \wedge A$.

   $\underline{A \to A \vee B}$ Its interpretation is $!A^{\mathrm{q}} \multimap !A^{\mathrm{q}} \oplus !B^{\mathrm{q}}$. We have $!A^{\mathrm{q}} \vdash !A^{\mathrm{q}}$, so $!A^{\mathrm{q}} \vdash !A^{\mathrm{q}} \oplus !B^{\mathrm{q}}$ by $\oplus$R. Analogously for $A \wedge B \to A$.

   $\underline{A \vee B \to B \vee A}$ Its interpretation is $!(!A^{\mathrm{q}} \oplus !B^{\mathrm{q}}) \multimap !B^{\mathrm{q}} \oplus !A^{\mathrm{q}}$. We have $!A^{\mathrm{q}} \vdash !A^{\mathrm{q}}$ and $!B^{\mathrm{q}} \vdash !B^{\mathrm{q}}$, so $!A^{\mathrm{q}} \vdash !B^{\mathrm{q}} \oplus !A^{\mathrm{q}}$ and $!B^{\mathrm{q}} \vdash !B^{\mathrm{q}} \oplus !A^{\mathrm{q}}$ by $\oplus$R, thus $!A^{\mathrm{q}} \oplus !B^{\mathrm{q}} \vdash !B^{\mathrm{q}} \oplus !A^{\mathrm{q}}$ by $\oplus$L. Analogously for $A \wedge B \to B \wedge A$.

   $\underline{A[t/x] \to \exists x\, A}$ Its interpretation is $!A[t/x]^{\mathrm{q}} \multimap \exists x\, !A^{\mathrm{q}}$. We have $!A[t/x]^{\mathrm{q}} \vdash !A[t/x]^{\mathrm{q}}$, so $!A[t/x]^{\mathrm{q}} \vdash \exists x\, !A^{\mathrm{q}}$ by $\exists$R and $A[t/x]^{\mathrm{q}} \equiv A^{\mathrm{q}}[t/x]$. Analogously for $\forall x\, A \to A[t/x]$.

   $\underline{\perp \to A}$ Its interpretation is $!0 \multimap A^{\mathrm{q}}$. By 0L we have $0 \vdash A^{\mathrm{q}}$.

   $\underline{A \to B,\, B \to C \,/\, A \to C}$ Its interpretation is $!A^{\mathrm{q}} \multimap B^{\mathrm{q}}, !B^{\mathrm{q}} \multimap C^{\mathrm{q}} \,/\, !A^{\mathrm{q}} \multimap C^{\mathrm{q}}$. From $!A^{\mathrm{q}} \vdash B^{\mathrm{q}}$ and $!B^{\mathrm{q}} \vdash C^{\mathrm{q}}$ we get $!A^{\mathrm{q}} \vdash C^{\mathrm{q}}$ by !R and cut. Analogously for $A,\, A \to B \,/\, B$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ Its interpretation is $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \multimap C^{\mathrm{q}} \,/\, !A^{\mathrm{q}} \multimap (!B^{\mathrm{q}} \multimap C^{\mathrm{q}})$. First we prove $(*)$ $!A^{\mathrm{q}}, !B^{\mathrm{q}} \vdash !(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}})$: from $A^{\mathrm{q}} \vdash A^{\mathrm{q}}$ and $B^{\mathrm{q}} \vdash B^{\mathrm{q}}$ we get $!A^{\mathrm{q}}, !B^{\mathrm{q}} \vdash A^{\mathrm{q}}$ and $!A^{\mathrm{q}}, !B^{\mathrm{q}} \vdash B^{\mathrm{q}}$ by !L and wkn, so $(*)$ by &L and !L. From $(*)$ and the interpretation of the premise, that is $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \vdash C^{\mathrm{q}}$, we get $!A^{\mathrm{q}}, !B^{\mathrm{q}} \vdash C^{\mathrm{q}}$ by cut.

$\underline{A \to (B \to C) \,/\, A \wedge B \to C}$ Its interpretation is $!A^{\mathrm{q}} \multimap (!B^{\mathrm{q}} \multimap C^{\mathrm{q}}) \,/\, !(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \multimap C^{\mathrm{q}}$. First we prove $(*_1)$ $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \vdash !A^{\mathrm{q}}$: we have $A^{\mathrm{q}} \vdash A^{\mathrm{q}}$, so $A^{\mathrm{q}} \,\&\, B^{\mathrm{q}} \vdash A^{\mathrm{q}}$ by &L, thus $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \vdash !A^{\mathrm{q}}$ by !L and !R. Analogously, $(*_2)$ $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \vdash !B^{\mathrm{q}}$. From $(*_1)$, $(*_2)$ and the interpretation of the premise, that is $!A^{\mathrm{q}}, !B^{\mathrm{q}} \vdash C^{\mathrm{q}}$, we get $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}), !(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \vdash C^{\mathrm{q}}$ by cut twice, and so $!(A^{\mathrm{q}} \,\&\, B^{\mathrm{q}}) \vdash C^{\mathrm{q}}$ by con.

$\underline{A \to B \,/\, C \vee A \to C \vee B}$ Its interpretation is $!A^{\mathrm{q}} \multimap B^{\mathrm{q}} \,/\, !(!C^{\mathrm{q}} \oplus !A^{\mathrm{q}}) \multimap !C^{\mathrm{q}} \oplus !B^{\mathrm{q}}$. From the interpretation of the premise, that is $!A^{\mathrm{q}} \vdash B^{\mathrm{q}}$, we get $(*_1)$ $!A^{\mathrm{q}} \vdash !C^{\mathrm{q}} \oplus !B^{\mathrm{q}}$ by !R and $\oplus$R. From $!C^{\mathrm{q}} \vdash !C^{\mathrm{q}}$ we get $(*_2)$ $!C^{\mathrm{q}} \vdash !C^{\mathrm{q}} \oplus !B^{\mathrm{q}}$ by $\oplus$R. From $(*_1)$ and $(*_2)$ we get $!C^{\mathrm{q}} \oplus !A^{\mathrm{q}} \vdash !C^{\mathrm{q}} \oplus !B^{\mathrm{q}}$ by $\oplus$L.

$\underline{A \to B \,/\, \exists x\, A \to B}$ Its interpretation is $!A^{\mathrm{q}} \multimap B \,/\, !\exists x\, !A^{\mathrm{q}} \multimap B$. From the interpretation of the premise, that is $!A^{\mathrm{q}} \vdash B$, we get $\exists x\, !A^{\mathrm{q}} \vdash B^{\mathrm{q}}$ by $\exists$L and $x \notin \mathrm{FV}(B) = \mathrm{FV}(B^{\mathrm{q}})$. Analogously for $A \to B \,/\, A \to \forall x\, B$.

$\underline{A_{\mathrm{at}}(\lambda \underline{x} \,.\, \underline{t}(\underline{x})) \leftrightarrow A_{\mathrm{at}}(\underline{t})}$ This axiom is of the form $A_{\mathrm{at}} \leftrightarrow B_{\mathrm{at}}$, so its interpretation is $(!A^{\mathrm{q}}_{\mathrm{at}} \multimap B^{\mathrm{q}}_{\mathrm{at}}) \,\&\, (!B^{\mathrm{q}}_{\mathrm{at}} \multimap A^{\mathrm{q}}_{\mathrm{at}})$, and follows from the corresponding axiom in ILL^{ω} (we write $A^{\mathrm{q}}_{\mathrm{at}}$ instead of A_{at} to include both the cases $A^{\mathrm{q}}_{\mathrm{at}} \equiv A_{\mathrm{at}}$ and $A^{\mathrm{q}}_{\mathrm{at}} \equiv 0$). Analogously for $A_{\mathrm{at}}\big((\lambda \underline{x} \,.\, \underline{t}(\underline{x}))\underline{s}\big) \leftrightarrow A_{\mathrm{at}}(\underline{t}(\underline{s}))$, $A_{\mathrm{at}}(\underline{s} \vee_{\mathrm{t}} \underline{t}) \leftrightarrow A_{\mathrm{at}}(\underline{s})$, $A_{\mathrm{at}}(\underline{s} \vee_{\mathrm{f}} \underline{t}) \leftrightarrow A_{\mathrm{at}}(\underline{t})$, $x = x$, $x = y \to y = x$, $\neg(\mathrm{t} = \mathrm{f})$ and $x = \mathrm{t} \vee x = \mathrm{f}$.

$\underline{x = y \wedge A_{\mathrm{at}}[x/z] \to A_{\mathrm{at}}[y/z]}$ Its interpretation is $(*)$ $!(x = y \,\&\, A_{\mathrm{at}}[x/z]^{\mathrm{q}}) \multimap A_{\mathrm{at}}[y/z]^{\mathrm{q}}$. From $!(x = y \,\&\, A_{\mathrm{at}}[x/z]^{\mathrm{q}}) \vdash !(x = y)$, $!(x = y \,\&\, A_{\mathrm{at}}[x/z]^{\mathrm{q}}) \vdash A_{\mathrm{at}}[y/z]^{\mathrm{q}}$ and the axiom $!(x = y), A^{\mathrm{q}}_{\mathrm{at}}[x/z] \vdash B^{\mathrm{q}}_{\mathrm{at}}[y/z]$ we get $(*)$ by cut and con. Analogously for $x = y \wedge y = z \to x = z$.

2. $\underline{A \to A \wedge A}$ Its interpretation is $!(A^{\mathrm{t}} \multimap A^{\mathrm{t}} \otimes A^{\mathrm{t}})$. From $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ we get $A^{\mathrm{t}}, A^{\mathrm{t}} \vdash A^{\mathrm{t}} \otimes A^{\mathrm{t}}$ by $\otimes$R, so $A^{\mathrm{t}} \vdash A^{\mathrm{t}} \otimes A^{\mathrm{t}}$ by con and remark 2. Analogously for $A \vee A \to A$.

$\underline{A \wedge B \to A}$ Its interpretation is $!(A^{\mathrm{t}} \otimes B^{\mathrm{t}} \multimap A^{\mathrm{t}})$. From $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ we get $A^{\mathrm{t}}, B^{\mathrm{t}} \vdash A^{\mathrm{t}}$ by wkn and remark 2, so $A^{\mathrm{t}} \otimes B^{\mathrm{t}} \vdash A^{\mathrm{t}}$ by $\otimes$L. Analogously for $A \to A \vee B$.

$\underline{A \wedge B \to B \wedge A}$ Its interpretation is $!(A^{\mathrm{t}} \otimes B^{\mathrm{t}} \multimap B^{\mathrm{t}} \otimes A^{\mathrm{t}})$. From $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ and $B^{\mathrm{t}} \vdash B^{\mathrm{t}}$ we get $A^{\mathrm{t}}, B^{\mathrm{t}} \vdash A^{\mathrm{t}} \otimes B^{\mathrm{t}}$ by $\otimes$R, that is $B^{\mathrm{t}}, A^{\mathrm{t}} \vdash A^{\mathrm{t}} \otimes B^{\mathrm{t}}$, so $B^{\mathrm{t}} \otimes A^{\mathrm{t}} \vdash A^{\mathrm{t}} \otimes B^{\mathrm{t}}$ by $\otimes$L. Analogously for $A \vee B \to B \vee A$.

$\underline{\bot \to A}$ Its interpretation is $!(0 \multimap A^{\mathrm{t}})$. We have $0 \vdash A^{\mathrm{t}}$ by 0L.

$\underline{\forall x\, A \to A[t/x]}$ Its interpretation is $!(!\forall x\, A^{\mathrm{t}} \multimap A[t/x]^{\mathrm{t}})$. We have $A^{\mathrm{t}} \vdash A^{\mathrm{t}}$, so $\forall x\, A^{\mathrm{t}} \vdash A^{\mathrm{t}}$ by $\forall$L, thus $\forall x\, A^{\mathrm{t}} \vdash A[t/x]^{\mathrm{t}}$ by point 6 of lemma 13.6 and $A[t/x]^{\mathrm{t}} \equiv A^{\mathrm{t}}[t/x]$. Analogously for $A[t/x] \to \exists x\, A$.

$\underline{A \to B,\, B \to C \,/\, A \to C}$ Its interpretation is $!(A \multimap B), !(B \multimap C) \,/\, !(A \multimap C)$. From $A \vdash B$ and $B \vdash C$ we get $A \vdash C$ by cut. Analogously for $A,\, A \to B \,/\, B$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ Its interpretation is $!(A^{\mathrm{t}} \otimes B^{\mathrm{t}} \multimap C^{\mathrm{t}}) \,/\, !(A^{\mathrm{t}} \multimap !(B^{\mathrm{t}} \multimap C^{\mathrm{t}}))$. From $A^{\mathrm{t}} \otimes B^{\mathrm{t}} \vdash C^{\mathrm{t}}$ we get $A^{\mathrm{t}}, B^{\mathrm{t}} \vdash C^{\mathrm{t}}$ by point 2 of lemma 13.6, so $A^{\mathrm{t}} \vdash B^{\mathrm{t}} \multimap C^{\mathrm{t}}$ by $\multimap$R, thus $A^{\mathrm{t}} \vdash !(B^{\mathrm{t}} \multimap C^{\mathrm{t}})$ by remark 2 and !R. Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, C \vee A \to C \vee B}$ Its interpretation is $!(A^{\mathrm{t}} \multimap B^{\mathrm{t}}) \,/\, !(C^{\mathrm{t}} \oplus A^{\mathrm{t}} \multimap C^{\mathrm{t}} \oplus B^{\mathrm{t}})$. From $C^{\mathrm{t}} \vdash C^{\mathrm{t}}$ and $A^{\mathrm{t}} \vdash B^{\mathrm{t}}$ we get $C^{\mathrm{t}} \vdash B^{\mathrm{t}} \oplus C^{\mathrm{t}}$ and $A^{\mathrm{t}} \vdash B^{\mathrm{t}} \oplus C^{\mathrm{t}}$ by $\oplus$R, so $C^{\mathrm{t}} \oplus A^{\mathrm{t}} \vdash C^{\mathrm{t}} \oplus B^{\mathrm{t}}$ by $\oplus$L.

$\underline{A \to B \,/\, A \to \forall x\, B}$ Its interpretation is $!(A^{\mathrm{t}} \multimap B^{\mathrm{t}}) \,/\, !(A^{\mathrm{t}} \multimap !\forall x\, B^{\mathrm{t}})$. From $A^{\mathrm{t}} \vdash B^{\mathrm{t}}$ we get $A^{\mathrm{t}} \vdash \forall x\, B^{\mathrm{t}}$ by $\forall$R and $x \notin \mathrm{FV}(A) = \mathrm{FV}(A^{\mathrm{t}})$, so $A^{\mathrm{t}} \vdash !\forall x\, B^{\mathrm{t}}$ by !R and remark 2.

$\underline{A_{\mathrm{at}}(\lambda \underline{x} \,.\, \underline{t}(\underline{x})) \leftrightarrow A_{\mathrm{at}}(\underline{t})}$ This axiom is of the form $A_{\mathrm{at}} \leftrightarrow B_{\mathrm{at}}$, so its interpretation is $!(A^{\mathrm{t}}_{\mathrm{at}} \multimap B^{\mathrm{t}}_{\mathrm{at}}) \otimes !(B^{\mathrm{t}}_{\mathrm{at}} \multimap A^{\mathrm{t}}_{\mathrm{at}})$, and follows from the corresponding axiom in ILL^{ω}. Analogously for $A_{\mathrm{at}}\big((\lambda \underline{x} \,.\, \underline{t}(\underline{x}))\underline{s}\big) \leftrightarrow A_{\mathrm{at}}(\underline{t}(\underline{s}))$, $A_{\mathrm{at}}(\underline{s} \vee_{\mathrm{t}} \underline{t}) \leftrightarrow A_{\mathrm{at}}(\underline{s})$, $A_{\mathrm{at}}(\underline{s} \vee_{\mathrm{f}} \underline{t}) \leftrightarrow A_{\mathrm{at}}(\underline{t})$, $x = x$, $x = y \to y = x$, $\neg(\mathrm{t} = \mathrm{f})$ and $x = \mathrm{t} \vee x = \mathrm{f}$.

$\underline{x = y \wedge y = z \to x = z}$ Its interpretation is $!(!(x = y) \otimes !(y = z) \multimap !(x = z))$. From the axiom $!(x = y), !(y = z) \vdash x = z$ we get $!(x = y) \otimes !(y = z) \vdash x = z$ by point 2 of lemma 13.6, so $!(x = y) \otimes !(y = z) \vdash !(x = z)$ by !R.

$\underline{x = y \wedge A_{\mathrm{at}}[x/z] \to A_{\mathrm{at}}[y/z]}$ Its interpretation is $!(!(x = y) \otimes A_{\mathrm{at}}[x/z]^{\mathrm{t}} \multimap A_{\mathrm{at}}[y/z]^{\mathrm{t}})$, which follows from the corresponding axiom in ILL^{ω} and $\otimes$L.

3. In the following, if $\Gamma \equiv A_1, \ldots, A_n$, then $\bigwedge \Gamma^{\mathrm{i}} :\equiv A^{\mathrm{i}}_1 \wedge \cdots \wedge A^{\mathrm{i}}_n$.

   $\underline{\Gamma, 0 \vdash A}$ Its interpretation is the provable $\bigwedge \Gamma^{\mathrm{i}} \wedge \bot \to A^{\mathrm{i}}$. Analogously for $\Gamma \vdash \top$.

   $\underline{!(x = y), A_{\mathrm{at}}[x/z] \vdash A_{\mathrm{at}}[y/z]}$ Its interpretation is $x = y \wedge A_{\mathrm{at}}[x/z]^{\mathrm{i}} \to A_{\mathrm{at}}[y/z]^{\mathrm{i}}$ which is provable by $A_{\mathrm{at}}[x/z]^{\mathrm{i}} \equiv A^{\mathrm{i}}_{\mathrm{at}}[x/z]$ and $A_{\mathrm{at}}[y/z]^{\mathrm{i}} \equiv A^{\mathrm{i}}_{\mathrm{at}}[y/z]$. Analogously for the remaining axioms.

   $\underline{\Gamma \vdash B \,/\, \Gamma \vdash \forall x\, B}$ Its interpretation is $\bigwedge \Gamma^{\mathrm{i}} \to B \,/\, \bigwedge \Gamma^{\mathrm{i}} \to \forall x\, B^{\mathrm{i}}$ which is provable by $x \notin \mathrm{FV}(\Gamma) = \mathrm{FV}(\bigwedge \Gamma^{\mathrm{i}})$. Analogously for the remaining rules.

13.12. Now we present a modified realisability of intuitionistic linear logic without truth lr, and with truth lrt. The interest of lrt is that the compositions $\mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ and $\mathrm{i} \circ \mathrm{lrt} \circ \mathrm{t}$ will tell us how to hardwire q-truth and t-truth in a proof interpretation.

13.13 Definition.

1. *Modified realisability*, in the context of IL^{ω}, is defined like in definition 3.4, except for

$$\begin{aligned}(A \vee B)_{\mathrm{mr}}(c^{\mathrm{b}}, \underline{a}, \underline{b}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \vee_c B_{\mathrm{mr}}(\underline{b}) \\ &\equiv (c = \mathrm{t} \to A_{\mathrm{mr}}(\underline{a})) \wedge (c = \mathrm{f} \to B_{\mathrm{mr}}(\underline{b}))\end{aligned}$$

instead of

$$\begin{aligned}(A \vee B)_{\mathrm{mr}}(c^0, \underline{a}, \underline{b}) &:\equiv A_{\mathrm{mr}}(\underline{a}) \vee_c B_{\mathrm{mr}}(\underline{b}) \\ &\equiv (c =_0 0 \to A_{\mathrm{mr}}(\underline{a})) \wedge (c \neq_0 0 \to B_{\mathrm{mr}}(\underline{b})).\end{aligned}$$

Analogously for mrq and mrt [22, proof of definitions 2.1 and 2.3].

2. *Linear modified realisability* lr [16, definition 1] assigns to each formula A of ILL^ω the formula $A_{\mathrm{lr}}(\underline{x}; \underline{y})$ defined by

$$\begin{aligned}(A_{\mathrm{at}})_{\mathrm{lr}}(;) &:\equiv A_{\mathrm{at}}, \\ (A \otimes B)_{\mathrm{lr}}(\underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{lr}}(\underline{a}; \underline{b}) \otimes B_{\mathrm{lr}}(\underline{c}; \underline{d}), \\ (A \mathbin{\&} B)_{\mathrm{lr}}(\underline{a}, \underline{c}; e^{\mathrm{b}}, \underline{b}, \underline{d}) &:\equiv A_{\mathrm{lr}}(\underline{a}; \underline{b}) \oplus_e B_{\mathrm{lr}}(\underline{c}; \underline{d}), \\ (A \oplus B)_{\mathrm{lr}}(e^{\mathrm{b}}, \underline{a}, \underline{c}; \underline{b}, \underline{d}) &:\equiv A_{\mathrm{lr}}(\underline{a}; \underline{b}) \oplus_e B_{\mathrm{lr}}(\underline{c}; \underline{d}), \\ (A \multimap B)_{\mathrm{lr}}(\underline{C}, \underline{B}; \underline{a}, \underline{d}) &:\equiv A_{\mathrm{lr}}(\underline{a}; \underline{B}\underline{a}\underline{d}) \multimap B_{\mathrm{lr}}(\underline{C}\underline{a}; \underline{d}), \\ (\forall x\, A)_{\mathrm{lr}}(\underline{A}; x, \underline{b}) &:\equiv A_{\mathrm{lr}}(\underline{A}x; \underline{b}), \\ (\exists x\, A)_{\mathrm{lr}}(x, \underline{a}; \underline{b}) &:\equiv A_{\mathrm{lr}}(\underline{a}; \underline{b}), \\ (!A)_{\mathrm{lr}}(\underline{a};) &:\equiv !\forall \underline{b}\, A_{\mathrm{lr}}(\underline{a}; \underline{b}).\end{aligned}$$

By $(A_{\mathrm{at}})_{\mathrm{lr}}(;)$ we mean $(A_{\mathrm{at}})_{\mathrm{lr}}(\underline{a}; \underline{b})$ with the tuples $\underline{a}$ and $\underline{b}$ empty. Analogously for $(!A)_{\mathrm{lr}}(\underline{a};)$.

3. *Linear modified realisability with truth* lrt [22, definition 3.3] is defined analogously to lr except for

$$(!A)_{\mathrm{lrt}}(\underline{a};) :\equiv !\forall \underline{b}\, A_{\mathrm{lrt}}(\underline{a}; \underline{b}) \otimes !A.$$

13.14 Theorem (soundness)**.** Let $A_1, \ldots, A_n, B$ be formulas of ILL^ω, $\Gamma \equiv A_1, \ldots, A_n$ and $\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) :\equiv (A_1)_{\mathrm{lr}}(\underline{a}_1; \underline{b}_1), \ldots, (A_n)_{\mathrm{lr}}(\underline{a}_n; \underline{b}_n)$, where $\underline{a} \equiv \underline{a}_1, \ldots, \underline{a}_n$ and $\underline{b} \equiv \underline{b}_1, \ldots, \underline{b}_n$. If $\Gamma \vdash B$ in ILL^ω, then we can extract from such a proof terms $\underline{s}, \underline{t}$ such that $\Gamma_{\mathrm{lr}}(\underline{a}; \underline{s}) \vdash B_{\mathrm{lr}}(\underline{t}; \underline{d})$ in ILL^ω, $\mathrm{FV}(\underline{s}) \subseteq \mathrm{FV}(\Gamma) \cup \mathrm{FV}(B) \cup \{\underline{a}, \underline{d}\}$ and $\mathrm{FV}(\underline{t}) \subseteq \mathrm{FV}(\Gamma) \cup \mathrm{FV}(B) \cup \{\underline{a}\}$ [16, theorem 1]. Analogously for lrt [22, theorem 3.5]. The terms constructed in the following proof for lr and lrt are the same [22, follows from the proof of theorem 3.5].

13.15 Proof. Let us make some remark.

1. We adopt here (with the proper adaptations, including an analogous unified treatment of variants without truth and with truth by means of $\mathrm{t} \in \{\mathrm{id}, \top\}$) the remarks made in the beginning of proof 3.12.

2. We will treat lr and lrt in a unified manner in the following way. Let id and 1 be functions defined by $A^{\mathrm{id}} :\equiv A$ and $A^1 :\equiv 1$, where A is a formula of ILL^ω, and let $\mathrm{t} \in \{\mathrm{id}, 1\}$. We redefine lr by changing its clause on ! to

$$(!A)_{\mathrm{lr}}(\underline{x};) :\equiv !\forall \underline{y}\, A_{\mathrm{lrt}}(\underline{x}; \underline{y}) \otimes (!A)^{\mathrm{t}}.$$

Then this redefined lr reduces to:

(a) the old lr when $\mathrm{t} = 1$;

(b) lrt when $\mathrm{t} = \mathrm{id}$.

By reducing we mean, for example, $\mathsf{ILL}^\omega \vdash A_{\mathrm{lr}}(\underline{x}; \underline{y}) \multimap\!\!\circ\, A_{\mathrm{lrt}}(\underline{x}; \underline{y})$ in the latter case. We prove the soundness theorem for the redefined lr, hence proving the theorem for the original lr and lrt. Moreover, the terms working for them will not depend on t, so they are the same.

Let us prove the theorem by induction on the derivation of $\Gamma \vdash B$.

$\underline{\Gamma, 0 \vdash A}$ We have

$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), 0 \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}),$$
$$\underline{t}_{\underline{b}} :\equiv \underline{\mathcal{O}}, \quad \underline{t}_{\underline{d}} :\equiv \underline{\mathcal{O}}.$$

Analogously for $A_{\mathrm{at}} \vdash A_{\mathrm{at}}$ and $\Gamma \vdash \top$.

$\underline{\Gamma \vdash A,\ \Delta, A \vdash B \,/\, \Gamma, \Delta \vdash B}$ We have

$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}),$$
$$\Delta_{\mathrm{lr}}(\underline{e}; \underline{f}), A_{\mathrm{lr}}(\underline{c}; \underline{d}) \vdash B_{\mathrm{lr}}(\underline{g}; \underline{h}),$$
$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), \Delta_{\mathrm{lr}}(\underline{e}; \underline{f}) \vdash B_{\mathrm{lr}}(\underline{g}; \underline{h}),$$
$$\underline{t}_{\underline{b}} :\equiv \underline{r}_{\underline{b}}[\underline{s}_{\underline{d}}[\underline{r}_{\underline{c}}/\underline{c}]/\underline{d}], \quad \underline{t}_{\underline{f}} :\equiv \underline{s}_{\underline{f}}[\underline{r}_{\underline{c}}/\underline{c}], \quad \underline{t}_{\underline{g}} :\equiv \underline{s}_{\underline{g}}[\underline{r}_{\underline{c}}/\underline{c}].$$

To see that the terms work, we take $\underline{d} = \underline{s}_{\underline{d}}[\underline{r}_{\underline{c}}/\underline{c}]$ in the interpretation of the first premise (note that this substitution does not change $\underline{r}_{\underline{c}}$ because $\underline{d} \notin \mathrm{FV}(\underline{r}_{\underline{c}})$) and $\underline{c} = \underline{r}_{\underline{c}}$ in the interpretation of the second premise. Then in the interpretation of both premise we get $A_{\mathrm{lr}}(\underline{r}_{\underline{c}}; \underline{s}_{\underline{d}}[\underline{r}_{\underline{c}}/\underline{c}])$ and we can cut it out. Note $\underline{h} \notin \mathrm{FV}(\underline{t}\underline{g})$ because $\underline{h} \notin \mathrm{FV}(\underline{s}_{\underline{g}}) \cup \mathrm{FV}(\underline{r}_{\underline{c}})$.

$\underline{\Gamma \vdash A,\ \Delta \vdash B \,/\, \Gamma, \Delta \vdash A \otimes B}$ We have

$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}),$$
$$\Delta_{\mathrm{lr}}(\underline{e}; \underline{f}) \vdash B_{\mathrm{lr}}(\underline{g}; \underline{h}),$$
$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), \Delta_{\mathrm{lr}}(\underline{e}; \underline{f}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}) \otimes B_{\mathrm{lr}}(\underline{g}; \underline{h}),$$
$$\underline{t}_{\underline{b}} :\equiv \underline{r}_{\underline{b}}, \quad \underline{t}_{\underline{f}} :\equiv \underline{s}_{\underline{f}}, \quad \underline{t}_{\underline{c}} :\equiv \underline{r}_{\underline{c}}, \quad \underline{t}_{\underline{g}} :\equiv \underline{s}_{\underline{g}}.$$

Analogously for $\Gamma, A, B \vdash C \,/\, \Gamma, A \oplus B \vdash C$.

$\underline{\Gamma, A \vdash B \,/\, \Gamma, A_1 \,\&\, A_2 \vdash B}$ We have

$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), A_{\mathrm{lr}}(\underline{c}; \underline{d}) \vdash C_{\mathrm{lr}}(\underline{e}; \underline{f}),$$
$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), A_{\mathrm{lr}}(\underline{c}; \underline{d}) \oplus_i B_{\mathrm{lr}}(\underline{g}; \underline{h}) \vdash C_{\mathrm{lr}}(\underline{e}; \underline{f}),$$
$$\underline{t}_{\underline{b}} :\equiv \underline{s}_{\underline{b}}, \quad \underline{t}_{\underline{d}} :\equiv \underline{s}_{\underline{d}}, \quad t_i :\equiv \mathrm{t}, \quad \underline{t}_{\underline{h}} :\equiv \underline{\mathcal{O}}, \quad \underline{t}_{\underline{e}} :\equiv \underline{s}_{\underline{e}}.$$

To see that the terms work, we use point 10 of lemma 13.6. Analogously for $\Gamma, B \vdash C \,/\, \Gamma, A \,\&\, B \vdash C$ and $\Gamma \vdash A_i \,/\, \Gamma \vdash A_1 \oplus A_2$.

$\underline{\Gamma \vdash A,\ \Gamma \vdash B \,/\, \Gamma \vdash A \,\&\, B}$ We have

$$\begin{gathered}
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}), \\
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash B_{\mathrm{lr}}(\underline{e}; \underline{f}), \\
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}) \oplus_g B_{\mathrm{lr}}(\underline{e}; \underline{f}), \\
\underline{t}_{\underline{b}} :\equiv \underline{r}_{\underline{b}} \oplus_g \underline{s}_{\underline{b}}, \quad \underline{t}_{\underline{c}} :\equiv \underline{r}_{\underline{c}}, \quad \underline{t}_{\underline{e}} :\equiv \underline{s}_{\underline{e}}.
\end{gathered}$$

To see that the terms work, we use point 11 of lemma 13.6.

$\underline{\Gamma, A \vdash C,\ \Gamma, B \vdash C \,/\, \Gamma, A \oplus B \vdash C}$ We have

$$\begin{gathered}
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), A_{\mathrm{lr}}(\underline{c}; \underline{d}) \vdash C_{\mathrm{lr}}(\underline{e}; \underline{f}), \\
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), B_{\mathrm{lr}}(\underline{g}; \underline{h}) \vdash C_{\mathrm{lr}}(\underline{e}; \underline{f}), \\
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), A_{\mathrm{lr}}(\underline{c}; \underline{d}) \oplus_i B_{\mathrm{lr}}(\underline{g}; \underline{h}) \vdash C_{\mathrm{lr}}(\underline{e}; \underline{f}), \\
\underline{t}_{\underline{b}} :\equiv \underline{r}_{\underline{b}} \oplus_i \underline{s}_{\underline{b}}, \quad \underline{t}_{\underline{d}} :\equiv \underline{r}_{\underline{d}}, \quad \underline{t}_{\underline{h}} :\equiv \underline{s}_{\underline{h}}, \quad \underline{t}_{\underline{e}} :\equiv \underline{r}_{\underline{e}} \oplus_i \underline{s}_{\underline{e}}.
\end{gathered}$$

To see that the terms work, we use point 12 of lemma 13.6.

$\underline{\Gamma \vdash A,\ \Delta, B \vdash C \,/\, \Gamma, \Delta, A \multimap B \vdash C}$ We have

$$\begin{gathered}
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{d}), \\
\Delta_{\mathrm{lr}}(\underline{e}; \underline{f}), B_{\mathrm{lr}}(\underline{g}; \underline{h}) \vdash C_{\mathrm{lr}}(\underline{i}; \underline{j}), \\
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), \Delta_{\mathrm{lr}}(\underline{e}; \underline{f}), A_{\mathrm{lr}}(\underline{c}; \underline{Dch}) \multimap B_{\mathrm{lr}}(\underline{Gc}; \underline{h}) \vdash C_{\mathrm{lr}}(\underline{i}; \underline{j}),
\end{gathered}$$

$$\begin{gathered}
\underline{t}_{\underline{b}} :\equiv \underline{r}_{\underline{b}}[\underline{Dr}_{\underline{c}}(\underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}])/\underline{d}], \quad \underline{t}_{\underline{f}} :\equiv \underline{s}_{\underline{f}}[\underline{Gr}_{\underline{c}}/\underline{g}], \\
\underline{t}_{\underline{c}} :\equiv \underline{r}_{\underline{c}}, \quad \underline{t}_{\underline{h}} :\equiv \underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}], \quad \underline{t}_{\underline{i}} :\equiv \underline{s}_{\underline{i}}[\underline{Gr}_{\underline{c}}/\underline{g}].
\end{gathered}$$

Let us see that the terms work. By induction hypothesis we have (13.1) and (13.2), and we want to prove (13.3):

$$\Gamma_{\mathrm{lr}}(\underline{a}; \underline{r}_{\underline{b}}) \vdash A_{\mathrm{lr}}(\underline{r}_{\underline{c}}; \underline{d}), \tag{13.1}$$

$$\Delta_{\mathrm{lr}}(\underline{e}; \underline{s}_{\underline{f}}), B_{\mathrm{lr}}(\underline{g}; \underline{s}_{\underline{h}}) \vdash C_{\mathrm{lr}}(\underline{s}_{\underline{i}}; \underline{j}), \tag{13.2}$$

$$\begin{gathered}
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{r}_{\underline{b}}[\underline{Dr}_{\underline{c}}(\underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}])/\underline{d}]), \Delta_{\mathrm{lr}}(\underline{e}; \underline{s}_{\underline{f}}[\underline{Gr}_{\underline{c}}/\underline{g}]), \\
A_{\mathrm{lr}}(\underline{r}_{\underline{c}}; \underline{Dr}_{\underline{c}}(\underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}])) \multimap B_{\mathrm{lr}}(\underline{Gr}_{\underline{c}}; \underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}]) \vdash C_{\mathrm{lr}}(\underline{s}_{\underline{i}}[\underline{Gr}_{\underline{c}}/\underline{g}]; \underline{j}).
\end{gathered} \tag{13.3}$$

Taking $\underline{d} = \underline{Dr}_{\underline{c}}(\underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}])$ in (13.1) and $\underline{g} = \underline{Gr}_{\underline{c}}$ in (13.2), we get

$$\begin{gathered}
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{r}_{\underline{b}}[\underline{Dr}_{\underline{c}}(\underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}])/\underline{d}]) \vdash A_{\mathrm{lr}}(\underline{r}_{\underline{c}}; \underline{Dr}_{\underline{c}}(\underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}])), \\
\Delta_{\mathrm{lr}}(\underline{e}; \underline{s}_{\underline{f}}[\underline{Gr}_{\underline{c}}/\underline{g}]), B_{\mathrm{lr}}(\underline{Gr}_{\underline{c}}; \underline{s}_{\underline{h}}[\underline{Gr}_{\underline{c}}/\underline{g}]) \vdash C_{\mathrm{lr}}(\underline{s}_{\underline{i}}[\underline{Gr}_{\underline{c}}/\underline{g}]; \underline{j}),
\end{gathered}$$

and from here we get (13.3).

$\underline{\Gamma, A \vdash B \,/\, \Gamma \vdash A \multimap B}$ We have

$$\begin{gathered}
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}), A_{\mathrm{lr}}(\underline{c}; \underline{d}) \vdash B_{\mathrm{lr}}(\underline{e}; \underline{f}), \\
\Gamma_{\mathrm{lr}}(\underline{a}; \underline{b}) \vdash A_{\mathrm{lr}}(\underline{c}; \underline{Dcf}) \multimap B_{\mathrm{lr}}(\underline{Ec}; \underline{f}), \\
\underline{t}_{\underline{b}} :\equiv \underline{s}_{\underline{b}}, \quad \underline{t}_{\underline{D}} :\equiv \lambda \underline{c}, \underline{f} \,.\, \underline{s}_{\underline{d}}, \quad \underline{t}_{\underline{E}} :\equiv \lambda \underline{c} \,.\, \underline{s}_{\underline{e}}.
\end{gathered}$$

$\underline{\Gamma \vdash A \,/\, \Gamma \vdash \forall x\, A}$ We have

$$\begin{gathered}\Gamma_{\mathrm{lr}}(\underline{a};\underline{b}) \vdash A_{\mathrm{lr}}(\underline{c};\underline{d}),\\ \Gamma_{\mathrm{lr}}(\underline{a};\underline{b}) \vdash A_{\mathrm{lr}}(\underline{C}x;\underline{d}),\\ \underline{t}_{\underline{b}} :\equiv \underline{s}_{\underline{b}}, \quad \underline{t}_{\underline{C}} :\equiv \lambda x\,.\,\underline{s}_{\underline{c}}.\end{gathered}$$

$\underline{\Gamma \vdash A[t/x] \,/\, \exists x\, A}$ We have

$$\begin{gathered}\Gamma_{\mathrm{lr}}(\underline{a};\underline{b}) \vdash A[t/x]_{\mathrm{lr}}(\underline{c};\underline{d}),\\ \Gamma_{\mathrm{lr}}(\underline{a};\underline{b}) \vdash A_{\mathrm{lr}}(\underline{c};\underline{d}),\\ \underline{t}_{\underline{b}} :\equiv \underline{s}_{\underline{b}}, \quad t_x :\equiv t, \quad \underline{t}_{\underline{c}} :\equiv \underline{s}_{\underline{c}}.\end{gathered}$$

To see that the terms work, we use $A_{\mathrm{lr}}(\underline{x};\underline{y})[t/z] \equiv A[t/z]_{\mathrm{lr}}(\underline{x};\underline{y})$. Analogously for $\Gamma, A \vdash B \,/\, \Gamma, \exists x\, A \vdash B$ and $\Gamma, A[t/x] \vdash B \,/\, \Gamma, \forall x\, A \vdash B$.

$\underline{!\Gamma \vdash A \,/\, !\Gamma \vdash !A}$ We have

$$\begin{gathered}!\forall \underline{b}\, \Gamma_{\mathrm{lr}}(\underline{a};\underline{b}) \otimes !\Gamma^{\mathrm{t}} \vdash A_{\mathrm{lr}}(\underline{c};\underline{d}),\\ !\forall \underline{b}\, \Gamma_{\mathrm{lr}}(\underline{a};\underline{b}) \otimes !\Gamma^{\mathrm{t}} \vdash !\forall \underline{d}\, A_{\mathrm{lr}}(\underline{c};\underline{d}) \otimes !A^{\mathrm{t}},\\ \underline{t}_{\underline{c}} :\equiv \underline{s}_{\underline{c}}.\end{gathered}$$

To see that the terms work, in the case $\mathrm{t} = \mathrm{id}$ we use that by hypothesis we proved $!\Gamma \vdash A$, so $!\Gamma \vdash !A$. Analogously for $\Gamma, A \vdash B \,/\, \Gamma, !A \vdash B$.

$\underline{\Gamma, !A, !A \vdash B \,/\, \Gamma, !A \vdash B}$ We have

$$\begin{gathered}\Gamma_{\mathrm{lr}}(\underline{a};\underline{b}), !\forall \underline{d}\, A_{\mathrm{lr}}(\underline{c};\underline{d}) \otimes !A^{\mathrm{t}}, !\forall \underline{f}\, A_{\mathrm{lr}}(\underline{e};\underline{f}) \otimes !A^{\mathrm{t}} \vdash B_{\mathrm{lr}}(\underline{g};\underline{h}),\\ \Gamma_{\mathrm{lr}}(\underline{a};\underline{b}), !\forall \underline{d}\, A_{\mathrm{lr}}(\underline{c};\underline{d}) \otimes !A^{\mathrm{t}}, \vdash B_{\mathrm{lr}}(\underline{g};\underline{h}),\\ \underline{t}_{\underline{b}} :\equiv \underline{s}_{\underline{b}}[\underline{c}/\underline{e}], \quad \underline{t}_{\underline{g}} :\equiv \underline{s}_{\underline{g}}[\underline{c}/\underline{e}].\end{gathered}$$

To see that the terms work, we take $\underline{e} = \underline{c}$ in the interpretation of the premise so that $!\forall \underline{d}\, A_{\mathrm{lr}}(\underline{c};\underline{d}) \otimes !A^{\mathrm{t}}$ and $!\forall \underline{e}\, A_{\mathrm{lr}}(\underline{e};\underline{f}) \otimes !A^{\mathrm{t}}$ become equivalent, and then we contract them using the fact that they can be regarded as banged formulas since $\vdash !D \otimes !E \multimapboth !(D \,\&\, E)$. Analogously for $\Gamma \vdash B \,/\, \Gamma, !A \vdash B$.

$\underline{\vdash A_{\mathrm{at}}(\underline{s} \oplus_{\mathrm{t}} \underline{t}) \multimapboth A_{\mathrm{at}}(\underline{s})}$ We have

$$\vdash (A_{\mathrm{at}}(\underline{s} \oplus_{\mathrm{t}} \underline{t}) \multimap A_{\mathrm{at}}(\underline{s})) \oplus_a (A_{\mathrm{at}}(\underline{s}) \multimap A_{\mathrm{at}}(\underline{s} \oplus_{\mathrm{t}} \underline{t})).$$

Analogously for $\vdash A_{\mathrm{at}}(\underline{s} \oplus_{\mathrm{f}} \underline{t}) \multimapboth A_{\mathrm{at}}(\underline{t})$ and the axioms of λ-abstraction.

$\underline{!(x = y), !(y = z) \vdash x = y}$ We have

$$!(x = y) \otimes !(x = y)^{\mathrm{t}}, !(y = z) \otimes !(y = z)^{\mathrm{t}} \vdash x = z.$$

To see that the interpretation is provable, we use point 2 of lemma 13.6. Analogously for $\vdash x = x$, $!(x = y) \vdash y = x$, $!(x = y), A_{\mathrm{at}}(x) \vdash A_{\mathrm{at}}(y)$, and $\Gamma, !(\mathrm{t} = \mathrm{f}) \vdash 0$.

$\underline{!(x = \mathrm{t}) \oplus !(x = \mathrm{f})}$ We have

$$(!(x = \mathrm{t}) \otimes !(x = \mathrm{t})^{\mathrm{t}}) \oplus_a (!(x = \mathrm{f}) \otimes !(x = \mathrm{f})^{\mathrm{t}}),$$
$$t_a :\equiv x.$$

13.16. To motivate our first heuristic, we are going to factorise in proposition 13.17 mrq and mrt in terms of lrt and Girard's embeddings q, t and i: $\mathrm{mrq} = \forall \circ \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ and $\mathrm{mrt} = \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{t}$. Figure 13.3 illustrates these factorisations. Then by tracking where $\forall \circ \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ and $\mathrm{i} \circ \mathrm{lrt} \circ \mathrm{t}$ are adding copies we read where q-truth and t-truth variants should have copies added.

$$\begin{tikzcd}
\mathsf{IL}^\omega \arrow[rr, "\mathrm{mrq}"] \arrow[dd, "\mathrm{q}"'] & & \mathsf{IL}^\omega \\
& & \mathsf{IL}^\omega \arrow[u, "\forall"'] \\
\mathsf{ILL}^\omega \arrow[rr, "\mathrm{lrt}"'] & & \mathsf{ILL}^\omega \arrow[u, "\mathrm{i}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{IL}^\omega \arrow[r, "\mathrm{mrt}"] \arrow[d, "\mathrm{t}"'] & \mathsf{IL}^\omega \\
\mathsf{ILL}^\omega \arrow[r, "\mathrm{lrt}"'] & \mathsf{ILL}^\omega \arrow[u, "\mathrm{i}"']
\end{tikzcd}$$

Figure 13.3: factorisations $\mathrm{mrq} = \forall \circ \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ and $\mathrm{mrt} = \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{t}$.

13.17 Proposition (factorisations $\mathrm{mrq} = \forall \circ \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ and $\mathrm{mrt} = \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{t}$).

1. For all formulas A of IL^ω we have $\mathsf{ILL}^\omega \vdash !A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \multimapboth !\forall \underline{b}\, (A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a}; \underline{b})$, therefore $\mathsf{IL}^\omega \vdash A_{\mathrm{mrq}}(\underline{a}) \leftrightarrow \forall \underline{b}\, (A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a}; \underline{b})^{\mathrm{i}}$ [22, proposition 3.7].

2. For all formulas A of IL^ω we have $\mathsf{ILL}^\omega \vdash A_{\mathrm{mrt}}(\underline{a})^{\mathrm{t}} \multimapboth (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)$, therefore $\mathsf{IL}^\omega \vdash A_{\mathrm{mrt}}(\underline{a}) \multimapboth (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)^{\mathrm{i}}$ [22, proposition 3.7].

13.18 Proof.

1. The proof is by induction on the structure of A. Let us only do the case of $A \to B$ to illustrate the need for $\forall \underline{b}$ in the claim, and the case of $\exists x\, A$ to illustrate the need for ! in the claim.

   $\underline{A \to B}$ Using $\vdash !\forall \underline{x}\, A \multimapboth !\forall \underline{x}\, !A$ in the first and sixth equivalences, $\vdash !(!A \multimap B) \multimapboth !(!A \multimap !B)$ in the second and fifth equivalences, $\vdash !(A \,\&\, B) \multimapboth !A \otimes !B$ in the third equivalence, induction hypothesis in the fourth equivalence, and $\vdash (A \multimap \forall \underline{x}\, B) \multimapboth \forall \underline{x}\, (A \multimap B)$ (with $\underline{x} \notin \mathrm{FV}(A)$) in the seventh equivalence, we get

$$
\begin{aligned}
!(A \to B)_{\mathrm{mrq}}(\underline{C})^{\mathrm{q}} &\equiv \\
!\forall \underline{a}\,(A_{\mathrm{mrq}}(\underline{a}) \wedge A \to B_{\mathrm{mrq}}(\underline{C}\underline{a}))^{\mathrm{q}} &\equiv \\
!\forall \underline{a}\,\big(!(A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \mathbin{\&} A^{\mathrm{q}}) \multimap B_{\mathrm{mrq}}(\underline{C}\underline{a})^{\mathrm{q}}\big) &\multimapboth \\
!\forall \underline{a}\,!\big(!(A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \mathbin{\&} A^{\mathrm{q}}) \multimap B_{\mathrm{mrq}}(\underline{C}\underline{a})^{\mathrm{q}}\big) &\multimapboth \\
!\forall \underline{a}\,!\big(!(A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \mathbin{\&} A^{\mathrm{q}}) \multimap !B_{\mathrm{mrq}}(\underline{C}\underline{a})^{\mathrm{q}}\big) &\multimapboth \\
!\forall \underline{a}\,!\big(!A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \otimes !A^{\mathrm{q}} \multimap !B_{\mathrm{mrq}}(\underline{C}\underline{a})^{\mathrm{q}}\big) &\multimapboth \\
!\forall \underline{a}\,!(!\forall \underline{b}\,(A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a};\underline{b}) \otimes !A^{\mathrm{q}} \multimap !\forall \underline{d}\,(B^{\mathrm{q}})_{\mathrm{lrt}}(\underline{C}\underline{a};\underline{d})) &\multimapboth \\
!\forall \underline{a}\,!(!\forall \underline{b}\,(A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a};\underline{b}) \otimes !A^{\mathrm{q}} \multimap \forall \underline{d}\,(B^{\mathrm{q}})_{\mathrm{lrt}}(\underline{C}\underline{a};\underline{d})) &\multimapboth \\
!\forall \underline{a}\,(!\forall \underline{b}\,(A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a};\underline{b}) \otimes !A^{\mathrm{q}} \multimap \forall \underline{d}\,(B^{\mathrm{q}})_{\mathrm{lrt}}(\underline{C}\underline{a};\underline{d})) &\multimapboth \\
!\forall \underline{a},\underline{d}\,(!\forall \underline{b}\,(A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a};\underline{b}) \otimes !A^{\mathrm{q}} \multimap (B^{\mathrm{q}})_{\mathrm{lrt}}(\underline{C}\underline{a};\underline{d})) &\equiv \\
!\forall \underline{a},\underline{d}\,(!A^{\mathrm{q}} \multimap B^{\mathrm{q}})_{\mathrm{lrt}}(\underline{C};\underline{a},\underline{d}) &\equiv \\
!\forall \underline{a},\underline{d}\,((A \to B)^{\mathrm{q}})_{\mathrm{lrt}}(\underline{C};\underline{a},\underline{d}). &
\end{aligned}
$$

$\underline{\exists x\, A}$ Using $\vdash !A \multimapboth !!A$ in the first equivalence, $\vdash !(A \mathbin{\&} B) \multimapboth !A \otimes !B$ in the second equivalence, and induction hypothesis in the third equivalence, we get

$$
\begin{aligned}
!(\exists x\, A)_{\mathrm{mrq}}(x,\underline{a})^{\mathrm{q}} &\equiv \\
!(A_{\mathrm{mrq}}(\underline{a}) \wedge A)^{\mathrm{q}} &\equiv \\
!(A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \mathbin{\&} A^{\mathrm{q}}) &\multimapboth \\
!!(A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \mathbin{\&} A^{\mathrm{q}}) &\multimapboth \\
!(!A_{\mathrm{mrq}}(\underline{a})^{\mathrm{q}} \otimes !A^{\mathrm{q}}) &\multimapboth \\
!(!\forall \underline{b}\,(A^{\mathrm{q}})_{\mathrm{lrt}}(\underline{a};\underline{b}) \otimes !A^{\mathrm{q}}) &\equiv \\
!(\exists x\, !A^{\mathrm{q}})_{\mathrm{lrt}}(x,\underline{a};\,) &\equiv \\
!((\exists x\, A)^{\mathrm{q}})_{\mathrm{lrt}}(x,\underline{a};\,). &
\end{aligned}
$$

2. The proof is by induction on the structure of A. Let us see only the most difficult cases.

$\underline{A \vee B}$ Using $\vdash !A \otimes !B \multimapboth !(A \mathbin{\&} B)$ in first equivalence, induction hypothesis in the second equivalence, $\vdash C \multimapboth !C$ with $C \equiv (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};\underline{b})$ (since we can prove by induction on the structure of A that for all formulas A of IL^ω there exists a formula B of IL^ω such that $\vdash (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};\underline{b}) \multimapboth !B$ in ILL^ω [22, point (b) of the proof of proposition 3.7]) in third and fifth equivalences, and $\vdash C \multimapboth !C$ with $C \equiv (!(x = \mathrm{t}) \multimap !A) \mathbin{\&} (!(e = \mathrm{f}) \multimap !B)$ [22, point (a) of the proof of proposition 3.7] in the fourth equivalence, we get

$$
\begin{aligned}
(A \vee B)_{\mathrm{mrt}}(e,\underline{a},\underline{c})^{\mathrm{t}} &\equiv \\
\big((e = \mathrm{t} \to A_{\mathrm{mrt}}(\underline{a})) \wedge (e = \mathrm{f} \to B_{\mathrm{mrt}}(\underline{c}))\big)^{\mathrm{t}} &\equiv \\
!(!(e = \mathrm{t}) \multimap A_{\mathrm{mrt}}(\underline{a})^{\mathrm{t}}) \otimes !(!(e = \mathrm{f}) \multimap B_{\mathrm{mrt}}(\underline{c})^{\mathrm{t}}) &\multimapboth \\
!\big((!(e = \mathrm{t}) \multimap A_{\mathrm{mrt}}(\underline{a})^{\mathrm{t}}) \mathbin{\&} (!(e = \mathrm{f}) \multimap B_{\mathrm{mrt}}(\underline{c})^{\mathrm{t}})\big) &\multimapboth
\end{aligned}
$$

$$
\begin{aligned}
!\big((!(e = \mathrm{t}) \multimap (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)) \,\&\, (!(e = \mathrm{f}) \multimap (B^{\mathrm{t}})_{\mathrm{lrt}}(\underline{c};))\big) &\circ\!\!-\!\!\circ \\
!\big((!(e = \mathrm{t}) \multimap !(A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)) \,\&\, (!(e = \mathrm{f}) \multimap !(B^{\mathrm{t}})_{\mathrm{lrt}}(\underline{c};))\big) &\circ\!\!-\!\!\circ \\
(!(e = \mathrm{t}) \multimap !(A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)) \,\&\, (!(e = \mathrm{f}) \multimap !(B^{\mathrm{t}})_{\mathrm{lrt}}(\underline{c};)) &\circ\!\!-\!\!\circ \\
(!(e = \mathrm{t}) \multimap (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)) \,\&\, (!(e = \mathrm{f}) \multimap (B^{\mathrm{t}})_{\mathrm{lrt}}(\underline{c};)) &\equiv \\
(A^{\mathrm{t}} \oplus B^{\mathrm{t}})_{\mathrm{lrt}}(e, \underline{a}; \underline{c}) &\equiv \\
((A \vee B)^{\mathrm{t}})_{\mathrm{lrt}}(e, \underline{a}; \underline{c}).&
\end{aligned}
$$

$\underline{\forall x\, A}$ Using the truth of mrt and that $\mathsf{IL}^\omega \vdash A \leftrightarrow B$ implies $\mathsf{ILL}^\omega \vdash A^{\mathrm{t}} \circ\!\!-\!\!\circ B^{\mathrm{t}}$ (by applying the soundness theorem of t to $\mathsf{IL}^\omega \vdash A \to B$ and $\mathsf{IL}^\omega \vdash B \to A$) in the first equivalence, and induction hypothesis in the second equivalence, we get

$$
\begin{aligned}
(\forall x\, A)_{\mathrm{mrt}}(\underline{A})^{\mathrm{t}} &\circ\!\!-\!\!\circ \\
((\forall x\, A)_{\mathrm{mrt}}(\underline{A}) \wedge \forall x\, A)^{\mathrm{t}} &\equiv \\
(\forall x\, A_{\mathrm{mrt}}(\underline{A}x) \wedge \forall x\, A)^{\mathrm{t}} &\equiv \\
!\forall x\, A_{\mathrm{mrt}}(\underline{A}x)^{\mathrm{t}} \otimes !\forall x\, A^{\mathrm{t}} &\circ\!\!-\!\!\circ \\
!\forall x\, (A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{A}x;) \otimes !\forall x\, A^{\mathrm{t}} &\equiv \\
(!\forall x\, A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{A};) &\equiv \\
((\forall x\, A)^{\mathrm{t}})_{\mathrm{lrt}}(\underline{A};).&
\end{aligned}
$$

13.19 Remark. The factorisation of mrq may seem different from the factorisation of mrt because of the bangs and the quantifications $\forall \underline{b}$. But the difference is only apparent because they also appear in the factorisation of mrt behind the notation since:

1. A^{t} is equivalent to a banged formula (remark 2 in proof 13.11) [22, remark 3.8];
2. the reason why there is no second tuple of variables $\underline{b}$ is because it is quantified in $A_{\mathrm{mrt}}(\underline{a})$ and in $(A^{\mathrm{t}})_{\mathrm{lrt}}(\underline{a};)$ [22, remark 3.8].

13.20.

1. Since $\mathrm{mrq} = \forall \circ \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ and mrq has q-truth, then the composition $\forall \circ \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{q}$ is adding copies of the original formulas in the right clauses. Let us see which clauses are those:
   (a) q adds bangs in the clauses of $\vee$, (premise of) $\to$, and $\exists$;
   (b) lrt adds copies of the original formulas whenever it finds a bang;
   (c) i keeps the copies.

   In conclusion, the composition adds copies of the original formulas in the clauses of $\vee$, (premise of) $\to$, and $\exists$ [22, section 3.5].
2. Analogously, the composition $\mathrm{mrt} = \mathrm{i} \circ \mathrm{lrt} \circ \mathrm{t}$ adds copies of the original formulas in the clauses of A_{at}, $\to$ and $\forall$ [22, section 3.5].

This leads us to our first heuristic on how to hardwire truth.

13.21 Heuristic. If we have a proof interpretation, then we should try to hardwire in it

1. q-truth by adding copies of the original formulas in the clauses of $\vee$, (premise of) $\to$, and $\exists$ [22, section 3.5].

2. t-truth by adding copies of the original formulas in the clauses of A_{at}, $\to$ and $\forall$ [22, section 3.5].

13.22. By applying this heuristic to mr, br, DN, B and | we get their variants with q- and t-truth.

13.23 Remark. For most proof interpretations there is no need to add a copy in the clause of A_{at}. For mrt there is no need to add a copy in the clause of $\forall$ (because $\forall x\, A_{\mathrm{mrt}}(\underline{A})$ implies $\forall x\, A$ by truth [22, page 592]). For other proof interpretations we really need to add a copy in the clause of $\forall$ (as discussed in paragraph 6.4 for DNt).

13.3 Heuristic 2

13.24. Our second heuristic on how to hardwire truth consists in first hardwiring q-truth (copying what is done for mrq) and then upgrading to t-truth. To see how to do this upgrade, we see how mrq can be upgraded to mrt.

13.25 Proposition (factorisation mrt = mrq $\wedge$ id)**.** For all formulas A of IL^ω we have $\mathsf{IL}^\omega \vdash A_{\mathrm{mrt}}(\underline{a}) \leftrightarrow A_{\mathrm{mrq}}(\underline{a}) \wedge A$ [22, theorem 2.5].

13.26 Proof. The proof is by induction on the structure of A. Let us see the case of $\to$; the cases of A_{at}, $\wedge$, $\vee$, $\forall$ and $\exists$ are analogous. Using the induction hypothesis in the first equivalence we get

$$\begin{aligned}
(A \to B)_{\mathrm{mrt}}(\underline{B}) &\equiv \\
\forall \underline{a}\, (A_{\mathrm{mrt}}(\underline{a}) \to B_{\mathrm{mrt}}(\underline{B}\underline{a})) \wedge (A \to B) &\leftrightarrow \\
\forall \underline{a}\, (A_{\mathrm{mrq}}(\underline{a}) \wedge A \to B_{\mathrm{mrq}}(\underline{B}\underline{a}) \wedge B) \wedge (A \to B) &\leftrightarrow \\
\forall \underline{a}\, (A_{\mathrm{mrq}}(\underline{a}) \wedge A \to B_{\mathrm{mrq}}(\underline{B}\underline{a})) \wedge (A \to B) &\equiv \\
(A \to B)_{\mathrm{mrq}}(\underline{B}) \wedge (A \to B)&.
\end{aligned}$$

13.27. There is another way of reading the factorisation mrt = mrq$\wedge$id: it suggests that if we have a proof interpretation with q-truth Iq (that may not have truth for all formulas), we may try to upgrade it to a proof interpretation with t-truth It (that will have truth for all formulas) by defining $A_{\mathrm{It}} :\equiv A_{\mathrm{Iq}} \wedge A$ (then, indeed, $A_{\mathrm{It}} \to A$ for all formulas A). This leads us to our second heuristic on how to hardwire truth.

13.28 Heuristic. If we have a proof interpretation I, we should:

1. start by hardwiring q-truth, copying what is done for mrq, getting Iq;

2. then upgrade to t-truth It by defining $A_{\mathrm{It}} :\equiv A_{\mathrm{Iq}} \wedge A$.

13.29. By applying this heuristic to mr, br, DN, B and | we get their variants with q- and t-truth.

13.4 Heuristic 3

13.30. Given a proof interpretation I of HA^ω into itself, defined by recursion on the structure of formulas by

$$
\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{I}} &:\equiv \ldots, \\
(A \wedge B)_{\mathrm{I}} &:\equiv \ldots, \\
(A \vee B)_{\mathrm{I}} &:\equiv \ldots, \\
(A \to B)_{\mathrm{I}} &:\equiv \ldots, \\
(\forall x\, A)_{\mathrm{I}} &:\equiv \ldots, \\
(\exists x\, A)_{\mathrm{I}} &:\equiv \ldots,
\end{aligned}
$$

we can try to hardwire in it t-truth by adding copies of the formulas under interpretation in all clauses:

$$
\begin{aligned}
(A_{\mathrm{at}})_{\mathrm{It}} &:\equiv \ldots \wedge A_{\mathrm{at}}, \\
(A \wedge B)_{\mathrm{It}} &:\equiv \ldots \wedge (A \wedge B), \\
(A \vee B)_{\mathrm{It}} &:\equiv \ldots \wedge (A \vee B), \\
(A \to B)_{\mathrm{It}} &:\equiv \ldots \wedge (A \to B), \\
(\forall x\, A)_{\mathrm{It}} &:\equiv \ldots \wedge \forall x\, A, \\
(\exists x\, A)_{\mathrm{It}} &:\equiv \ldots \wedge \exists x\, A.
\end{aligned}
$$

The resulting It trivially has the truth property $A_{\mathrm{It}} \to A$ for all formulas A. But the questions is: is It sound? We are going to study its soundness in the following way.

1. First we define an extension $\mathsf{HA}^\omega_{\mathrm{c}}$ of HA^ω and two translations t and o, and we prove the factorisation $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$ (where I is extended to $\mathsf{HA}^\omega_{\mathrm{c}}$) illustrated in the right of figure 13.4.

2. Then we study the soundness of t, I (extended to $\mathsf{HA}^\omega_{\mathrm{c}}$) and o. When we know that t, I and o are sound, then we conclude that $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$ is sound.

We also study along the same lines how to hardwire q-truth in I, using a translation q instead of t, as illustrated in the left of figure 13.4.

$$
\begin{tikzcd}
\mathsf{HA}^\omega \arrow[r, "\mathrm{q}"] \arrow[rrr, bend right, "\mathrm{Iq}"'] & \mathsf{HA}^\omega_{\mathrm{c}} \arrow[r, "\mathrm{I}"] & \mathsf{HA}^\omega_{\mathrm{c}} \arrow[r, "\mathrm{o}"] & \mathsf{HA}^\omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{HA}^\omega \arrow[r, "\mathrm{t}"] \arrow[rrr, bend right, "\mathrm{It}"'] & \mathsf{HA}^\omega_{\mathrm{c}} \arrow[r, "\mathrm{I}"] & \mathsf{HA}^\omega_{\mathrm{c}} \arrow[r, "\mathrm{o}"] & \mathsf{HA}^\omega
\end{tikzcd}
$$

Figure 13.4: factorisations $\mathrm{Iq} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{q}$ and $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$.

13.31 Definition. We defined the theory $\mathsf{HA}^\omega_{\mathrm{c}}$ [22, section 3.5] as follows.

1. The language of $\mathsf{HA}^\omega_{\mathrm{c}}$ is the language of HA^ω enriched with a fresh atomic formula A_{c}, called *copy* of A, for each non-atomic formula A of HA^ω. For an atomic formula A_{at} of HA^ω we take $(A_{\mathrm{at}})_{\mathrm{c}} :\equiv A_{\mathrm{at}}$.

2. We extend the notion of free variables to copies by $\mathrm{FV}(A_{\mathrm{c}}) := \mathrm{FV}(A)$, and the notion of substitution by $A_{\mathrm{c}}[t/x] :\equiv A[t/x]_{\mathrm{c}}$.
3. The axioms and rules of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ are the ones of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ (but based on the language of $\mathsf{HA}^{\omega}_{\mathrm{c}}$) enriched with the axioms $(A_1)_{\mathrm{c}} \to \cdots \to (A_n)_{\mathrm{c}}$ (associating $\to$ to the right) for each theorem $A_1 \to \cdots \to A_n$ (possibly with $n = 1$) of HA^{ω}.

13.32. Informally, for each theorem $A_1, \ldots, A_{n-1} \vdash A_n$ of HA^{ω}, we add the corresponding axiom $(A_1)_{\mathrm{c}}, \ldots, (A_{n-1})_{\mathrm{c}} \vdash (A_n)_{\mathrm{c}}$. But to avoid using the deduction theorem (that fails in some versions of Heyting arithmetic), we prefer to state this with all formulas on the right, that is in the form $A_1 \to \cdots \to A_n$ and $(A_1)_{\mathrm{c}} \to \cdots \to (A_n)_{\mathrm{c}}$.

13.33 Definition.

1. To each formula A of HA^{ω} we assign the formula A^{q} of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ [22, section 3.5] defined by recursion on the structure of A by

$$\begin{aligned}
A_{\mathrm{at}}^{\mathrm{q}} &:\equiv A_{\mathrm{at}}, \\
(A \wedge B)^{\mathrm{q}} &:\equiv A^{\mathrm{q}} \wedge B^{\mathrm{q}}, \\
(A \vee B)^{\mathrm{q}} &:\equiv (A^{\mathrm{q}} \wedge A_{\mathrm{c}}) \vee (B^{\mathrm{q}} \wedge B_{\mathrm{c}}), \\
(A \to B)^{\mathrm{q}} &:\equiv A^{\mathrm{q}} \wedge A_{\mathrm{c}} \to B^{\mathrm{q}}, \\
(\forall x\, A)^{\mathrm{q}} &:\equiv \forall x\, A^{\mathrm{q}}, \\
(\exists x\, A)^{\mathrm{q}} &:\equiv \exists x\, (A^{\mathrm{q}} \wedge A_{\mathrm{c}}).
\end{aligned}$$

2. To each formula A of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ we assign the formula A^{t} of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ [22, section 3.5] defined by recursion on the structure of A by

$$\begin{aligned}
A_{\mathrm{at}}^{\mathrm{t}} &:\equiv A_{\mathrm{at}}, \\
(A \wedge B)^{\mathrm{t}} &:\equiv (A^{\mathrm{t}} \wedge B^{\mathrm{t}}) \wedge (A \wedge B)_{\mathrm{c}}, \\
(A \vee B)^{\mathrm{t}} &:\equiv (A^{\mathrm{t}} \vee B^{\mathrm{t}}) \wedge (A \vee B)_{\mathrm{c}}, \\
(A \to B)^{\mathrm{t}} &:\equiv (A^{\mathrm{t}} \to B^{\mathrm{t}}) \wedge (A \to B)_{\mathrm{c}}, \\
(\forall x\, A)^{\mathrm{t}} &:\equiv \forall x\, A^{\mathrm{t}} \wedge (\forall x\, A)_{\mathrm{c}}, \\
(\exists x\, A)^{\mathrm{t}} &:\equiv \exists x\, A^{\mathrm{t}} \wedge (\exists x\, A)_{\mathrm{c}}.
\end{aligned}$$

3. To each formula A of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ we assign the formula A^{o} [22, section 3.5] of HA^{ω} defined by recursion on the structure of A by

$$\begin{aligned}
(A_{\mathrm{c}})^{\mathrm{o}} &:\equiv A, \\
(A \wedge B)^{\mathrm{o}} &:\equiv A^{\mathrm{o}} \wedge B^{\mathrm{o}}, \\
(A \vee B)^{\mathrm{o}} &:\equiv A^{\mathrm{o}} \vee B^{\mathrm{o}}, \\
(A \to B)^{\mathrm{o}} &:\equiv A^{\mathrm{o}} \to B^{\mathrm{o}}, \\
(\forall x\, A)^{\mathrm{o}} &:\equiv \forall x\, A^{\mathrm{o}}, \\
(\exists x\, A)^{\mathrm{o}} &:\equiv \exists x\, A^{\mathrm{o}}.
\end{aligned}$$

13.34. The letter "o" in the symbol for the translation o comes from "original" since this translation replaces copies A_{c}s by their originals As.

13.35. In point 3 of the definition 13.33 (by recursion on the structure of A) we took the base case to be the case of formulas of the form A_{c}. This is correct because all atomic formulas B of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ are of the form A_{c}: either B is one of the old atomic formulas of HA^{ω} and so $B_{\mathrm{c}} :\equiv B$, or B is one of the new atomic formulas A_{c} introduced by a non-atomic formula A of HA^{ω}.

13.36.

1. The formula A^{q} is obtained from A by adding copies in $\vee$, (the premise of) $\to$ and $\exists$.
2. The formula A^{t} is obtained from A by replacing each subformula B by $B \wedge B_{\mathrm{c}}$, that is by "duplicating" with copies every subformula. Actually, in the case of A_{at}, since $A_{\mathrm{at}} \equiv (A_{\mathrm{at}})_{\mathrm{c}}$, we just take $A^{\mathrm{t}}_{\mathrm{at}} \equiv A_{\mathrm{at}}$ instead of $A^{\mathrm{t}}_{\mathrm{at}} \equiv A_{\mathrm{at}} \wedge (A_{\mathrm{at}})_{\mathrm{c}}$.
3. The formula A^{o} is obtained from A by replacing each subformula of the form B_{c} by B, that is by replacing every copy by its original.

13.37. As a curiosity we can mention $\mathrm{o} \circ \mathrm{q} = \mathrm{id}$ and $\mathrm{o} \circ \mathrm{t} = \mathrm{id}$ (that is $\mathsf{HA}^{\omega} \vdash (A^{\mathrm{q}})^{\mathrm{o}} \leftrightarrow A$ and $\mathsf{HA}^{\omega} \vdash (A^{\mathrm{t}})^{\mathrm{o}} \leftrightarrow A$): for example, t maps each subformula S to $S \wedge S_{\mathrm{c}}$, and then o "undoes" this by mapping $S \wedge S_{\mathrm{c}}$ to $S \wedge S$, that is S.

13.38. The It defined above has copies in all clauses, but for example mrt, DNt and Bt only have copies in some clauses. So it is natural to ask if It is the "right" t-truth variant It′ of I. Indeed, it is in the sense that It and It′ are equivalent: since $A_{\mathrm{It}'} \to A$, then $A_{\mathrm{It}'} \leftrightarrow A_{\mathrm{It}'} \wedge A$, so modulo equivalence we can add copies in all clauses of It′, resulting in our It.

13.39 Theorem (soundness)**.**

1. If $\mathsf{HA}^{\omega} \vdash A$, then $\mathsf{HA}^{\omega}_{\mathrm{c}} \vdash A^{\mathrm{q}}$ [22, section 3.5].
2. If $\mathsf{HA}^{\omega} \vdash A$, then $\mathsf{HA}^{\omega}_{\mathrm{c}} \vdash A^{\mathrm{t}}$ [22, section 3.5].
3. If $\mathsf{HA}^{\omega}_{\mathrm{c}} \vdash A$, then $\mathsf{HA}^{\omega} \vdash A^{\mathrm{o}}$.

13.40 Proof. First, let us make some remarks.

1. We adopt here (with the proper adaptations) the remarks made in the beginning of proof 3.12.
2. The interpretation A^{t} of an axiom A is of the form $\ldots \wedge A_{\mathrm{c}}$. So to prove that A^{t} is interpretable we have in particular to prove A_{c}: since A is provable, then A_{c} is an axiom. Analogously, when proving the interpretation of a rule $A \,/\, B$ we have to prove B_{c}: since A is provable, then B is provable, so B_{c} is an axiom. Since the argument is always the same, we will systematically omit A_{c}. When we do it, we write "$\equiv$" instead of $\equiv$.
3. For all formulas A of HA^{ω} we have $\mathsf{HA}^{\omega} \vdash A^{\mathrm{t}} \to A_{\mathrm{c}}$.

Let us the theorem by induction on the derivation of A.

1. $\underline{A \to A \vee B}$ We have

$$(A \to A \vee B)^{\mathrm{q}} \equiv A^{\mathrm{q}} \wedge A_{\mathrm{c}} \to (A^{\mathrm{q}} \wedge A_{\mathrm{c}}) \vee (B^{\mathrm{q}} \wedge B_{\mathrm{c}}).$$

Analogously for $A \to A \wedge A$, $A \vee A \to A$, $A \wedge B \to A$, $A \wedge B \to B \wedge A$, $A \vee B \to B \vee A$ and $\bot \to A$.

$\underline{A[t/x] \to \exists x\, A}$ We have

$$(A[t/x] \to \exists x\, A)^{\mathrm{q}} \equiv A[t/x]^{\mathrm{q}} \wedge A[t/x]_{\mathrm{c}} \to \exists x\, (A^{\mathrm{q}} \wedge A_{\mathrm{c}}).$$

Here we use $A[t/x]^{\mathrm{q}} \equiv A^{\mathrm{q}}[t/x]$. Analogously for $\forall x\, A \to A[t/x]$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ We have

$$(A \wedge B \to C)^{\mathrm{q}} \equiv (A^{\mathrm{q}} \wedge B^{\mathrm{q}}) \wedge (A \wedge B)_{\mathrm{c}} \to C^{\mathrm{q}},$$
$$(A \to (B \to C))^{\mathrm{q}} \equiv A^{\mathrm{q}} \wedge A_{\mathrm{c}} \to (B^{\mathrm{q}} \wedge B_{\mathrm{c}} \to C^{\mathrm{q}}).$$

The non-trivial part is that we need $A_{\mathrm{c}} \wedge B_{\mathrm{c}} \to (A \wedge B)_{\mathrm{c}}$: it follows from $A_{\mathrm{c}} \to (B_{\mathrm{c}} \to (A \wedge B)_{\mathrm{c}})$, which is an axiom since $A \to (B \to A \wedge B)$ is a theorem. Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, C \vee A \to C \vee B}$ We have

$$(A \to B)^{\mathrm{q}} \equiv A^{\mathrm{q}} \wedge A_{\mathrm{c}} \to B^{\mathrm{q}},$$

$$(C \vee A \to C \vee B)^{\mathrm{q}} \equiv$$
$$((C^{\mathrm{q}} \wedge C_{\mathrm{c}}) \vee (A^{\mathrm{q}} \wedge A_{\mathrm{c}})) \wedge (C \vee A)_{\mathrm{c}} \to (C^{\mathrm{q}} \wedge C_{\mathrm{c}}) \vee (B^{\mathrm{q}} \wedge B_{\mathrm{c}}).$$

The non-trivial part is that we need $A_{\mathrm{c}} \to B_{\mathrm{c}}$: it is an axiom since we are assuming that $A \to B$ is a theorem. Analogously for A, $A \to B \,/\, B$ and $A \to B$, $B \to C \,/\, A \to C$.

$\underline{A \to B \,/\, \exists x\, A \to B}$ We have

$$(A \to B)^{\mathrm{q}} \equiv A^{\mathrm{q}} \wedge A_{\mathrm{c}} \to B^{\mathrm{q}},$$
$$(\exists x\, A \to B)^{\mathrm{q}} \equiv \exists x\, (A^{\mathrm{q}} \wedge A_{\mathrm{c}}) \wedge (\exists x\, A)_{\mathrm{c}} \to B^{\mathrm{q}}.$$

Here we use $x \notin \mathrm{FV}(B) = \mathrm{FV}(B^{\mathrm{q}})$. Analogously for $A \to B \,/\, A \to \forall x\, B$.

$\underline{A_{\mathrm{at}}(\Pi xy) \leftrightarrow A_{\mathrm{at}}(x)}$ We have

$$(A_{\mathrm{at}}(\Pi xy) \leftrightarrow A_{\mathrm{at}}(x))^{\mathrm{q}} \equiv$$
$$(A_{\mathrm{at}}(\Pi xy) \wedge A_{\mathrm{at}}(\Pi xy) \to A_{\mathrm{at}}(x)) \wedge (A_{\mathrm{at}}(x) \wedge A_{\mathrm{at}}(x) \to A_{\mathrm{at}}(\Pi xy)).$$

Analogously for the axioms of Σ, $\underline{\mathrm{R}}$, $=_0$ and S.

$\underline{A[0/x],\, A \to A[\mathrm{S}x/x] \,/\, A}$ We have

$$A[0/x]^{\mathrm{q}},$$
$$(A \to A[\mathrm{S}x/x])^{\mathrm{q}} \equiv A^{\mathrm{q}} \wedge A_{\mathrm{c}} \to A[\mathrm{S}x/x]^{\mathrm{q}},$$
$$A^{\mathrm{q}}.$$

Here we use $A[0/x]^{\mathrm{q}} \equiv A^{\mathrm{q}}[0/x]$, $A[\mathrm{S}x/x]^{\mathrm{q}} \equiv A^{\mathrm{q}}[\mathrm{S}x/x]$, and that A_{c} is an axiom since we are assuming that A is a theorem.

2. $\underline{A \to A \wedge A}$ We have

$$(A \to A \wedge A)^{\mathrm{t}} \text{ “}\equiv\text{” } A^{\mathrm{t}} \to (A^{\mathrm{t}} \wedge A^{\mathrm{t}}) \wedge (A \wedge A)_{\mathrm{c}}.$$

The only non-trivial verification is $A^{\mathrm{t}} \to (A \wedge A)_{\mathrm{c}}$: it follows from remark 3 and $A_{\mathrm{c}} \to (A \wedge A)_{\mathrm{c}}$, which is an axiom since $A \to A \wedge A$ is a theorem. Analogously for $A \vee A \to A$, $A \wedge B \to A$, $A \to A \vee B$, $A \wedge B \to B \wedge A$, $A \vee B \to B \vee A$ and $\perp \to A$.

$\underline{A[t/x] \to \exists x\, A}$ We have

$$(A[t/x] \to \exists x\, A)^{\mathrm{t}} \text{ “}\equiv\text{” } A[t/x]^{\mathrm{t}} \to \exists x\, A^{\mathrm{t}} \wedge (\exists x\, A)_{\mathrm{c}}.$$

We have $A[t/x]^{\mathrm{t}} \to \exists x\, A^{\mathrm{t}}$ by $A[t/x]^{\mathrm{t}} \equiv A^{\mathrm{t}}[t/x]$. We have $A[t/x]^{\mathrm{t}} \to (\exists x\, A)_{\mathrm{c}}$ by remark 3 and $A[t/x]_{\mathrm{c}} \to (\exists x\, A)_{\mathrm{c}}$, which is an axiom since $A[t/x] \to \exists x\, A$ is a theorem. Analogously for $\forall x\, A \to A[t/x]$.

$\underline{A \to B \,/\, C \vee A \to C \vee B}$ We have

$$(A \to B)^{\mathrm{t}} \text{ “}\equiv\text{” } A^{\mathrm{t}} \to B^{\mathrm{t}},$$
$$(C \vee A \to C \vee B)^{\mathrm{t}} \text{ “}\equiv\text{” } (C^{\mathrm{t}} \vee A^{\mathrm{t}}) \wedge (C \vee A)_{\mathrm{c}} \to (C^{\mathrm{t}} \vee B^{\mathrm{t}}) \wedge (C \vee B)_{\mathrm{c}}.$$

The only non-trivial verification is $(C \vee A)_{\mathrm{c}} \to (C \vee B)_{\mathrm{c}}$: it is an axiom because we are assuming that $C \vee A \to C \vee B$ is a theorem. Analogously for A, $A \to B \,/\, B$ and $A \to B$, $B \to C \,/\, A \to C$.

$\underline{A \wedge B \to C \,/\, A \to (B \to C)}$ We have

$$(A \wedge B \to C)^{\mathrm{t}} \text{ “}\equiv\text{” } (A^{\mathrm{t}} \wedge B^{\mathrm{t}}) \wedge (A \wedge B)_{\mathrm{c}} \to C^{\mathrm{t}}, \tag{13.4}$$
$$(A \to (B \to C))^{\mathrm{t}} \text{ “}\equiv\text{” } A^{\mathrm{t}} \to (B^{\mathrm{t}} \to C^{\mathrm{t}}) \wedge (B \to C)_{\mathrm{c}}.$$

First we prove $(*)$ $A^{\mathrm{t}} \wedge B^{\mathrm{t}} \to (A \wedge B)_{\mathrm{c}}$: we have $A^{\mathrm{t}} \wedge B^{\mathrm{t}} \to A_{\mathrm{c}} \wedge B_{\mathrm{c}}$ by remark 3, and we have $A_{\mathrm{c}} \wedge B_{\mathrm{c}} \to (A \wedge B)_{\mathrm{c}}$ because $A_{\mathrm{c}} \to (B_{\mathrm{c}} \to (A \wedge B)_{\mathrm{c}})$ is an axiom since $A \to (B \to A \wedge B)$ is a theorem. By $(*)$ we can dismiss $(A \wedge B)_{\mathrm{c}}$ in (13.4). Then the only non-trivial verification is $A^{\mathrm{t}} \to (B \to C)_{\mathrm{c}}$: it follows from remark 3 and $A_{\mathrm{c}} \to (B \to C)_{\mathrm{c}}$, which is an axiom since $A \to (B \to C)$ is a theorem. Analogously for $A \to (B \to C) \,/\, A \wedge B \to C$.

$\underline{A \to B \,/\, A \to \forall x\, B}$ We have

$$(A \to B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \to B^{\mathrm{t}},$$
$$(A \to \forall x\, B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \to \forall x\, B^{\mathrm{t}} \wedge (\forall x\, B)_{\mathrm{c}}.$$

Here we use $x \notin \mathrm{FV}(A) = \mathrm{FV}(A^{\mathrm{t}})$. The part $A^{\mathrm{t}} \to (\forall x\, B)_{\mathrm{c}}$ follows from remark 3 and $A_{\mathrm{c}} \to (\forall x\, B)_{\mathrm{c}}$, which is an axiom since we are assuming that $A \to \forall x\, B$ is a theorem. Analogously for $A \to B \,/\, \exists x\, A \to B$.

$\underline{A_{\mathrm{at}}(\Pi xy) \leftrightarrow A_{\mathrm{at}}(x)}$ We have

$$(A_{\mathrm{at}}(\Pi xy) \leftrightarrow A_{\mathrm{at}}(x))^{\mathrm{t}} \text{ “}\equiv\text{”}$$
$$(A_{\mathrm{at}}(\Pi xy) \to A_{\mathrm{at}}(x)) \wedge (A_{\mathrm{at}}(\Pi xy) \to A_{\mathrm{at}}(x))_{\mathrm{c}} \wedge$$
$$(A_{\mathrm{at}}(x) \to A_{\mathrm{at}}(\Pi xy)) \wedge (A_{\mathrm{at}}(x) \to A_{\mathrm{at}}(\Pi xy))_{\mathrm{c}}.$$

Analogously for the axioms of Σ, $\underline{\mathrm{R}}$, $=_0$ and S.

$\underline{A[0/x],\, A \to A[\mathrm{S}x/x] \,/\, A}$ We have

$$\begin{gathered} A[0/x]^{\mathrm{t}}, \\ (A \to A[\mathrm{S}x/x])^{\mathrm{t}} \text{ “}\equiv\text{” } A^{\mathrm{t}} \to A[\mathrm{S}x/x]^{\mathrm{t}}, \\ A^{\mathrm{t}}. \end{gathered}$$

Here we use $A[0/x]^{\mathrm{t}} \equiv A^{\mathrm{t}}[0/x]$ and $A[\mathrm{S}x/x]^{\mathrm{t}} \equiv A^{\mathrm{t}}[\mathrm{S}x/x]$.

3. $\underline{A \vee B \to B \vee A}$ We have

$$(A \vee B \to B \vee A)^{\mathrm{o}} \equiv A^{\mathrm{o}} \vee B^{\mathrm{o}} \to B^{\mathrm{o}} \vee A^{\mathrm{o}}.$$

Analogously for $A \to A \wedge A$, $A \vee A \to A$, $A \wedge B \to A$, $A \to A \vee B$, $A \wedge B \to B \wedge A$, $\bot \to A$ and for the axioms of $=_0$, S, Π, Σ and $\underline{\mathrm{R}}$.

$\underline{\forall x\, A \to A[t/x]}$ We have

$$(\forall x\, A \to A[t/x])^{\mathrm{o}} \equiv \forall x\, A^{\mathrm{o}} \to A[t/x]^{\mathrm{o}}.$$

Here we use $A[t/x]^{\mathrm{o}} \equiv A^{\mathrm{o}}[t/x]$. Analogously for $A[t/x] \to \exists x\, A$.

$\underline{A \wedge B \to B \,/\, A \to (B \to C)}$ We have

$$\begin{gathered} (A \wedge B \to C)^{\mathrm{o}} \equiv A^{\mathrm{o}} \wedge B^{\mathrm{o}} \to C^{\mathrm{o}}, \\ (A \to (B \to C))^{\mathrm{o}} \equiv A^{\mathrm{o}} \to (B^{\mathrm{o}} \to C^{\mathrm{o}}). \end{gathered}$$

Analogously for $A,\, A \to B \,/\, B$, $A \to B,\, B \to C \,/\, A \to C$, $A \to (B \to C) \,/\, A \wedge B \to C$ and $A \to B \,/\, C \vee A \to C \vee B$.

$\underline{A \to B \,/\, A \to \forall x\, B}$ We have

$$\begin{gathered} (A \to B)^{\mathrm{o}} \equiv A^{\mathrm{o}} \to B^{\mathrm{o}}, \\ (A \to \forall x\, B)^{\mathrm{o}} \equiv A^{\mathrm{o}} \to \forall x\, B^{\mathrm{o}}. \end{gathered}$$

Here we $x \notin \mathrm{FV}(A) = \mathrm{FV}(A^{\mathrm{o}})$. Analogously for $A \to B \,/\, \exists x\, A \to B$.

$\underline{A[0/x],\, A \to A[\mathrm{S}x/x] \,/\, A}$ We have

$$\begin{gathered} A[0/x]^{\mathrm{o}}, \\ (A \to A[\mathrm{S}x/x])^{\mathrm{o}} \equiv A^{\mathrm{o}} \to A[\mathrm{S}x/x]^{\mathrm{o}}, \\ A^{\mathrm{o}}. \end{gathered}$$

Here we use $A[0/x]^{\mathrm{o}} \equiv A^{\mathrm{o}}[0/x]$ and $A[\mathrm{S}x/x]^{\mathrm{o}} \equiv A^{\mathrm{o}}[\mathrm{S}x/x]$.

$\underline{(A_1)_{\mathrm{c}} \to \cdots \to (A_n)_{\mathrm{c}}}$ We have

$$((A_1)_{\mathrm{c}} \to \cdots \to (A_n)_{\mathrm{c}})^{\mathrm{o}} \equiv A_1 \to \cdots \to A_n.$$

The formula $A_1 \to \cdots \to A_n$ is provable because the axiom $(A_1)_{\mathrm{c}} \to \cdots \to (A_n)_{\mathrm{c}}$ was introduced by the theorem $A_1 \to \cdots \to A_n$.

13.41. Our third heuristic applies to proof interpretations I that are "well-behaved" in a certain sense, and gives variants with q-truth Iq and t-truth It. In the next definition we define the exact meaning of "well-behaved", and the variants Iq and It.

13.42 Definition. Let I be a proof interpretation of HA^ω into HA^ω that assigns to each formula A of HA^ω to the formula $A_\mathrm{I}(\underline{a})$ of HA^ω with distinguished variables $\underline{a}$.

1. We say that I is *definable by recursion* (on the structure of formulas) if and only if there exist functions f_{A_at}, $f_\wedge$, $f_\vee$, $f_\to$, $f_\forall$ and $f_\exists$ such that for all formulas A_at, A and B of HA^ω we have

$$\begin{aligned}
(A_\mathrm{at})_\mathrm{I}(\underline{a}) &\equiv f_{A_\mathrm{at}}(A_\mathrm{at}),\\
(A \wedge B)_\mathrm{I}(\underline{c}) &\equiv f_\wedge(A_\mathrm{I}(\underline{a}), B_\mathrm{I}(\underline{b})),\\
(A \vee B)_\mathrm{I}(\underline{c}) &\equiv f_\vee(A_\mathrm{I}(\underline{a}), B_\mathrm{I}(\underline{b})),\\
(A \to B)_\mathrm{I}(\underline{c}) &\equiv f_\to(A_\mathrm{I}(\underline{a}), B_\mathrm{I}(\underline{b})),\\
(\forall x\, A)_\mathrm{I}(\underline{c}) &\equiv f_\forall(A_\mathrm{I}(\underline{a}), x),\\
(\exists x\, A)_\mathrm{I}(\underline{c}) &\equiv f_\exists(A_\mathrm{I}(\underline{a}), x).
\end{aligned}$$

2. If I is definable by recursion as above, then we define Iq by

$$\begin{aligned}
(A_\mathrm{at})_\mathrm{Iq}(\underline{a}) &:\equiv f_{A_\mathrm{at}}(A_\mathrm{at}),\\
(A \wedge B)_\mathrm{Iq}(\underline{c}) &:\equiv f_\wedge(A_\mathrm{Iq}(\underline{a}), B_\mathrm{Iq}(\underline{b})),\\
(A \vee B)_\mathrm{Iq}(\underline{c}) &:\equiv f_\vee(A_\mathrm{Iq}(\underline{a}) \wedge A, B_\mathrm{Iq}(\underline{b}) \wedge B),\\
(A \to B)_\mathrm{Iq}(\underline{c}) &:\equiv f_\to(A_\mathrm{Iq}(\underline{a}) \wedge A, B_\mathrm{Iq}(\underline{b})),\\
(\forall x\, A)_\mathrm{Iq}(\underline{c}) &:\equiv f_\forall(A_\mathrm{Iq}(\underline{a}), x),\\
(\exists x\, A)_\mathrm{Iq}(\underline{c}) &:\equiv f_\exists(A_\mathrm{Iq}(\underline{a}) \wedge A, x),
\end{aligned}$$

   and It by

$$\begin{aligned}
(A_\mathrm{at})_\mathrm{It}(\underline{a}) &:\equiv f_{A_\mathrm{at}}(A_\mathrm{at}),\\
(A \wedge B)_\mathrm{It}(\underline{c}) &:\equiv f_\wedge(A_\mathrm{It}(\underline{a}), B_\mathrm{It}(\underline{b})) \wedge (A \wedge B),\\
(A \vee B)_\mathrm{It}(\underline{c}) &:\equiv f_\vee(A_\mathrm{It}(\underline{a}), B_\mathrm{It}(\underline{b})) \wedge (A \vee B),\\
(A \to B)_\mathrm{It}(\underline{c}) &:\equiv f_\to(A_\mathrm{It}(\underline{a}), B_\mathrm{It}(\underline{b})) \wedge (A \to B),\\
(\forall x\, A)_\mathrm{It}(\underline{c}) &:\equiv f_\forall(A_\mathrm{It}(\underline{a}), x) \wedge \forall x\, A,\\
(\exists x\, A)_\mathrm{It}(\underline{c}) &:\equiv f_\exists(A_\mathrm{It}(\underline{a}), x) \wedge \exists x\, A.
\end{aligned}$$

3. We say that I is *well-behaved* (with respect to truth) if and only if:

   (a) I is definable by recursion as above;

   (b) I leaves invariant atomic formulas, that is $(A_\mathrm{at})_\mathrm{I}() \equiv A_\mathrm{at}$, or in other words, $f_{A_\mathrm{at}}(B) \equiv B$ and no variables are distinguished;

   (c) I leaves invariant implications between atomic formulas, that is $(A_\mathrm{at} \to B_\mathrm{at})_\mathrm{I}() \equiv A_\mathrm{at} \to B_\mathrm{at}$, or in other words, $f_\to(A_\mathrm{at}, B_\mathrm{at}) \equiv A_\mathrm{at} \to B_\mathrm{at}$ and no variables are distinguished;

(d) I respects conjunctions, that is $(A \wedge B)_{\mathrm{I}}(\underline{a}, \underline{b}) \equiv A_{\mathrm{I}}(\underline{a}) \wedge B_{\mathrm{I}}(\underline{b})$, or in other words, $f_{\wedge}(C, D) \equiv C \wedge D$ and the distinguished variables are the ones of C and D;

(e) the functions commute with o, that is $f_{\wedge}(A, B)^{\mathrm{o}} \equiv f_{\wedge}(A^{\mathrm{o}}, B^{\mathrm{o}})$, $f_{\forall}(A, x)^{\mathrm{o}} \equiv f_{\forall}(A^{\mathrm{o}}, x)$ and analogously for $f_{A_{\mathrm{at}}}$, $f_{\vee}$, $f_{\rightarrow}$ and $f_{\exists}$;

(f) I is sound under the addition of arbitrary predicate symbols;

(g) the soundness theorem of I is provable by induction on the length of derivations.

13.43. In the next theorem we show that if a proof interpretation I is well-behaved, then the process of hardwiring q- and t-truth in I, obtaining Iq and It, can be decomposed in three steps: first q or t, second I, and third o, as illustrated in figure 13.4.

13.44 Theorem (factorisations $\mathrm{Iq} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{q}$ and $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$)**.** If I is a well-behaved proof interpretation of HA^{ω} into itself, then for all formulas A of HA^{ω} we have

1. $A_{\mathrm{Iq}}(\underline{a}) \equiv (A^{\mathrm{q}})_{\mathrm{I}}(\underline{a})^{\mathrm{o}}$;

2. $A_{\mathrm{It}}(\underline{a}) \equiv (A^{\mathrm{t}})_{\mathrm{I}}(\underline{a})^{\mathrm{o}}$.

13.45 Proof. The proof is by induction on the structure of A.

1. $\underline{A_{\mathrm{at}}}$ We have

$$\begin{aligned}(A_{\mathrm{at}}^{\mathrm{q}})_{\mathrm{I}}()^{\mathrm{o}} &\equiv \\ (A_{\mathrm{at}})_{\mathrm{I}}()^{\mathrm{o}} &\equiv \\ A_{\mathrm{at}}^{\mathrm{o}} &\equiv \\ (A_{\mathrm{at}})_{\mathrm{Iq}}().\end{aligned}$$

$\underline{\wedge}$ Using induction hypothesis in the fourth equality, we get

$$\begin{aligned}((A \wedge B)^{\mathrm{q}})_{\mathrm{I}}(\underline{a}, \underline{b})^{\mathrm{o}} &\equiv \\ (A^{\mathrm{q}} \wedge B^{\mathrm{q}})_{\mathrm{I}}(\underline{a}, \underline{b})^{\mathrm{o}} &\equiv \\ ((A^{\mathrm{q}})_{\mathrm{I}}(\underline{a}) \wedge (B^{\mathrm{q}})_{\mathrm{I}}(\underline{b}))^{\mathrm{o}} &\equiv \\ (A^{\mathrm{q}})_{\mathrm{I}}(\underline{a})^{\mathrm{o}} \wedge (B^{\mathrm{q}})_{\mathrm{I}}(\underline{b})^{\mathrm{o}} &\equiv \\ A_{\mathrm{Iq}}(\underline{a}) \wedge B_{\mathrm{Iq}}(\underline{b}) &\equiv \\ (A \wedge B)_{\mathrm{Iq}}(\underline{a}, \underline{b}).\end{aligned}$$

$\underline{\vee}$ Using induction hypothesis in the seventh equality, we get

$$\begin{aligned}
((A \vee B)^{\mathrm{q}})_{\mathrm{I}}(\underline{c})^{\mathrm{o}} &\equiv \\
((A^{\mathrm{q}} \wedge A_{\mathrm{c}}) \vee (B^{\mathrm{q}} \wedge B_{\mathrm{c}}))_{\mathrm{I}}(\underline{c})^{\mathrm{o}} &\equiv \\
f_{\vee}((A^{\mathrm{q}} \wedge A_{\mathrm{c}})_{\mathrm{I}}(\underline{a}), (B^{\mathrm{q}} \wedge B_{\mathrm{c}})_{\mathrm{I}}(\underline{b}))^{\mathrm{o}} &\equiv \\
f_{\vee}((A^{\mathrm{q}})_{\mathrm{I}}(\underline{a}) \wedge (A_{\mathrm{c}})_{\mathrm{I}}(), (B^{\mathrm{q}})_{\mathrm{I}}(\underline{b}) \wedge (B_{\mathrm{c}})_{\mathrm{I}}())^{\mathrm{o}} &\equiv \\
f_{\vee}((A^{\mathrm{q}})_{\mathrm{I}}(\underline{a}) \wedge A_{\mathrm{c}}, (B^{\mathrm{q}})_{\mathrm{I}}(\underline{b}) \wedge B_{\mathrm{c}})^{\mathrm{o}} &\equiv \\
f_{\vee}\big(((A^{\mathrm{q}})_{\mathrm{I}}(\underline{a}) \wedge A_{\mathrm{c}})^{\mathrm{o}}, ((B^{\mathrm{q}})_{\mathrm{I}}(\underline{b}) \wedge B_{\mathrm{c}})^{\mathrm{o}}\big) &\equiv \\
f_{\vee}((A^{\mathrm{q}})_{\mathrm{I}}(\underline{a})^{\mathrm{o}} \wedge A, (B^{\mathrm{q}})_{\mathrm{I}}(\underline{b})^{\mathrm{o}} \wedge B) &\equiv \\
f_{\vee}(A_{\mathrm{Iq}}(\underline{a}) \wedge A, B_{\mathrm{Iq}}(\underline{b}) \wedge B) &\equiv \\
(A \vee B)_{\mathrm{Iq}}(\underline{c}).&
\end{aligned}$$

Analogously for $\to$, $\forall$ and $\exists$.

2. $\underline{A_{\mathrm{at}}}$ Analogously to the case of Iq.

$\underline{\wedge}$ Using induction hypothesis in the sixth equality, we get

$$\begin{aligned}
((A \wedge B)^{\mathrm{t}})_{\mathrm{I}}(\underline{a}, \underline{b})^{\mathrm{o}} &\equiv \\
((A^{\mathrm{t}} \wedge B^{\mathrm{t}}) \wedge (A \wedge B)_{\mathrm{c}})_{\mathrm{I}}(\underline{a}, \underline{b})^{\mathrm{o}} &\equiv \\
\big((A^{\mathrm{t}} \wedge B^{\mathrm{t}})_{\mathrm{I}}(\underline{a}, \underline{b}) \wedge ((A \wedge B)_{\mathrm{c}})_{\mathrm{I}}()\big)^{\mathrm{o}} &\equiv \\
\big((A^{\mathrm{t}} \wedge B^{\mathrm{t}})_{\mathrm{I}}(\underline{a}, \underline{b}) \wedge (A \vee B)_{\mathrm{c}}\big)^{\mathrm{o}} &\equiv \\
\big(((A^{\mathrm{t}})_{\mathrm{I}}(\underline{a}) \wedge (B^{\mathrm{t}})_{\mathrm{I}}(\underline{b})) \wedge (A \vee B)_{\mathrm{c}}\big)^{\mathrm{o}} &\equiv \\
((A^{\mathrm{t}})_{\mathrm{I}}(\underline{a})^{\mathrm{o}} \wedge (B^{\mathrm{t}})_{\mathrm{I}}(\underline{b})^{\mathrm{o}}) \wedge (A \wedge B) &\equiv \\
(A_{\mathrm{It}}(\underline{a}) \wedge B_{\mathrm{It}}(\underline{b})) \wedge (A \wedge B) &\equiv \\
(A \wedge B)_{\mathrm{It}}(\underline{a}, \underline{b}).&
\end{aligned}$$

$\underline{\vee}$ Using the definition in the seventh equality, we get

$$\begin{aligned}
((A \vee B)^{\mathrm{t}})_{\mathrm{I}}(\underline{c})^{\mathrm{o}} &\equiv \\
((A^{\mathrm{t}} \vee B^{\mathrm{t}}) \wedge (A \vee B)_{\mathrm{c}})_{\mathrm{I}}(\underline{c})^{\mathrm{o}} &\equiv \\
\big((A^{\mathrm{t}} \vee B^{\mathrm{t}})_{\mathrm{I}}(\underline{c}) \wedge ((A \vee B)_{\mathrm{c}})_{\mathrm{I}}()\big)^{\mathrm{o}} &\equiv \\
\big((A^{\mathrm{t}} \vee B^{\mathrm{t}})_{\mathrm{I}}(\underline{c}) \wedge (A \vee B)_{\mathrm{c}}\big)^{\mathrm{o}} &\equiv \\
\big(f_{\vee}((A^{\mathrm{t}})_{\mathrm{I}}(\underline{a}), (B^{\mathrm{t}})_{\mathrm{I}}(\underline{b})) \wedge (A \vee B)_{\mathrm{c}}\big)^{\mathrm{o}} &\equiv \\
f_{\vee}((A^{\mathrm{t}})_{\mathrm{I}}(\underline{a}), (B^{\mathrm{t}})_{\mathrm{I}}(\underline{b}))^{\mathrm{o}} \wedge (A \vee B) &\equiv \\
f_{\vee}((A^{\mathrm{t}})_{\mathrm{I}}(\underline{a})^{\mathrm{o}}, (B^{\mathrm{t}})_{\mathrm{I}}(\underline{b})^{\mathrm{o}}) \wedge (A \vee B) &\equiv \\
f_{\vee}(A_{\mathrm{It}}(\underline{a}), B_{\mathrm{It}}(\underline{b})) \wedge (A \vee B) &\equiv \\
(A \vee B)_{\mathrm{It}}(\underline{c}).&
\end{aligned}$$

Analogously for $\to$, $\forall$ and $\exists$.

13.46. Now let us motivate our third heuristic on how to hardwire truth. If I is a well-behaved proof interpretation of HA^{ω} into itself, then I extends to a sound proof interpretation of $\mathsf{HA}^{\omega}_{\mathrm{c}}$ into itself because:

1. the soundness of I is not spoiled by the new atomic formulas A_{c};

2. we can extend the soundness proof of I to include the new axioms $(A_1)_{\mathrm{c}} \to \cdots \to (A_n)_{\mathrm{c}}$ (because the interpretations of these axioms are the axioms themselves, so we add the cases of these axioms to the soundness proof by induction on the length of derivations).

So q, t, o and I (extended to $\mathsf{HA}^{\omega}_{\mathrm{c}}$) are sound, thus the compositions $\mathrm{Iq} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{q}$ and $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$ are also sound. This leads us to our third heuristic on how to hardwire truth.

13.47 Heuristic. If I is a well-behaved proof interpretation, then I has the sound variants with truth $\mathrm{Iq} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{q}$ and $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$.

13.48. By applying this heuristic to mr, br, DN, B and | we get their variants with q- and t-truth.

This heuristic also suggests that there is no sound Gödel's functional interpretation with truth (as discussed in paragraph 5.9): D is not well-behaved because it is seemly not sound under the addition of arbitrary predicate symbols P since the proof of the soundness theorem of D needs characteristic terms χ_P. Analogously for MD.

13.5 Conclusion

13.49. We saw that we hardwire truth in a proof interpretation I, getting Iq and It, by adding copies of the formulas under interpretation in some clauses of the definition of I. The questions here are: in which clauses? and are Iq and It sound? We answered these questions with three heuristics.

Heuristic 1 By moving back and forth between IL^{ω} and ILL^{ω} via Girard's embeddings, we saw that

1. to hardwire q-truth we should add copies in the clauses of $\vee$, (premise of) $\to$, and $\exists$;
2. to hardwire t-truth we should add copies in the clauses of A_{at}, $\to$ and $\forall$.

Heuristic 2 Motivated by the factorisation $A_{\mathrm{mrt}}(\underline{a}) \leftrightarrow A_{\mathrm{mrq}}(\underline{a}) \wedge A$, we saw that to hardwire truth we should

1. start by hardwiring q-truth imitating the way in which it is done for mrq;
2. then upgrade to t-truth by defining $A_{\mathrm{It}} :\equiv A_{\mathrm{Iq}} \wedge A$.

Heuristic 3 By studying the soundness of the factors in the factorisations $\mathrm{Iq} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{q}$ and $\mathrm{It} = \mathrm{o} \circ \mathrm{I} \circ \mathrm{t}$, we saw that if I is well-behaved, then Iq and It are sound.

Chapter 14

Copies of classical logic in intuitionistic logic

14.1 Introduction

14.1. Let us recall that a negative translation N is a embedding of CL in IL, in the sense of having the following two properties:

Soundness theorem if $\mathsf{CL} + \Gamma \vdash A$, then $\mathsf{IL} + \Gamma^{\mathrm{N}} \vdash A^{\mathrm{N}}$;

Characterisation theorem $\mathsf{CL} \vdash A \leftrightarrow A^{\mathrm{N}}$.

14.2. The image $\operatorname{im} \mathrm{N}$ of a negative translation N is a copy of CL in IL because, as the equivalence

$$\mathsf{CL} \vdash A \;\Leftrightarrow\; \mathsf{IL} \vdash A^{\mathrm{N}}$$

shows, the formulas $A^{\mathrm{N}} \in \operatorname{im} \mathrm{N}$ mirror inside IL the behaviour of CL.

14.3. All the usual negative translations GG, Ko, Kr and Ku give (modulo equivalence in IL) the same copy of CL in IL: the negative fragment NF. This lead us to the question: is NF the only copy? In this chapter we are going to answer this question:

1. we present three different copies;

2. we characterise why GG, Ko, Kr and Ku give the same copy.

14.2 Definitions

14.4. We give an abstract definition of negative translation. Its actually a simple and natural definition: we simply ask for the soundness and characterisation theorems to hold true. Then we define a copy of CL in IL as being the image of a negative translation.

We also define two special behaviours of negative translations: acting as the identity on NF, and translating into NF. They will be used to characterise the negative translations that give the copy NF.

The definition are modulo IL, informally meaning that we identify formulas equivalent in IL.

14.5 Definition. Let M and N functions mapping formulas of CL to formulas of IL, and Γ and Δ be sets of formulas of IL.

1. (a) We call *soundness theorem* of N to the following condition: for all formulas A of CL and for all sets Γ of formulas of CL, if $\mathsf{CL} + \Gamma \vdash A$ then $\mathsf{CL} + \Gamma^{\mathrm{N}} \vdash A^{\mathrm{N}}$ [18, definition 1].

   (b) We call *characterisation theorem* of N to the following condition: for all formulas A of CL we have $\mathsf{CL} \vdash A \leftrightarrow A^{\mathrm{N}}$ [18, definition 1].

   (c) We say that N is a *negative translation* if and only if the soundness theorem of N and the characterisation theorem of N hold true [18, definition 1].

   (d) We call *image* of N, and denote by $\operatorname{im} \mathrm{N}$, to the set of all formulas A^{N} (with A ranging over the formulas of CL).

2. (a) We say that Γ and Δ are *equal (modulo* IL*)* if and only if:

      i. Γ is *contained (modulo* IL*)* in Δ, that is for all $A \in \Gamma$ there exists $B \in \Delta$ such that $\mathsf{IL} \vdash A \leftrightarrow B$;

      ii. Δ is *contained (modulo* IL*)* in Γ, that is for all $B \in \Delta$ there exists $A \in \Gamma$ such that $\mathsf{IL} \vdash A \leftrightarrow B$.

   (b) We say that M and N are *equal (modulo* IL*)* if and only if for all formulas A of CL we have $\mathsf{IL} \vdash A^{\mathrm{M}} \leftrightarrow A^{\mathrm{N}}$ [18, definition 3].

3. We say that Γ is a *copy* of CL in IL if and only if there exists a negative translation N such that Γ and $\operatorname{im} \mathrm{N}$ are equal (modulo IL).

4. (a) We say that N *translates into* NF *(modulo* IL*)* if and only if for all formulas A of CL there exists $B \in \mathrm{NF}$ such that $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow B$ [18, definition 5].

   (b) We say that N *acts as the identity on* NF *(modulo* IL*)* if and only if for all $A \in \mathrm{NF}$ we have $\mathsf{IL} \vdash A \leftrightarrow A^{\mathrm{N}}$ [18, definition 5].

14.6 Example. We can prove that the functions Ko, GG, Ku and Kr:

1. are negative translations;

2. have images $\operatorname{im} \mathrm{Ko}$, $\operatorname{im} \mathrm{GG}$, $\operatorname{im} \mathrm{Ku}$ and $\operatorname{im} \mathrm{Kr}$ equal (modulo IL) to NF;

3. are pairwise equal (modulo IL);

4. translate into NF (modulo IL);

5. act as the identity on NF (modulo IL).

(These claims are easy to check for GG, and they also apply to Ko, Ku and Kr by proposition 2.7.)

14.3 Three different copies

14.7. We saw that the usual negative translations Ko, GG, Ku and Kr all give the same image, that is the same copy of CL in IL, namely the negative fragment NF. This raises the question: is NF the only copy? The answer is no, and we show this by proving that

$$\mathrm{NF}, \qquad \mathrm{NF} \vee F := \{A \vee F : A \in \mathrm{NF}\}, \qquad \mathrm{NF}[F/\bot] := \{A[F/\bot] : A \in \mathrm{NF}\}$$

(where F is a suitable formula) are three different copies.

14.8 Definition. Let us fix a formula F of IL. We define the functions N_1 and N_2, mapping formulas of CL to formulas of IL, by:

1. $A^{\mathrm{N}_1} :\equiv A^{\mathrm{GG}} \vee F$ [18, definition 6];
2. $A^{\mathrm{N}_2} :\equiv A^{\mathrm{GG}}[F/\bot]$ [8, section 2.3] [40, definition 6].

14.9. As a curiosity, we can mention the following. Let minimal logic ML be IL without the ex falso quodlibet $\bot \to A$ [78, definition 3.2 in chapter 2].

FD The Friedman-Dragalin translation FD [17, section 1] [12, page 463 of the translation] (also known as Friedman's A-translation) assigns to each formula A of IL the formula A^{FD} obtained from A by simultaneously replacing $\bot$ by F and all atomic subformulas A_{at} by $A_{\mathrm{at}} \vee F$. It is sound in the following sense: if $\mathsf{IL} \vdash A$, then $\mathsf{ML} \vdash A^{\mathrm{FD}}$ [17, theorem 1.2] [12, page 463 of the translation].

FD′ There is a variant FD′ [4, lemma 2.1] where we only replace $\bot$ by F, but it is only sound in the following sense: if $\mathsf{ML} \vdash A$, then $\mathsf{ML} \vdash A^{\mathrm{FD}'}$ [4, lemma 2.1].

The curiosity is that we have the factorisations $\mathrm{N}_2 = \mathrm{FD} \circ \mathrm{GG}$ and $\mathrm{N}_2 = \mathrm{FD}' \circ \mathrm{GG}$: for all formulas A of CL we have $\mathsf{ML} \vdash A^{\mathrm{N}_2} \leftrightarrow (A^{\mathrm{GG}})^{\mathrm{FD}}$ and $A^{\mathrm{N}_2} \equiv (A^{\mathrm{GG}})^{\mathrm{FD}'}$ [18, proposition 11].

14.10 Theorem.

1. The function N_1 has a soundness theorem for all formulas F of IL. Analogously for N_2 [18, theorem 8.1].
2. The function N_1 has a characterisation theorem if and only if $\mathsf{CL} \vdash \neg F$. Analogously for N_2 [18, theorem 8.2].
3. The sets $\operatorname{im} \mathrm{N}_1$ and $\mathrm{NF} \vee F := \{A \vee F : A \in \mathrm{NF}\}$ are equal (modulo IL), and the sets $\operatorname{im} \mathrm{N}_2$ and $\mathrm{NF}[F/\bot] := \{A[F/\bot] : A \in \mathrm{NF}\}$ are equal (modulo IL).
4. The image $\operatorname{im} \mathrm{N}_1$ is equal to NF (modulo IL) if and only if $\mathsf{IL} \vdash \neg F$. Analogously for N_2 [18, theorem 8.3].
5. The images $\operatorname{im} \mathrm{N}_1$ and $\operatorname{im} \mathrm{N}_2$ are not equal (modulo IL) if $\mathsf{CL} \vdash \neg F$ and $\mathsf{IL} \nvdash \neg F$ [18, proposition 9].
6. There exists a formula F of CL such that $\mathsf{CL} \vdash \neg F$ but $\mathsf{IL} \nvdash \neg F$.

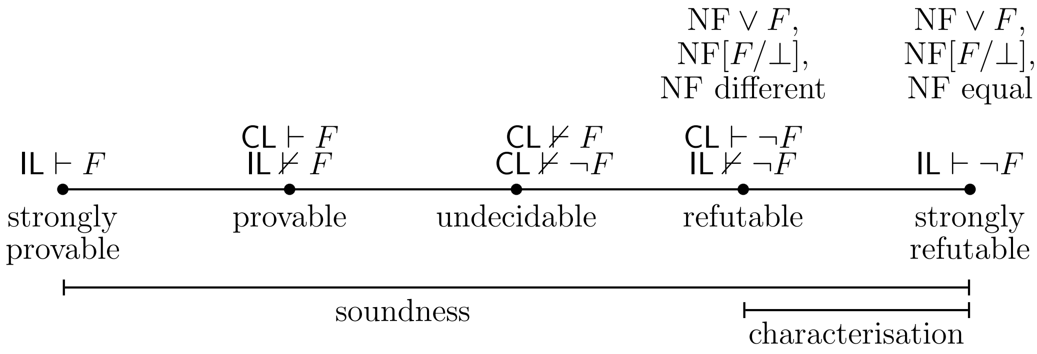

Figure 14.1: summary of theorem 14.10.

So for a formula F of CL such that $\mathsf{CL} \vdash \neg F$ but $\mathsf{IL} \nvdash \neg F$, the sets NF, NF $\vee F$ and NF$[F/\bot]$ are pairwise not equal (modulo IL) copies of CL in IL. This is summarised in figure 14.1.

14.11 Proof.

1. $\underline{\mathrm{N}_1}$ Let us assume $\mathsf{CL} + \Gamma \vdash A$ and prove $\mathsf{IL} + \Gamma^{\mathrm{N}_1} \vdash A^{\mathrm{N}_1}$. From the assumption we get $\mathsf{CL} + A_1 + \cdots + A_n \vdash A$ for some $A_1, \ldots, A_n \in \Gamma$. By the soundness theorem of GG we get $\mathsf{IL} + A_1^{\mathrm{GG}} + \cdots + A_n^{\mathrm{GG}} \vdash A^{\mathrm{GG}}$. Then $\mathsf{IL} + A_1^{\mathrm{GG}} \vee F + \cdots + A_n^{\mathrm{GG}} \vee F \vdash A^{\mathrm{GG}} \vee F$, that is $\mathsf{IL} + A_1^{\mathrm{N}_1} + \cdots + A_n^{\mathrm{N}_1} \vdash A^{\mathrm{N}_1}$. We conclude $\mathsf{IL} + \Gamma^{\mathrm{N}_1} \vdash A^{\mathrm{N}_1}$.

   $\underline{\mathrm{N}_2}$ Let us assume $\mathsf{CL} + \Gamma \vdash A$ and prove $\mathsf{IL} + \Gamma^{\mathrm{N}_2} \vdash A^{\mathrm{N}_2}$. We can prove that GG has a soundness theorem from CL into ML: if $\mathsf{CL} + \Gamma \vdash A$, then $\mathsf{ML} + \Gamma^{\mathrm{GG}} \vdash A^{\mathrm{GG}}$ [78, theorem 3.5 in chapter 2]. So by the assumption we get $\mathsf{ML} + \Gamma^{\mathrm{GG}} \vdash A^{\mathrm{GG}}$. Since $\bot$ is treated as an arbitrary propositional letter in ML [78, NB after the definition 3.2 in chapter 2], then $\mathsf{ML} + \Gamma^{\mathrm{GG}}[F/\bot] \vdash A^{\mathrm{GG}}[F/\bot]$, that is $\mathsf{ML} + \Gamma^{\mathrm{N}_2} \vdash A^{\mathrm{N}_2}$. We conclude $\mathsf{IL} + \Gamma^{\mathrm{N}_2} \vdash A^{\mathrm{N}_2}$.

2. Let us do the proof for N_2; the case of N_1 is analogous.

   $\underline{\Rightarrow}$ If N_2 has a characterisation theorem, then $\mathsf{CL} \vdash \bot^{\mathrm{N}_2} \leftrightarrow \bot$, that is $\mathsf{CL} \vdash F \leftrightarrow \bot$, so $\mathsf{CL} \vdash \neg F$.

   $\underline{\Leftarrow}$ If $\mathsf{CL} \vdash \neg F$, then $\mathsf{CL} \vdash A^{\mathrm{GG}}[F/\bot] \leftrightarrow A^{\mathrm{GG}}$, that is $\mathsf{CL} \vdash A^{\mathrm{N}_2} \leftrightarrow A^{\mathrm{GG}}$, so $\mathsf{CL} \vdash A^{\mathrm{N}_2} \leftrightarrow A$ by the characterisation theorem of GG, concluding that N_2 has a characterisation theorem.

3. Let us prove that $\operatorname{im} \mathrm{N}_2$ and NF$[F/\bot]$ are equal (modulo IL).

   $\underline{\subseteq}$ If $A^{\mathrm{GG}}[F/\bot] \in \operatorname{im} \mathrm{N}_2$, then $A^{\mathrm{GG}}[F/\bot] \in \mathrm{NF}[F/\bot]$ because $A^{\mathrm{GG}} \in \mathrm{NF}$.

   $\underline{\supseteq}$ If $A[F/\bot] \in \mathrm{NF}[F/\bot]$ where $A \in \mathrm{NF}$, then $\mathsf{IL} \vdash A[F/\bot] \leftrightarrow A^{\mathrm{GG}}[F/\bot]$ where $A^{\mathrm{GG}}[F/\bot] \in \operatorname{im} \mathrm{N}_2$, because we can prove $\mathsf{ML} \vdash A \leftrightarrow A^{\mathrm{GG}}$ by induction on the structure of $A \in \mathrm{NF}$.

   Analogously for N_1.

4. Let us do the proof for N_2; the case of N_1 is analogous. Since $\operatorname{im}\mathrm{GG}$ is equal (modulo IL) to NF, it suffices to prove: $\operatorname{im}\mathrm{N}_2$ and $\operatorname{im}\mathrm{GG}$ are equal (modulo IL) if and only if $\mathsf{IL} \vdash \neg F$. In turn, to prove this it suffices by $(*_1)$ below to prove: N_2 and GG are not equal (modulo IL).

   Let us prove the following for negative translations M and N: $(*_1)$ $\operatorname{im}\mathrm{M}$ and $\operatorname{im}\mathrm{N}$ are equal (modulo IL) if and only if N and M are equal (modulo IL). The right-to-left implication is trivial, so let us prove the left-to-right implication. Let us assume that $\operatorname{im}\mathrm{M}$ and $\operatorname{im}\mathrm{N}$ are equal (modulo IL), take an arbitrary formula A of CL and prove $(*_2)$ $\mathsf{IL} \vdash A^{\mathrm{M}} \leftrightarrow A^{\mathrm{N}}$. By the assumption there exists a formula B of CL such that $(*_3)$ $\mathsf{IL} \vdash A^{\mathrm{M}} \leftrightarrow B^{\mathrm{N}}$. By the characterisation theorems of M and N we get $\mathsf{CL} \vdash A \leftrightarrow B$, so $\mathsf{CL} + B \vdash A$ and $\mathsf{CL} + A \vdash B$. By the soundness theorems of N we get $\mathsf{IL} + A^{\mathrm{N}} \vdash B^{\mathrm{N}}$ and $\mathsf{IL} + B^{\mathrm{N}} \vdash A^{\mathrm{N}}$, so $(*_4)$ $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow B^{\mathrm{N}}$ by the deduction theorem of IL. From $(*_3)$ and $(*_4)$ we get $(*_2)$.

   Taking $A \equiv \bot$ in the left-to-right implication of the last equivalence below, we get

$$
\begin{aligned}
\operatorname{im}\mathrm{N}_2 \text{ and } \operatorname{im}\mathrm{GG} \text{ are equal (modulo } \mathsf{IL}) &\text{ if and only if}\\
\text{for all formulas } A \text{ of } \mathsf{CL} \text{ we have } \mathsf{IL} \vdash A^{\mathrm{N}_2} \leftrightarrow A^{\mathrm{GG}} &\text{ if and only if}\\
\text{for all formulas } A \text{ of } \mathsf{CL} \text{ we have } \mathsf{IL} \vdash A^{\mathrm{GG}}[F/\bot] \leftrightarrow A^{\mathrm{GG}} &\text{ if and only if}\\
\mathsf{IL} \vdash \neg F&.
\end{aligned}
$$

5. Let us assume $\mathsf{CL} \vdash \neg F$ and $\mathsf{IL} \nvdash \neg F$. Let P be a fresh (that is different from $\bot$ and not occurring in F) nullary predicate symbol. By $(*_1)$ it suffices to prove $\mathsf{IL} \nvdash P^{\mathrm{N}_1} \leftrightarrow P^{\mathrm{N}_2}$, that is $\mathsf{IL} \nvdash \neg\neg P \vee F \leftrightarrow ((P \to F) \to F)$. We do so by presenting a Kripke model forcing $(P \to F) \to F$ but not forcing $\neg\neg P \vee F$.

   $\underline{\mathcal{K}}$ There exists a Kripke model $\mathcal{K}$ with a bottom node, forcing $\neg F$ and P. Indeed, just take a classical model $\mathcal{K}$ (which forces $\neg F$ since $\mathsf{CL} \vdash \neg F$), consider it as a one-node Kripke model and force P in the model [80].

   $\underline{\mathcal{L}}$ There exists a Kripke model $\mathcal{L}$ with a bottom node and forcing F and $\neg P$. Indeed, there exists a Kripke model $\mathcal{L}'$ with some node n not forcing F (because $\mathsf{IL} \nvdash F$) and forcing $\neg P$ (because P is fresh), so we take $\mathcal{L}$ to be $\mathcal{L}'$ restricted to the nodes above or equal to n.

   $\underline{\mathcal{M}}$ We can assume (renaming elements if needed) that the domains of the bottom nodes of $\mathcal{K}$ and $\mathcal{L}$ share a node d. Let $\mathcal{M}$ be the Kripke model constructed from $\mathcal{K}$ and $\mathcal{L}$ by connecting their bottom nodes to a new node 0 with domain $\{d\}$ (so that the domains of $\mathcal{M}$ are monotone as required to be Kripke model), as illustrated in figure 14.2. The node 0 does not force $((P \to F) \to F) \to \neg\neg P \vee F$ because 0:

   (a) forces $(P \to F) \to F$ since $\mathcal{K}$ does not force $P \to F$, $\mathcal{L}$ forces P, and 0 does not force $P \to F$ (otherwise $\mathcal{K}$ would force $P \to F$);

   (b) does not force $\neg\neg P$ since $\mathcal{L}$ forces $\neg P$;

   (c) does not force F since $\mathcal{K}$ forces $\neg F$.

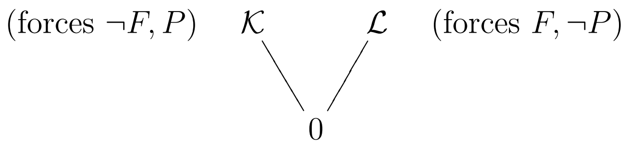

Figure 14.2: the Kripke model $\mathcal{M}$.

6. Let $P(x)$ be a unary predicate symbol, $F' :\equiv \forall x\, \neg\neg P(x) \to \neg\neg\forall x\, P(x)$ (an instance of the double negation shift) and $F :\equiv \neg F'$ [62]. We have $\mathsf{CL} \vdash \neg F$, but $\mathsf{IL} \nvdash \neg F$ because $\mathsf{IL} \vdash \neg F \leftrightarrow F'$ and $\mathsf{IL} \nvdash F'$ [79, page 166].

14.12. To be sure, by now we have three copies of CL in IL that are not equal (modulo IL): NF, $\mathrm{NF} \vee F$ and $\mathrm{NF}[F/\bot]$ (with $\mathsf{CL} \vdash \neg F$ and $\mathsf{IL} \nvdash F$). This is illustrated in figure 14.3.

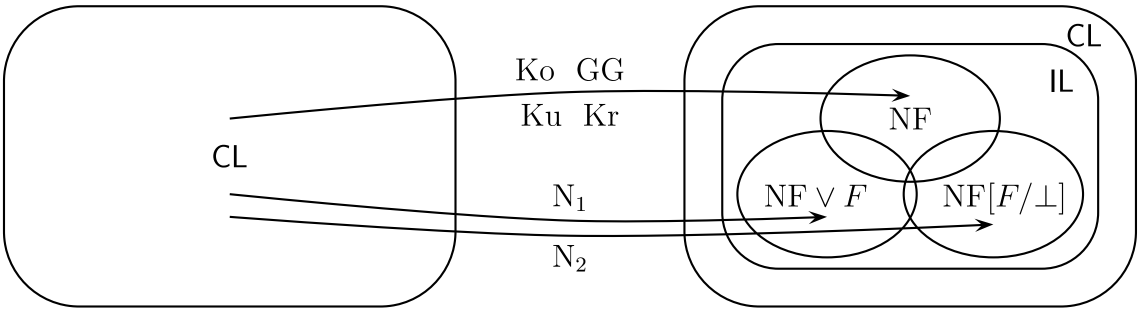

Figure 14.3: the three copies NF, $\mathrm{NF} \vee F$ and $\mathrm{NF}[F/\bot]$ of CL in IL.

14.4 Characterisation

14.13. We saw that there are different copies of CL in IL, but all the usual negative translations (Ko, GG, Ku and Kr) give the same copy NF. Is this just a coincidence, or is there a reason why the usual negative translations all give the same copy? In the next theorem we determine why the usual negative translations give the same copy: because they are all "well-behaved" with respect to NF, in the sense of

1. translating into NF (modulo IL);

2. acting as the identity on NF (modulo IL).

The latter property is of especial importance because it plays a central role in proving point 1 of theorem 2.17.

14.14 Theorem. Let N be a negative translation. The following conditions are equivalent:

1. $\operatorname{im} \mathrm{N}$ and NF are equal (modulo IL);

2. N and GG are equal (modulo IL) [18, theorem 13];

3. N translates into NF (modulo IL) [75, section 1.10.1] [18, theorem 13];

4. N acts as the identity on NF (modulo IL) [18, theorem 13].

14.15 Proof.

$1 \Leftrightarrow 2$ We already saw this equivalence in point 4 of proof 14.11.

$2 \Rightarrow 3$ Follows from GG translating into NF (modulo IL).

$3 \Rightarrow 4$ We assume that N translates into NF (modulo IL), take an arbitrary $A \in \mathrm{NF}$ and prove $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow A$. By the characterisation theorem of N we have $\mathsf{CL} \vdash A^{\mathrm{N}} \leftrightarrow A$ where, by our assumption, $A^{\mathrm{N}} \leftrightarrow A$ is equivalent in IL to a formula in NF. By point 1 of theorem 2.17 we get $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow A$.

$4 \Rightarrow 2$ We assume that N acts as the identity on NF (modulo IL), take an arbitrary formula A of CL and prove $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow A^{\mathrm{GG}}$. By the characterisation theorem of GG we have $\mathsf{CL} \vdash A \leftrightarrow A^{\mathrm{GG}}$, so $\mathsf{CL} + A \vdash A^{\mathrm{GG}}$ and $\mathsf{CL} + A^{\mathrm{GG}} \vdash A$. By the soundness theorem of N we get $\mathsf{IL} + A^{\mathrm{N}} \vdash (A^{\mathrm{GG}})^{\mathrm{N}}$ and $\mathsf{IL} + (A^{\mathrm{GG}})^{\mathrm{N}} \vdash A^{\mathrm{N}}$, so $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow (A^{\mathrm{GG}})^{\mathrm{N}}$ by the deduction theorem of IL. Since $A^{\mathrm{GG}} \in \mathrm{NF}$, then by our assumption we get $\mathsf{IL} \vdash A^{\mathrm{N}} \leftrightarrow A^{\mathrm{GG}}$.

14.5 Conclusion

14.16. We saw that all the usual negative translations (GG, Ko, Kr and Ku) give the same copy of CL in IL: NF. This raised the question: is NF the only copy? The answer is no, and we presented three different copies:

$$\mathrm{NF}, \qquad \mathrm{NF} \vee F, \qquad \mathrm{NF}[F/\bot].$$

14.17. The fact that there are different copies but still the usual negative translations all give the same copy raises another question: why do all the usual negative translations give the same copy? Our answer to this question is: because they are "well-behaved" with respect to the negative fragment (in the sense of translating into NF and acting as the identity on NF).

Part IV

Practical contributions

Chapter 15

“Finitary” infinite pigeonhole principles

15.1 Introduction

15.1. In 2007 and 2008, Terence Tao wrote on his blog essays [71, 73] about the finitisation of principles in analysis. To introduce Tao’s notion of finitisation, first we need to recall the notions of soft analysis and hard analysis.

<u>Soft analysis</u> It is the part of analysis that deals with infinite objects (such as sequences and σ-algebras) and their qualitative properties (such as convergence and compactness).

<u>Hard analysis</u> It is the part of analysis that deals with finite objects (such as finite sets and the value of convergent integrals) and their quantitative properties (such as the cardinality of finite sets and bounds).

<u>Finitisation</u> A finitisation of a soft analysis statement is an equivalent hard analysis statement.

Tao’s finitisations are usually achieved by strong ineffective methods: proof by contradiction and sequential compactness. An intuitive explanation for this is that Tao relies only on the truth of the statement that he finitises, so he needs to compensate by using strong methods. As a consequence, Tao does not get numerical bounds. In proof theory, we rely not on the truth but on a proof of the statement, so we can avoid the strong methods and get numerical bounds.

15.2. Tao has several reasons for being interested in finitisations.

<u>Green-Tao theorem</u> Tao achieved improved results by reducing soft analysis parts of proofs to their hard analysis skeleton. For example, the Green-Tao theorem (proving the existence of arbitrary long arithmetic sequences of primes) uses a sort of finitary ergodic theory [71, footnote 4].

<u>Exact relations</u> Tao believes that there is a close relation between the two types of analysis: a hard analysis statement is a soft analysis statement with the exact relations between objects made explicit; or in other words, a soft analysis

statement is a hard analysis statement with the exact relations between objects concealed by the use of infinitary notions [71].

For example, letting the sequence $(x_n)_{n\in\mathbb{N}\setminus\{0\}}$ be defined by $x_n := 1/n^2$, in table 15.1 we give the statement $x_n \to 0$ with the exact relation between quantities first hidden and then explicitly showed.

soft analysis	$\forall\varepsilon > 0\,\exists N\,\forall n > N\,(\lvert x_n - 0\rvert < \varepsilon)$	no relation between ε and N
hard analysis	$\forall\varepsilon > 0\,\forall n > 1/\sqrt{\varepsilon}\ (\lvert x_n - 0\rvert < \varepsilon)$	$N = 1/\sqrt{\varepsilon}$

Table 15.1: the statement $x_n := 1/n^2 \to 0$ with the exact relation between quantities hidden and shown.

Long/short range mathematics Soft analysis is good for "long-range" mathematics: it allows us to move faster by ignoring the exact relation between quantities. Hard analysis is good for "short-range" mathematics: it allows us to refine existing results by relating the exact quantities [71, footnote 4].

Best of both worlds There are connections between soft analysis and hard analysis that allow us to move back and forth between the two, taking advantage of both worlds.

In table 15.2 we give two examples of such connections [73, sections 3 and 6].

soft analysis	hard analysis	connection
ergodic theory	combinatorial number theory	Furstenberg correspondence principle
ergodic graph theory	graph theory	graph correspondence principle

Table 15.2: examples of connections between soft analysis and hard analysis.

15.3 Example. One of Tao's prime examples is an almost finitisation of the infinite pigeonhole principle. To present this example, first we need the following notions.

Weak convergence We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ *weakly converges* to an infinite subset I of $\mathbb{N}$ if and only if for all finite subsets B of $\mathbb{N}$ we have $A_n \cap B = I \cap B$ for n large enough [71].

Asymptotic stability near infinite sets We say that a function F, that takes as input finite subsets of $\mathbb{N}$ and outputs natural numbers, is *asymptotically stable near infinite sets*, and write $F \in$ ASNIS, if and only if it stabilises over all weakly convergent sequences [71].

Notation Let us denote the initial segment $\{0, 1, 2, \dots, n-1\}$ of $\mathbb{N}$ by n.

Now let us present Tao's example. The *infinite pigeonhole principle* IPP is the following principle.

> Every colouring of $\mathbb{N}$ with finitely many colours has an infinite colour class.

It is a soft analysis statement because:

1. it talks about infinite objects, namely a colouring of the natural numbers;
2. it talks about qualitative properties, namely a colour class being infinite.

The infinite pigeonhole principle IPP almost finitises into the *third "finitary" infinite pigeonhole principle* FIPP_3 (later on we will introduce the first two "finitary" infinite pigeonhole principles), that is the following principle.

> For every number of colours n and for every asymptotically stable near infinite sets function F, there exists an initial segment k of the natural numbers such that any colouring $f\colon k \to n$ of k with n colours has a "big" colour class $A = f^{-1}(c)$ in the sense of $|A| > F(A)$ [71]. In symbols:
>
> $$\forall n\,\forall F \in \text{ASNIS}\,\exists k\,\forall f\colon k \to n\,\exists c \in n\,\bigl(|f^{-1}(c)| > F(f^{-1}(c))\bigr). \tag{15.1}$$

It is almost a hard analysis statement because:

1. it talks about a finite object, namely the colouring f;
2. it talks about quantitative properties, namely the inequality $|A| > F(A)$;
3. but it is not completely finitary because it also talks about infinite objects and qualitative properties, namely the asymptotically stable near infinite sets function F (that is why "finitary" is written in quotation marks).

15.4. The story of Tao's finitary infinite pigeonhole principle is involved. There are three variants of the "finitary" infinite pigeonhole principle.

<u>FIPP_1</u> Analogous to (15.1) but with the class ASNIS replaced by a larger class AS;

<u>FIPP_2</u> Analogous to (15.1) but with ASNIS replaced by AS and only stating the existence of a "big" monochromatic set $A \subseteq f^{-1}(c)$;

<u>FIPP_3</u> The one from (15.1).

Initially Tao proposed FIPP_1 as an almost finitisation of IPP, but gave no proof of the equivalence between IPP and FIPP_1. When we tried to prove it, we were only able to show that IPP implies the weaker FIPP_2. It turned out that FIPP_1 is false (so not equivalent to the true IPP) and we gave a counter-example to it.

The principle FIPP_2 results from translating $(\mathsf{IPP}^{\mathrm{Ku}})^{\mathrm{D}}$ into Tao's language (that uses sets, set theoretic functions and asymptotic stability). Since D finitises a formula (although in the different sense of making explicit its computational content), we take this as evidence that FIPP_2 is a natural finitisation of IPP.

When Tao was made aware of our counter-example, he corrected FIPP_1 by reducing the class AS to ASNIS, arriving at FIPP_3.

15.5. Having two proposed finitisations FIPP_2 and FIPP_3 of IPP (FIPP_1 is excluded by the counter-example), we naturally ask how they compare. To do so, we try to determine which one is a more faithful finitisation of IPP. This leads us to discuss the notion of "faithfulness". Consider the following silly finitisation of IPP: $0 = 0$ is a finitisation of IPP because it is a hard analysis statement and is equivalent to IPP (since both IPP and $0 = 0$ happen to be true). The problem with this finitisation is that a proof of $\mathsf{IPP} \to 0 = 0$ does not even use IPP, and a proof of $0 = 0 \to \mathsf{IPP}$ is a proof of IPP from scratch. This because our setting theory (whatever it may be) already proves $0 = 0$. So we should study the provability of $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_{2/3}$ in theories T weaker than IPP and than $\mathsf{FIPP}_{2/3}$. Then we say that a finitisation $\mathsf{FIPP}_{2/3}$ of IPP is more faithful the weaker the theory T that proves $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_{2/3}$. To achieve this we turn to reverse mathematics and slide T along the "big five" subsystems of second order arithmetic that give us the scale of strength from figure 15.1. We are going to conclude $\mathsf{WKL}_0 \vdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ and

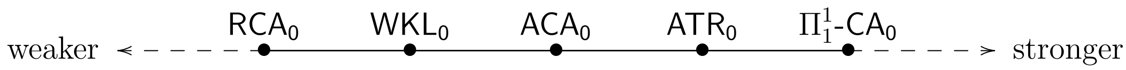

Figure 15.1: the "big five" subsystems of second order arithmetic.

$\mathsf{ACA}_0 \vdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$, suggesting that FIPP_2 is a more faithful finitisation of IPP than FIPP_3 (but to give a definite answer we would need to show $\mathsf{WKL}_0 \nvdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$). (Studying $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ in WKL_0 is of interest because $\mathsf{WKL}_0 \nvdash \mathsf{IPP}$ [38, corollary 6.5], but studying $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ in ACA_0 is more questionable because $\mathsf{ACA}_0 \vdash \mathsf{IPP}$ [64, lemma III.7.4] [21, page 359].)

15.2 Asymptotic stability

15.6. Now we introduce the notion of asymptotic stability. Roughly speaking, a function is asymptotically stable if it is sequentially continuous on the Cantor space $\mathcal{P}(\mathbb{N})$. Actually, Tao did not formulate the notion of asymptotic stability in this way, but rather talking about functions stabilising over sequences of subsets of $\mathbb{N}$. So first we give the definitions in Tao's terms, and then we recast them in terms of the Cantor space.

15.7 Definition. Let us denote by $\mathcal{P}(\mathbb{N})$ the set of all subsets of $\mathbb{N}$, by $\mathcal{P}_{\text{fin}}(\mathbb{N})$ the set of all finite subsets of $\mathbb{N}$ and by $\mathcal{P}_{\text{inf}}(\mathbb{N})$ the set of all infinite subsets of $\mathbb{N}$. Also, given a natural number n, let us denote by n the set $\{0, 1, 2, \ldots, n-1\}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence with terms in $\mathcal{P}_{\text{fin}}(\mathbb{N})$, $F \colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N}$ be a function and $I \in \mathcal{P}_{\text{inf}}(\mathbb{N})$.

1. We say that $(A_n)_{n\in\mathbb{N}}$ is a *nested* if and only if $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ [21, definition 5.1].

2. We say that $(A_n)_{n\in\mathbb{N}}$ *weakly converges to* I if and only if for all $B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$ we eventually have $A_n \cap B = I \cap B$ for all n sufficient large [71]. We say that $(A_n)_{n\in\mathbb{N}}$ *weakly converges* if and only if $(A_n)_{n\in\mathbb{N}}$ weakly converges to some $I \in \mathcal{P}_{\text{inf}}(\mathbb{N})$.

3. We say that F is *asymptotically stable*, and denote by $F \in \mathrm{AS}$, if and only if F stabilises (that is eventually becomes constant) over all nested sequences [71].

4. We say that F is *asymptotically stable near infinite sets*, and denote by $F \in \mathrm{ASNIS}$, if and only if F stabilises over all weakly convergent sequences [71].

15.8 Example.

1. The sequence $(A_n)_{n\in\mathbb{N}}$ defined by $A_n := \{0, \ldots, n\}$ is nested and asymptotically converges to $\mathbb{N}$.

2. The function $F\colon \mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \to \mathbb{N}$ defined by $F(A) := \min A$ (with the non-standard convention $\min \emptyset := 0$) satisfies $F \in \mathrm{AS}$ and $F \in \mathrm{ASNIS}$ [71].

15.9. Although we are going to use the previous definitions as Tao gave them, as a curiosity in the next proposition we recast these definitions in the more standard terms of Cantor space. Roughly speaking, weakly convergent sequences are the convergent sequences (in the Cantor space), and asymptotically stable functions are sequentially continuous functions (on the Cantor space). (To be sure, the point 1 of the proposition is really trivial: it simply remarks that "nested" and "non-decreasing" are synonymous.)

15.10 Proposition. Consider the Cantor space $\mathcal{P}(\mathbb{N})$ with the order given by set inclusion and the distance

$$d(A,B) := \begin{cases} 0 & \text{if } A = B \\ 1/n & \text{if } n = \mu n \in \mathbb{N} \,.\, A \cap n \neq B \cap n \end{cases}.$$

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence with terms in $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$, $F\colon \mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \to \mathbb{N}$ be a function and $I \in \mathcal{P}_{\mathrm{inf}}(\mathbb{N})$.

1. The sequence $(A_n)_{n\in\mathbb{N}}$ is nested if and only if $(A_n)_{n\in\mathbb{N}}$ is non-decreasing.

2. The sequence $(A_n)_{n\in\mathbb{N}}$ weakly converges to I if and only if $(A_n)_{n\in\mathbb{N}}$ converges (in the Cantor space) to I.

3. We have $F \in \mathrm{AS}$ if and only if F stabilises over all non-decreasing sequences with terms in $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$.

4. We have $F \in \mathrm{ASNIS}$ if and only if F stabilises over all convergent (in the Cantor space) sequences with terms in $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ and with infinite limit.

To better compare the points 3 and 4, we rewrite point 3:

3′. We have $F \in \mathrm{AS}$ if and only if F stabilises over all non-decreasing convergent (in the Cantor space) sequences with terms in $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ and with infinite limit.

15.11 Proof. Let us only prove the non-trivial points.

2. We have $A_n \cap k = I \cap k \;\Leftrightarrow\; d(A_n, I) \leq \frac{1}{k+1}$. A sequence $(A_n)_{n\in\mathbb{N}}$ weakly converges to I if and only if $\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \geq m\, (A_n \cap k = I \cap k)$, that is $\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \geq m\, \big(d(A_n, I) \leq \frac{1}{k+1}\big)$, that is $(A_n)_{n\in\mathbb{N}}$ converges (in the Cantor space) to I.

3′. To rewrite point 3, we use that a non-decreasing sequence converges (in the Cantor space) to its union, and if the union is finite then the sequence eventually becomes constant.

15.12. The next proposition clarifies the relation between nested sequences and weakly convergent sequences, and between AS and ASNIS. The proposition is pictured in figure 15.2.

15.13 Proposition. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence with terms in $\mathcal{P}_{\text{fin}}(\mathbb{N})$ and with union A, and let $F\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N}$ be a function.

1. (a) If $(A_n)_{n\in\mathbb{N}}$ is nested and A is finite, then $(A_n)_{n\in\mathbb{N}}$ is not weakly convergent [21, remark 6].
   (b) If $(A_n)_{n\in\mathbb{N}}$ is nested and A is infinite, then $(A_n)_{n\in\mathbb{N}}$ is weakly convergent to A [21, remark 6].
   (c) There is a weakly convergent $(A_n)_{n\in\mathbb{N}}$ that is not nested [72] [21, remark 6].
2. (a) If $F \in$ ASNIS, then $F \in$ AS [21, remark 6].
   (b) There is an $F \in$ AS such that $F \notin$ ASNIS [72] [21, remark 6].

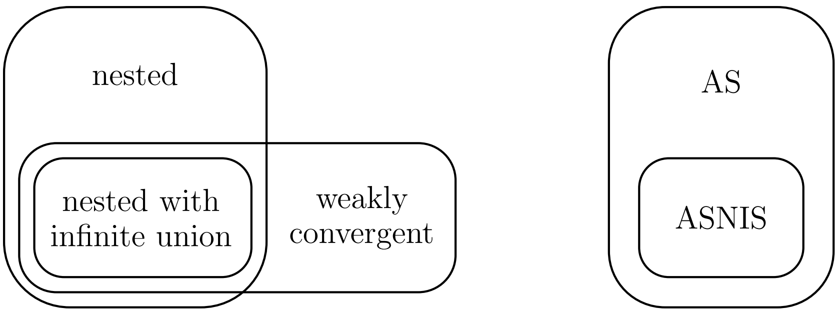

Figure 15.2: relation between nested sequences and weakly convergent sequences, and between AS and ASNIS.

15.14 Proof.

1. (a) If the sequence $(A_n)_{n\in\mathbb{N}}$ were weakly convergent, then it would converge to A, but the limit of a weakly convergent sequence is (by definition) infinite.
   (b) Let us take an arbitrary $B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$ and construct an $m \in \mathbb{N}$ such that for all $n \geq m$ we have $A_n \cap B = A \cap B$. Since $A \cap B$ is finite, say $A \cap B = \{x_1, \dots, x_k\}$. Each x_i is in some A_{m_i}, thus $\{x_1, \dots, x_k\} \subseteq A_{m_1} \cup \dots \cup A_{m_k}$. Since $(A_n)_{n\in\mathbb{N}}$ is nested, $A_{m_1} \cup \dots \cup A_{m_k} = A_m$ for $m := \max(m_1, \dots, m_k)$. Therefore $A \cap B \subseteq A_m$. So, for $n \geq m$, we have $A \cap B \subseteq A_m \cap B \subseteq A_n \cap B \subseteq A \cap B$ (using $A_m \subseteq A_n \subseteq A$), thus $A_n \cap B = A \cap B$.

(c) The sequence $A_n := \{0, \ldots, n\} \cup \{n+2\}$ weakly converges to $\mathbb{N}$ but it is not nested.

2. (a) An $F \in \mathrm{ASNIS}$ stabilises over nested sequences with infinite union (because they are weakly convergent) and over nested sequences with finite union (because such sequences eventually become constant). So $F \in \mathrm{AS}$.

   (b) In proof 15.36 we construct an $F \in \mathrm{AS}$ such that $F \notin \mathrm{ASNIS}$ (otherwise F would be a counterexample to the true FIPP_3).

15.3 "Finitary" infinite pigeonhole principles

15.15. Now we present the three "finitary" infinite pigeonhole principles FIPP_1, FIPP_2 and FIPP_3. Roughly speaking, they say that all colourings (with a finite number of colours) of sufficient long initial segments of $\mathbb{N}$ have "large" monochromatic sets A. They differ in whatever the "large" monochromatic sets are full colour classes or not, and in what "large" exactly means

15.16 Definition.

1. The *infinite pigeonhole principle* IPP is the following principle: for every number of colours n, any colouring $f\colon \mathbb{N} \to n$ of $\mathbb{N}$ with n colours has an infinite colour class $f^{-1}(c)$ with colour $c \in n$ [71] [21, definition 7.1]. In symbols:
$$\forall n\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, (|f^{-1}(c)| = \infty).$$

2. The *first "finitary" infinite pigeonhole principle* FIPP_1 is the following principle: for every number of colours n and for every asymptotically stable function F, there exists an initial segment k of the natural numbers such that any colouring $f\colon k \to n$ of k with n colours has a "big" colour class $A = f^{-1}(c)$ in the sense of $|A| > F(A)$ [71] [21, definition 7.2]. In symbols:
$$\forall n\, \forall F \in \mathrm{AS}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \big(|f^{-1}(c)| > F(f^{-1}(c))\big).$$

3. The *second "finitary" infinite pigeonhole principle* FIPP_2 is analogous to FIPP_1 but only claiming the existence of a "big" subset of a colour class [21, definition 7.3]. In symbols:
$$\forall n\, \forall F \in \mathrm{AS}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \exists A \subseteq f^{-1}(c)\, (|A| > F(A)).$$

4. The *third "finitary" infinite pigeonhole principle* FIPP_3 is analogous to FIPP_1 but replacing AS by ASNIS [71] [21, definition 7.4].

15.17. We can informally derive FIPP_2 from IPP in the following way [50, pages 35–37]. This derivation shows the that FIPP_2 can be obtained by first deriving $(\mathsf{IPP}^{\mathrm{Ku}})^{\mathrm{D}}$, then performing a majorisation-by-compactness argument, and finally translating everything into Tao's language (that talks about sets, set functions and asymptotically stability).

1. Consider any number of colours n and any colouring $f\colon \mathbb{N} \to n$ of $\mathbb{N}$ with n colours. By IPP we get an infinite colour class $f^{-1}(c)$ with colour $c \in n$. So given any l we can construct a monochromatic strictly increasing sequence $m_0, \ldots, m_l \in f^{-1}(c)$ coded by a single natural number m, getting (15.2) below (we discard the condition $m_i > m_{i-1}$ for $i = 0$ since m_{-1} is undefined).

2. We take m as a function g of l (by $\mathsf{QF\text{-}AC}$ to be in line with the characterisation theorem $\mathsf{PA}^\omega + \mathsf{QF\text{-}AC} \vdash A \leftrightarrow (A^{\mathrm{N}})^{\mathrm{D}}$ of D after $\mathrm{N} \in \{\mathrm{GG}, \mathrm{Ko}, \mathrm{Kr}, \mathrm{Ku}\}$ [55, section 5.1] [50, proposition 10.13], like the characterisation theorem of S), getting (15.3).

3. We take l as a function F of c and g (by $\mathsf{QF\text{-}AC}$), getting (15.4). This formula is essentially $(\mathsf{IPP}^{\mathrm{Ku}})^{\mathrm{D}}$ (written as a $\forall\exists$ formula, before a last application of $\mathsf{QF\text{-}AC}$ transforms it into a $\exists\forall$ formula).

4. Now we restrict ourselves to continuous functionals F, so g can be replaced by a long enough initial segment $\bar{m}$ of g coded by a natural number m, getting (15.5). (The restriction is without loss of generality because if (15.4) is false for a discontinuous F, then (15.3) is false, so (15.4) is false for the continuous $F(c, g) := \mu l \,.\, \neg\forall i \leq l \left(f(g(l)_i) = c \wedge g(l)_i > g(l)_{i-1}\right)$ [52].)

5. Consider the functional that assigns to f the least code of a pair (c, m) such that $c \in n$ and $(*)$ holds true. This functional is continuous because f is only evaluated at finitely many points, therefore it is bounded on the compact Cantor space $n^{\mathbb{N}}$. So there exists a strict bound k on m, thus we get (15.6).

6. Since f is evaluated only at the m_is and $m_i \leq m < k$, then we can restrict f to the set k getting (15.7).

7. Define the strictly increasing with colour c sequence $x_i := \bar{m}(F(c, \bar{m}))_i$ for $i = 0, \ldots, l$ where $l := F(c, \bar{m})$, and $A := \{x_0, \ldots, x_l\}$. Then $A \subseteq f^{-1}(c)$ and $|A| = l + 1 > F(c, \bar{m})$. By abuse of notation we replace $F(c, \bar{m})$ by $F(c, A)$. We get (15.8).

8. Since any $F\colon \mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \xrightarrow{\mathrm{cont}} \mathbb{N}$ can be "extended" to an $\bar{F}\colon n \times \mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \xrightarrow{\mathrm{cont}} \mathbb{N}$ by making $\bar{F}$ constant on the first argument, we get (15.9).

9. Finally, replacing the standard notion of continuity in the Cantor space by Tao's non-standard notion of asymptotic stability, we arrive at FIPP_2.

$$
\begin{aligned}
&\mathsf{IPP} \rightsquigarrow \\
\forall n\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, \forall l\, \exists m\, \forall i \le l\, (f(m_i) = c \wedge m_i > m_{i-1}) &\rightsquigarrow \quad (15.2)\\
\forall n\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, \exists g\colon \mathbb{N} \to \mathbb{N}\, \forall l\, \forall i \le l \\
\big(f(g(l)_i) = c \wedge g(l)_i > g(l)_{i-1}\big) &\rightsquigarrow \quad (15.3)\\
\forall n\, \forall f\colon \mathbb{N} \to n\, \forall F\colon n \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \exists c \in n\, \exists g\colon \mathbb{N} \to \mathbb{N}\, \forall i \le F(c,g) \\
\big(f\big(g(F(c,g))_i\big) = c \wedge g(F(c,g))_i > g(F(c,g))_{i-1}\big) &\rightsquigarrow \quad (15.4)\\
\forall n\, \forall f\colon \mathbb{N} \to n\, \forall F\colon n \times \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{cont}} \mathbb{N}\, \exists c \in n\, \exists m \\
\underbrace{\forall i \le F(c,\bar{m})\, \big(f\big(\bar{m}(F(c,\bar{m}))_i\big) = c \wedge \bar{m}(F(c,\bar{m}))_i > \bar{m}(F(c,\bar{m}))_{i-1}\big)}_{(*)} &\rightsquigarrow \quad (15.5)\\
\forall n\, \forall F\colon n \times \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{cont}} \mathbb{N}\, \exists k\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, \exists m < k\, \forall i \le F(c,\bar{m}) \\
\big(f\big(\bar{m}(F(c,\bar{m}))_i\big) = c \wedge \bar{m}(F(c,\bar{m}))_i > \bar{m}(F(c,\bar{m}))_{i-1}\big) &\rightsquigarrow \quad (15.6)\\
\forall n\, \forall F\colon n \times \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{cont}} \mathbb{N}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \exists m\, \forall i \le F(c,\bar{m}) \\
\big(f\big(\bar{m}(F(c,\bar{m}))_i\big) = c \wedge \bar{m}(F(c,\bar{m}))_i > \bar{m}(F(c,\bar{m}))_{i-1}\big) &\rightsquigarrow \quad (15.7)\\
\forall n\, \forall F\colon n \times \mathcal{P}_{\text{fin}}(\mathbb{N}) \xrightarrow{\text{cont}} \mathbb{N}\, \exists k\, \forall f\colon k \to n \\
\exists c \in n\, \exists A \subseteq f^{-1}(c)\, (|A| > F(c,A)) &\rightsquigarrow \quad (15.8)\\
\forall n\, \forall F\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \xrightarrow{\text{cont}} \mathbb{N}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \exists A \subseteq f^{-1}(c)\, (|A| > F(A)) &\rightsquigarrow \quad (15.9)\\
&\mathsf{FIPP}_2.
\end{aligned}
$$

15.18. Now we give two proofs that IPP and FIPP_2 are equivalent, not in the sense that both happen to be true but in the (non-rigorous) sense that from one we can derive the other by an argument where one plays a meaningful role in the deduction of the other. The proofs use different forms of compactness.

Heine-Borel compactness A set is Heine-Borel compact if and only if every open cover has a finite subcover. This is equivalently (in metric spaces) to every real-valued continuous function being bounded on the set [59].

Sequential compactness A set is sequentially compact if and only if every sequence with terms in the set has a convergent subsequence.

The first proof (proof 15.20) uses Heine-Borel compactness and the second proof (proof 15.21) uses sequential compactness. Regarding the equivalence of IPP and FIPP_3, we only give one proof (proof 15.21) using sequential compactness; we do not know a proof using Heine-Borel compactness. (Trying to adapt the proof 15.21 to FIPP_3 fails because even if we know $A = f^{-1}(c) \cap p$ in (15.12), in (15.13) we only get $A \subseteq f^{-1}(c) \cap k$, so A may not be the full colour class required by FIPP_3.)

15.19 Proposition.

1. $\mathsf{IPP} \Leftrightarrow \mathsf{FIPP}_2$ [48, for the way that $\Rightarrow$ is proved].

2. $\mathsf{IPP} \Leftrightarrow \mathsf{FIPP}_3$ [71].

15.20 Proof.

1. Let us abbreviate "$A \subseteq B$ and A is finite" by $A \subseteq_{\text{fin}} B$. We are going to prove $\mathsf{IPP} \Leftrightarrow (15.10) \Leftrightarrow (15.11) \Leftrightarrow (15.12) \Leftrightarrow (15.13) \Leftrightarrow \mathsf{FIPP}_2$, where

$$\forall n\, \forall F \in \mathrm{AS}\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, \exists A \subseteq_{\text{fin}} f^{-1}(c)\, (|A| > F(A)), \tag{15.10}$$

$$\forall n\, \forall F \in \mathrm{AS}\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, \exists p\, \exists A \subseteq f^{-1}(c) \cap p\, (|A| > F(A)), \tag{15.11}$$

$$\begin{gathered}\forall n\, \forall F \in \mathrm{AS}\, \exists k\, \forall f\colon \mathbb{N} \to n \\ \exists c \in n\, \exists p \le k\, \exists A \subseteq f^{-1}(c) \cap p\, (|A| > F(A)),\end{gathered} \tag{15.12}$$

$$\forall n\, \forall F \in \mathrm{AS}\, \exists k\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, \exists A \subseteq f^{-1}(c) \cap k\, (|A| > F(A)). \tag{15.13}$$

<u>$\mathsf{IPP} \Leftrightarrow (15.10)$</u>

<u>$\Rightarrow$</u> Let us assume IPP. Consider arbitrary n, $F \in \mathrm{AS}$ and $f\colon \mathbb{N} \to n$. By IPP, f has an infinite colour class $f^{-1}(c)$. The function F stabilises over the nested sequence $(f^{-1}(c) \cap k)_{k\in\mathbb{N}}$, but the cardinality of the elements of the sequence goes to infinite, so for k sufficient large and $A := f^{-1}(c) \cap k$ we have $|A| > F(A)$.

<u>$\Leftarrow$</u> Let us assume (15.10) and, by contradiction, $\neg\mathsf{IPP}$. Then there exists n and $f\colon \mathbb{N} \to n$ such that all colour classes $f^{-1}(c)$ are finite. So we can define the constant asymptotically stable function $F(A) := \max\{|f^{-1}(c)| : c \in n\}$. Then by (15.10) there exists $c \in n$ and $A \subseteq_{\text{fin}} f^{-1}(c)$ such that $|A| > F(A)$. But this leads to the contradiction $|f^{-1}(c)| \ge |A| > F(A) \ge |f^{-1}(c)|$.

<u>$(15.10) \Leftrightarrow (15.11)$</u> It follows from $\exists A \subseteq_{\text{fin}} f^{-1}(c)\,(\ldots)$ being equivalent to $\exists p\, \exists A \subseteq f^{-1}(c) \cap p\,(\ldots)$.

<u>$(15.11) \Leftrightarrow (15.12)$</u> The right-to-left implication is trivial, so let us see the left-to-right implication. Let us assume (15.11). Consider arbitrary n, $F \in \mathrm{AS}$. By (15.11), for all $f\colon \mathbb{N} \to n$ there exists $c \in n$, p and $A \subseteq f^{-1}(c) \cap p$ such that $|A| > F(A)$. So we can define $F(f) := \min\{q \le p : \exists c \in n\, \exists A \subseteq f^{-1}(c) \cap q\, (|A| > F(A))\}$ because the set is non-empty since p is in it. The colouring f is only evaluated on the set p, so F is continuous on the compact Cantor space $n^{\mathbb{N}}$, thus F is bounded by some k. Then we take $p := F(f) \le k$ in (15.12).

<u>$(15.12) \Leftrightarrow (15.13)$</u> It follows from $\exists p \le k\, \exists A \subseteq f^{-1}(c) \cap p\,(\ldots)$ being equivalent to $\exists A \subseteq f^{-1}(c) \cap k\,(\ldots)$.

<u>$(15.13) \Leftrightarrow \mathsf{FIPP}_2$</u> This equivalence follows from the fact that f is only evaluated on k, so it makes no difference for f to be defined on $\mathbb{N}$ or on k.

15.21 Proof. First we prove the following claim: assuming IPP, for all number of colours n and for all sequences $(f_k)_{k\in\mathbb{N}}$ of colourings $f_k\colon k \to n$ of k with n colours, there exists a colour $c \in n$ such that:

1. there exists a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ of $(f_k)_{k\in\mathbb{N}}$ such that the sequence $(A_{i_k})_{k\in\mathbb{N}}$ of colour classes $A_{i_k} := (f_{i_k})^{-1}(c)$ weakly converges to an infinite set I and $|A_{i_k}| \to \infty$;

2. there exists a sequence $(B_{i_k})_{k\in\mathbb{N}}$ of finite subsets B_{i_k} of A_{i_k} that is nested and $|B_{i_k}| \to \infty$.

Let us prove the claim. We extend each $f_k\colon k \to n$ to $\bar{f}_k\colon \mathbb{N} \to n+1$ by $\bar{f}_k(m) := n$ for $m \geq k$. The $\bar{f}_k$s belongs to the compact Cantor space $(n+1)^{\mathbb{N}}$, so there exists a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ that converges to some limit $f\colon \mathbb{N} \to n$, that is $\forall m\, \exists k_m\, \forall k \geq k_m\, (\bar{f}_{j_k}|_m = f|_m)$, and in particular $\forall m\, (\bar{f}_{j_{k_m}}|_m = f|_m)$. So f takes values even in n. Therefore f has an infinite colour class $f^{-1}(c)$ (with $c \in n$) by IPP. Then $\forall m\, ((\bar{f}_{j_{k_m}}|_m)^{-1}(c) = (f|_m)^{-1}(c))$ where $(f_{j_{k_m}}|_m)^{-1}(c) = (f_{j_{k_m}})^{-1}(c) \cap m$ and $(f|_m)^{-1}(c) = f^{-1}(c) \cap m$. From here, by taking $i_m := j_{k_m}$ and $I := f^{-1}(c)$ we get point 1, and by taking $B_{i_m} := (f_{j_{k_m}})^{-1}(c) \cap m$ we get point 2 (it may help to note that in the definition of weakly convergent sequence we can equivalently restrict the quantification over finite sets to initial segments m of $\mathbb{N}$).

Now we prove the theorem.

1. $\underline{\Rightarrow}$ Let us assume IPP and, by contradiction, $\neg\mathsf{FIPP}_2$. So there exists n and $F \in \mathrm{AS}$ such that for all k there exists $f_k\colon k \to n$ (that we take as a sequence $(f_k)_{k\in\mathbb{N}}$) such that no subset A of some $(f_k)^{-1}(c)$ satisfies $|A| > F(A)$. By point 2 there exists a nested sequence $(B_{i_k})_{k\in\mathbb{N}}$ of subsets of some $(f_{i_k})^{-1}(c)$, such that $|B_{i_k}| \to \infty$. Since F eventually stabilises over this nested sequence, then for some k we have $|B_{i_k}| > F(B_{i_k})$, contradicting that there is no such set A.

   $\underline{\Leftarrow}$ Let us assume FIPP_2 and, by contradiction, $\neg\mathsf{IPP}$. So there exists n and $f\colon \mathbb{N} \to n$ such that all colour classes of f are finite. Then we can define the constant asymptotically stable function $F(A) := \max\{|f^{-1}(c)| : c \in n\}$. By FIPP_2 applied to $f|_k$ there exists $A \subseteq (f|_k)^{-1}(c)$ (for some $c \in n$) such that $|A| > F(A)$. This leads to the contradiction $|f^{-1}(c)| \geq |(f|_k)^{-1}(c)| \geq |A| > F(A) \geq |f^{-1}(c)|$.

2. Analogously to the previous point (but using point 1 of the claim in the beginning of the proof instead of point 2).

15.4 Reverse mathematics

15.22. Reverse mathematics is a research project that attempts to measure the strength of theorems by determining which axioms are need to prove them. For example, over set theory ZF, to prove the theorem "every vector space has a basis" we need exactly the axiom of choice. However, reverse mathematics usually does not deal with set theory but with a hierarchy of five subsystems $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, $\mathsf{ACA_0}$, $\mathsf{ATR_0}$ and Π^1_1-$\mathsf{CA_0}$ of second order arithmetic $\mathsf{Z_2}$, pictured in figure 15.1. Informally, $\mathsf{Z_2}$ is Peano arithmetic extended to talk about sets of natural numbers, or equivalently, it is set theory restricted to natural numbers and sets of natural numbers. Here we are only going to need the first three systems $\mathsf{RCA_0}$, $\mathsf{WKL_0}$ and $\mathsf{ACA_0}$. In order to give a feeling of how powerful they are, we describe in table 15.3 their first and second order parts [64, remarks I.3.3, I.7.6 and I.10.5] [82].

15.23 Definition. Let us define the *second order arithmetic* $\mathsf{Z_2}$ [64, definition I.2.4].

	$\mathsf{RCA_0}$	$\mathsf{WKL_0}$	$\mathsf{ACA_0}$
first order part	Peano arithmetic with only Σ^0_1 induction	Peano arithmetic with only Σ^0_1 induction	full Peano arithmetic
second order part	only proves the existence of computable sets	proves the existence of some non-computable sets	proves the existence of all arithmetical sets

Table 15.3: first and second order parts of $\mathsf{RCA_0}$, $\mathsf{WKL_0}$ and $\mathsf{ACA_0}$.

1. The language of $\mathsf{Z_2}$ is the following.

   (a) The language of $\mathsf{Z_2}$ has the following symbols.

      i. *Number variables* (usually denoted by lower case letters).
      ii. *Set variables* (usually denoted by upper case letters).
      iii. The constant *zero* 0.
      iv. The constant *one* 1.
      v. The binary operation symbol *addition* $+$.
      vi. The binary operation symbol *multiplication* $\cdot$.
      vii. The binary relation *equality* $=$.
      viii. The binary relation *strict inequality* $<$.
      ix. The binary relation *membership* $\in$.
      x. The logical constants $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$, $\exists x$, $\exists X$, $\forall x$ and $\forall X$ (where x is a number variable and X is a set variable).

   (b) Terms are defined as follows.

      i. Variables and (non-logical) constants are terms.
      ii. If s and t are terms, then $s + t$ and $s \cdot t$ are terms.

   (c) Formulas are defined as follows.

      i. The expressions $s = t$, $s < t$ and $s \in X$ are atomic formulas (where s and t are numerical terms and X is a set variable).
      ii. Formulas are built from atomic formulas by means of $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$, $\exists x$, $\exists X$, $\forall x$ and $\forall X$.

2. The axioms of $\mathsf{Z_2}$ are the ones of CL, the usual axioms of equality and the ones given in table 15.4.

15.24. We have the following interpretation in mind for the language of $\mathsf{Z_2}$: numerical terms are intended to range in $\mathbb{N}$, and set variables are intended to range over all subsets of $\mathbb{N}$. Let us quickly comment on the role of the axioms of $\mathsf{Z_2}$.

1. The axioms of successor characterise the behaviour of $x \mapsto x + 1$.

2. The axioms of $+$ and $\cdot$ define these operations by recursion.

axioms of successor	$x + 1 \neq 0$ $x + 1 = y + 1 \to x = y$
axioms of +	$x + 0 = x$ $x + (y + 1) = (x + y) + 1$
axioms of $\cdot$	$x \cdot 0 = 0$ $x \cdot (y + 1) = x \cdot y + y$
axioms of <	$x \not< 0$ $x < y + 1 \leftrightarrow x < y \vee x = y$
induction axiom (schema)	$A(0) \wedge \forall x\, (A(x) \to A(x + 1)) \to \forall x\, A(x)$
comprehension axiom (schema)	$\exists X\, \forall x\, (x \in X \leftrightarrow A(x))$ $(x \notin \mathrm{FV}(A))$

Table 15.4: axioms of Z_2 (in addition to the ones of CL and the usual axioms of equality).

3. The axioms of < suffice (in the presence of a small amount of induction) to show that < is a strict total order (that is irreflexive, asymmetric, transitive and trichotomous) compatible with + and $\cdot$ (in the sense of $x < y \leftrightarrow x + z < y + z$, and $x < y \leftrightarrow xz < yz$ if $z \neq 0$) [64, lemma II.2.1].

4. The comprehension axiom for the formula $A(x)$ says that we can form the set $X = \{x \in \mathbb{N} \colon A(x)\}$.

15.25. We are not going to develop in detail Z_2 and its subsystems RCA_0, WKL_0 and ACA_0 because that by itself would be much longer than the use that we make of these systems. Instead, we quickly outline now what we need. In the presence of a small amount of induction and comprehension (for example, in RCA_0 from point 1 of definition 15.27), we can code finite sets and finite sequences of numbers by numbers [64, section II.2]. Furthermore, we can code functions $f \colon X \subseteq \mathbb{N} \to Y \subseteq \mathbb{N}$ by sets [64, definition II.3.1]. In particular, we can code *binary trees*, that is sets of sequences of 0s and 1s closed under initial segments. An *infinite path* through a tree is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that any initial segment of f is in the tree [64, definition I.6.5]. We can also formalise the notion $|A| = n$ [21, definition 4]. All this allows us to talk in RCA_0 about sequences, functions, trees, finite cardinals, and so on. To deal with sequences and codes, we need the following notation. Let s and t be finite sequences of natural numbers, and $f \colon \mathbb{N} \to \mathbb{N}$ be a function.

1. We denote by A_s the set $\{s(i) : i < \operatorname{lh} s\}$ coded by s [21, definition 2] (where $\operatorname{lh} s$ is the length of s and $s(i)$ is the i-th term of s [64, definition II.2.6]).

2. We denote by $\bar{s}n$ and $\bar{f}n$ the initial segment of length n of s and f, respectively.

3. We denote by Seq the set of all codes of finite sequences [64, definition II.2.6].

Given functions $f, g\colon \mathbb{N} \to \mathbb{N}$, sometimes we write $\bar{f}n = \bar{g}n$ (to fit the language of RCA_0) instead of $f|_n = g|_n$ (that is f and g agree on the first n natural numbers).

15.26. Now we are going to define the subsystems RCA_0, WKL_0 and ACA_0 of Z_2. These subsystems are obtained from Z_2 by restricting induction and comprehension to certain classes of formulas.

15.27 Definition. Let us define the notions of Σ^0_n, Π^0_n and Δ^0_n as usual but with bounded matrices (that is formulas without unbounded quantifications but possibly with the bounded quantifications $\forall x < t\, A :\equiv \forall x\,(x < t \to A)$ and $\exists x < t\, A :\equiv \exists x\,(x < t \wedge A)$ where the number variable x does not occur in the numerical term t) [64, definitions II.1.1 and II.1.2]. An *arithmetical formula* is a formula without set quantifiers [64, definition II.1.2].

1. The *recursive comprehension axiom* RCA_0 [64, definition I.7.4] is the subsystem of Z_2 obtained by restricting the second order induction axiom (schema) to Σ^0_1 formulas and restricting the comprehension axiom (schema) to Δ^0_1 formulas.

2. The *weak Kőnig's lemma* WKL_0 [64, definition I.10.1] is the subsystem of Z_2 obtained by adding to RCA_0 the following principle also called weak Kőnig's lemma: every infinite binary tree has an infinite path.

3. The *arithmetical comprehension axiom* ACA_0 [64, definition I.3.2] is the subsystem of Z_2 obtained by restricting the second order induction axiom (schema) and the comprehension axiom (schema) to arithmetical formulas.

15.28 Remark. We have $\mathsf{RCA}_0 \vdash A \; \substack{\Rightarrow \\ \not\Leftarrow} \; \mathsf{WKL}_0 \vdash A \; \substack{\Rightarrow \\ \not\Leftarrow} \; \mathsf{ACA}_0 \vdash A$ [64, remark I.10.2].

15.29 Example. First, we need to introduce some notions.

Countable field A *countable field* [64, definition II.9.1] is a set $K \subseteq \mathbb{N}$ together with operations $+_K$, $\cdot_K$ and $-_K$ over K satisfying the field axioms.

Polynomial A *polynomial* [64, definition II.9.1] over K is a finite sequence $\langle a_0, \dots, a_n \rangle$ of elements of K, usually written $a_n x^n + \cdots + a_1 x + a_0$.

Weak algebraic closure A *weak algebraic closure* [64, definition II.9.2] of K is a countable field L, together with a monomorphism $h\colon K \to L$, such that L is:

Algebraic closed every non-constant polynomial over L has a root in L;

Algebraic over K every element of L is a root of some non-zero polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ over K (more precisely, of the image $h(p)(x) := h(a_n)x^n + \cdots h(a_1)x + h(a_0)$ of p by h).

Strong algebraic closure A *strong algebraic closure* [64, definition III.3.1] of K is a weak algebraic closure L, h of K together with the image $h(K)$ of h. (The condition that $h(K)$ exists is not superfluous because in general RCA_0 does not prove its existence.)

Uniqueness We say that the algebraic closure of K is *unique* if and only if any for all algebraic closures L, h and L', h' of K, there exists an isomorphism $i\colon L \to L'$ such that $i \circ h = h'$.

The following result shows (by measuring their strength against $\mathsf{RCA_0}$, $\mathsf{WKL_0}$ and $\mathsf{ACA_0}$) that the existence of a weak algebraic closure is weaker than the existence and uniqueness of a weak algebraic closure, which in turn is weaker than the existence and uniqueness of a strong algebraic closure. $\mathsf{RCA_0}$ proves:

1. every countable field has a weak algebraic closure [64, theorem II.9.4];
2. $\mathsf{WKL_0}$ is equivalent to every countable field having a unique weak algebraic closure [64, theorem IV.5.1];
3. $\mathsf{ACA_0}$ is equivalent to every countable field having a unique strong algebraic closure [64, theorem III.3.2].

15.30 Example. First, we need to introduce some notions.

Complete separable metric space A *complete separable metric space* $\hat{A}$ [64, definition II.5.1] is (coded by) a non-empty set $A \subseteq \mathbb{N}$ together with a pseudometric $d\colon A \times A \to \mathbb{R}$ (defined like a metric but excluding the condition $(*)$ $d(x,y) = 0 \to x = y$).

Points The points of $\hat{A}$ are fast Cauchy sequences $x = (x_n)_{n\in\mathbb{N}}$ (that is $\forall m\, \forall n < m\, (d(x_m, x_n) \leq 2^{-n})$) with terms in A;

Metric The metric of $\hat{A}$ is $\hat{d}(x,y) := \lim d(x_n, y_n)$, for which $(*)$ holds by the definition $x = y :\equiv \hat{d}(x,y) = 0$.

Compact metric space A *compact metric space* $\hat{A}$ is a complete separable metric space such that there exists a sequence $(*)$ $((B_{ij})_{i\leq n_j})_{j\in\mathbb{N}}$ of finite sequences $(B_{ij})_{i\leq n_j}$ of open balls B_{ij} of radius 2^{-j} such that each $(B_{ij})_{i\leq n_j}$ covers $\hat{A}$ (formally, to avoid forming the sets B_{ij}, we work with the centre $c_{ij} \in \hat{A}$ of B_{ij}, so $(*)$ becomes $((c_{ij})_{i\leq n_j})_{j\in\mathbb{N}}$) [64, definition III.2.3].

The notions of open set [64, definition I.4.7] and continuous functions [64, definition I.4.6] have to be coded along similar lines, but let us skip this to keep the example light.

We discussed in paragraph 15.18 two forms of compactness: Heine-Borel compactness and sequential compactness. The following result shows (by measuring their strength against $\mathsf{WKL_0}$ and $\mathsf{ACA_0}$) that Heine-Borel compactness is weaker than sequential compactness. $\mathsf{RCA_0}$ proves:

1. $\mathsf{WKL_0}$ is equivalent to Heine-Borel compactness in the form "every countable open cover of a compact metric space has a finite subcover" and also in the form "every real-valued continuous function is bounded on a compact metric space" [64, theorem I.10.3];
2. $\mathsf{ACA_0}$ is equivalent to sequential compactness in the form "every sequence with terms in a compact metric space has a convergent subsequence" [64, theorem III.2.7].

15.5 Reverse mathematics of the "finitary" infinite pigeonhole principles

15.31. The principles FIPP_1, FIPP_2 and FIPP_3 talk about sequences $(A_n)_{n\in\mathbb{N}}$ with terms in $\mathcal{P}_{\text{fin}}(\mathbb{N})$, and functions $F\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N}$. But these objects do not fit the language of $\mathsf{RCA_0}$. So we rewrite the principles to make them fit the language in the following way.

1. Instead of talking about sequences $(A_n)_{n\in\mathbb{N}}$ of finite sets, we talk about sequences $(l_n)_{n\in\mathbb{N}}$ of codes of finite sets.

2. Instead of talking about functions $F\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N}$ that take finite sets as input, we talk about functions $F\colon \mathbb{N} \to \mathbb{N}$ that take codes of finite sets as input. Because the same set can have more than one code, but the value of F is suppose to depend only on the set and not on the chosen code, we assume that F is *extensional*, that is $\forall l, l' \in \mathrm{Seq}\,(A_l \to A_{l'} \to F(l) = F(l'))$ [21, definition 5.3].

To be sure, let us briefly rewrite the definitions 15.7 and 15.16 (omitting the references to the literature since they are the same as before).

15.32 Definition. Let $(l_n)_{n\in\mathbb{N}}$ be a sequence with terms in Seq, $F\colon \mathbb{N} \to \mathbb{N}$ be an extensional function and $I \in \mathcal{P}_{\text{inf}}(\mathbb{N})$. We define the following in $\mathsf{RCA_0}$ (where $f^{-1}(c)$ exists by Σ^0_0 comprehension).

1. We say that $(l_n)_{n\in\mathbb{N}}$ is a *nested* if and only if $\forall n\,(A_{l_n} \subseteq A_{l_{n+1}})$ and $\bigcup_{n\in\mathbb{N}}$ exists.

2. We say that $(l_n)_{n\in\mathbb{N}}$ *weakly converges* to I if and only if for all finite $B \subseteq \mathbb{N}$ we eventually have $A_{l_n} \cap B = I \cap B$ for all n sufficient large.

3. We write $F \in \mathrm{AS}$ if and only if F stabilises over all nested sequences.

4. We write $F \in \mathrm{ASNIS}$ if and only if F stabilises over all weakly convergent sequences.

5. The principles IPP, FIPP_1, FIPP_2 and FIPP_3 are, respectively,

$$\forall n\, \forall f\colon \mathbb{N} \to n\, \exists c \in n\, (|f^{-1}(c)| = \infty),$$
$$\forall n\, \forall F \in \mathrm{AS}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \exists l \in \mathrm{Seq}\,(A_l = f^{-1}(c) \wedge |A_l| > F(l)),$$
$$\forall n\, \forall F \in \mathrm{AS}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \exists l \in \mathrm{Seq}\,(A_l \subseteq f^{-1}(c) \wedge |A_l| > F(l)),$$
$$\forall n\, \forall F \in \mathrm{ASNIS}\, \exists k\, \forall f\colon k \to n\, \exists c \in n\, \exists l \in \mathrm{Seq}\,(A_l = f^{-1}(c) \wedge |A_l| > F(l)).$$

15.33. We adopt the following scheme in the proofs that follow.

<u>Mathematical argument</u> In a first part of the proofs we argue informally, that is without adhering to the language of $\mathsf{RCA_0}$ (by talking about sequences $(A_n)_{n\in\mathbb{N}}$ with terms in $\mathcal{P}_{\text{fin}}(\mathbb{N})$, and functions $F\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to A$) and to the axioms of $\mathsf{RCA_0}$ (by using induction and comprehension at will).

Logical argument If necessary, in a second part of the proof we say what needs to be added to or changed in the first part to formalise the proof in RCA_0.

We strive to get a balance between a clear presentation of the mathematical arguments and giving some detail about the logical arguments. Admittedly, the logical arguments are not so detailed here as in other parts of the text or in our paper [21]. This is by design, since filling the proofs with the technicalities of the logical arguments easily obscure the mathematical arguments, and anyway most of the technicalities are just bureaucratic verifications.

15.34. We start by giving a counterexample to FIPP_1, Tao's original proposed finitisation of IPP. This counterexample led us to propose FIPP_2 and Tao to propose FIPP_3.

15.35 Theorem. $\mathsf{RCA}_0 \vdash \neg\mathsf{FIPP}_1$ [21, theorem 15].

15.36 Proof. Let us show

$$\mathsf{RCA}_0 \vdash \exists n\, \exists F \in \mathrm{AS}\, \forall k\, \exists f_k \colon k \to n\, \forall c \in n \left(|f^{-1}(c)| > F(f^{-1}(c))\right).$$

Let us take $n := 2$ colours. Let $\mathbb{O}$ and $\mathbb{E}$ be the sets of odd and even natural numbers, respectively, and define $F(A) := \min(A \cap \mathbb{O}) + \min(A \cap \mathbb{E}) + 2$ with the non-standard convention $\min \emptyset := 0$. Note $F \in \mathrm{AS}$ because the function min stabilise over nested sequences. Finally, for each k let the colouring f_k assign to odd numbers the colour 0 and to even numbers the colour 1, except for the last two numbers $k-2$ and $k-1$ of the set k, to which we reverse the assignment of colours. We write in figure 15.3 the coloured sets k, with the numbers with colour 0 and 1 marked with one and two dots, respectively, and on the left of each set we write the value of F over its colour classes. We can see that the cardinality $|(f_k)^{-1}(c)|$ of each colour class is less than or equal than the value $F((f_k)^{-1}(c))$ of F over that colour class, because the former is less than or equal to k and the latter is greater than or equal to k.

$$\begin{array}{ccccccccccccc}
\dot{2} & \ddot{2} & & 0 = \{ & & & & & & & & \} & \\
\dot{2} & \ddot{2} & & 1 = \{ & \dot{0} & & & & & & & \} & \\
\dot{2} & \ddot{3} & & 2 = \{ & \dot{0}, & \ddot{1} & & & & & & \} & \\
\dot{4} & \ddot{3} & & 3 = \{ & \ddot{0}, & \ddot{1}, & \dot{2} & & & & & \} & \\
\dot{5} & \ddot{5} & & 4 = \{ & \ddot{0}, & \dot{1}, & \dot{2}, & \ddot{3} & & & & \} & \\
\dot{7} & \ddot{5} & & 5 = \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \ddot{3}, & \dot{4} & & & \} & \\
\dot{7} & \ddot{7} & & 6 = \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \dot{3}, & \dot{4}, & \ddot{5} & & \} & \\
\dot{9} & \ddot{7} & & 7 = \{ & \ddot{0}, & \dot{1}, & \ddot{2}, & \dot{3}, & \ddot{4}, & \ddot{5}, & \dot{6} & \} & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{array}$$

Figure 15.3: the colourings f_k and the value of F over their colour classes.

To formalise the proof in RCA_0, we replace our F by the extensional function $F \colon \mathbb{N} \to \mathbb{N}$ defined by Σ^0_1 comprehension by

$$F(l) := \begin{cases} \min(A_l \cap \mathbb{O}) + \min(A_l \cap \mathbb{E}) + 2 & \text{if } l \in \mathrm{Seq} \\ 0 & \text{if } l \notin \mathrm{Seq} \end{cases}.$$

15.37. With FIPP_1 out of the way, now we turn to FIPP_2 and FIPP_3. We want to determine which of the finitisations FIPP_2 and FIPP_3 of IPP is more faithful, that is which of the equivalences $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ and $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ is provable in a weaker theory. Our candidate theories are RCA_0, WKL_0 and ACA_0. Instead of directly comparing $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ and $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ with RCA_0, WKL_0 and ACA_0, we introduce the continuous uniform boundedness principle CUB as an intermediate step. This principle is a form of "fan principle", a type of principle prominent in intuitionistic mathematics.

15.38 Definition. Let $A(f, \underline{x})$ be a formula with a distinguished set variable f and with distinguished number variables $\underline{x}$. Let $n \notin \mathrm{FV}(A(f, \underline{x}))$.

1. Let us fix a number of colours n. We say that A is *continuous* with respect to $f, \underline{x}$, and write $\mathrm{cont}(A(f, \underline{x}))$, if and only if for all colourings $f\colon \mathbb{N} \to n$ of $\mathbb{N}$ with n colours and for any bound z on $\underline{x}$, if a colouring $g\colon \mathbb{N} \to n$ agrees with f over a long enough initial segment y of $\mathbb{N}$, then A does not "see the difference" between f by g for $\underline{x} \leq z$, in the sense of $\forall \underline{x} \leq z\, (A(f, \underline{x}) \leftrightarrow A(g, \underline{x}))$ [49] [21, definition 18.1]. In symbols:
$$\forall f\colon \mathbb{N} \to n\, \forall z\, \exists y\, \forall g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to \forall \underline{x} \leq z\, (A(f, \underline{x}) \leftrightarrow A(g, \underline{x}))\big).$$
2. The *continuous uniform boundedness principle* CUB is the following principle: for any number of colours n, if $\mathrm{cont}(A(f, \underline{x}))$ and for all colourings $f\colon \mathbb{N} \to n$ there exist $\underline{x}$ such that $A(f, \underline{x})$, then there exists a bound z on $\underline{x}$ uniformly on f [49] [21, definition 18.1]. In symbols:
$$\forall n\, \big(\mathrm{cont}(A(f, \underline{x})) \wedge \forall f\colon \mathbb{N} \to n\, \exists \underline{x}\, A(f, \underline{x}) \to \exists z\, \forall f\colon \mathbb{N} \to n\, \exists \underline{x} \leq z\, A(f, \underline{x})\big).$$
We denote by Σ^0_0-CUB the restriction of CUB to Σ^0_0 formulas, and analogously for Σ^0_1 and Π^0_1 formulas.

15.39. We can think of $\mathrm{cont}(A(f, \underline{x}))$ as the translation of the notion (15.14) below of a continuous functional $\phi\colon n^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}$ to formulas $A(f, \underline{x})$, resulting in (15.15), with some bounded collection $\forall z\, (\forall \underline{x} \leq z\, \exists y\, A(y) \to \exists y\, \forall \underline{x} \leq z\, \exists y' \leq y\, A(y'))$ hardwired in step (15.16):

$$\forall f\colon \mathbb{N} \to n\, \forall \underline{x}\, \exists y\, \forall g\colon \mathbb{N} \to n\, (\bar{f}y = \bar{g}y \to \phi(f, \underline{x}) = \phi(g, \underline{x})) \rightsquigarrow \tag{15.14}$$
$$\forall f\colon \mathbb{N} \to n\, \forall \underline{x}\, \exists y\, \forall g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to (A(f, \underline{x}) \leftrightarrow A(g, \underline{x}))\big) \leftrightarrow \tag{15.15}$$
$$\begin{aligned}&\forall f\colon \mathbb{N} \to n\, \forall z\, \forall \underline{x} \leq z\, \exists y\, \forall g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to (A(f, \underline{x}) \leftrightarrow A(g, \underline{x}))\big) \leftrightarrow \\ &\forall f\colon \mathbb{N} \to n\, \forall z\, \exists y\, \forall \underline{x} \leq z\, \exists y' \leq y\, \forall g\colon \mathbb{N} \to n \\ &\big(\bar{f}y' = \bar{g}y' \to (A(f, \underline{x}) \leftrightarrow A(g, \underline{x}))\big) \leftrightarrow \end{aligned} \tag{15.16}$$
$$\forall f\colon \mathbb{N} \to n\, \forall z\, \exists y\, \forall \underline{x} \leq z\, \forall g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to (A(f, \underline{x}) \leftrightarrow A(g, \underline{x}))\big) \leftrightarrow \mathrm{cont}(A(f, \underline{x})).$$

Then we can think of CUB as saying that if a formula $A(f, \underline{x})$ is continuous and "defines" a function $f \mapsto \underline{x}$ (that is $\forall f\colon \mathbb{N} \to n\, \exists \underline{x}\, A(f, \underline{x})$), then the function is bounded by some z on the compact Cantor space $n^{\mathbb{N}}$: $f \mapsto \underline{x} \leq z$ (that is $\forall f\colon \mathbb{N} \to n\, \exists \underline{x} \leq z\, A(f, \underline{x})$).

15.40. Now we show that the implications and equivalences of figure 15.4 hold in RCA_0. The implications in RCA_0 and WKL_0 are meaningful because these systems do not prove IPP [38, corollary 6.5], FIPP_2 and FIPP_3 [21, corollary 17], but the implication in ACA_0 is not so meaningful because ACA_0 proves IPP [64, lemma III.7.4] [21, page 359]. The equivalences calibrate CUB in terms of WKL_0 and ACA_0 (we could further add that RCA_0 plus unrestricted induction proves $\mathsf{Z}_2 \leftrightarrow \mathsf{CUB}$ [49] [21, theorem 22.3], but this does not fit so nicely our framework and is not needed). The figure suggests that FIPP_2 is a more faithful finitisation of IPP than FIPP_3 because the equivalence $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ is provable in WKL_0, while $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ is provable in ACA_0. However, we did not exclude that $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ may be provable in WKL_0, so we cannot come to a definitive conclusion.

$$\begin{tikzcd}
\mathsf{ACA}_0 \arrow[r, leftrightarrow] & \Pi^0_1\text{-}\mathsf{CUB} \arrow[r] & (\mathsf{IPP} \to \mathsf{FIPP}_3) \\
\mathsf{WKL}_0 \arrow[u, hook] \arrow[r, leftrightarrow] & \Sigma^0_0\text{-}\mathsf{CUB} \arrow[r] & (\mathsf{IPP} \to \mathsf{FIPP}_2) \\
\mathsf{RCA}_0 \arrow[u, hook] \arrow[rr] & & (\mathsf{FIPP}_{2/3} \to \mathsf{IPP})
\end{tikzcd}$$

Figure 15.4: reverse mathematics of "finitary" infinite pigeonhole principles.

15.41 Theorem.

1. $\mathsf{RCA}_0 \vdash \mathsf{FIPP}_2 \to \mathsf{IPP}$ [49] [21, theorem 16.1].
2. $\mathsf{RCA}_0 \vdash \mathsf{FIPP}_3 \to \mathsf{IPP}$ [21, theorem 16.1].

15.42 Proof.

1. Let us assume FIPP_2 and, by contradiction, $\neg\mathsf{IPP}$. So there exists n and $f\colon \mathbb{N} \to n$ such that all colour classes of f are finite. Thus we can define the function $F\colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N}$ by

$$F(A) := \begin{cases} |A| & \text{if } A \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases}.$$

   Note $F \in \mathrm{AS}$: given any nested sequence $(A_m)_{m\in\mathbb{N}}$ with union A, if A is finite then $(A_m)_{m\in\mathbb{N}}$ eventually becomes constant so F stabilises over $(A_m)_{m\in\mathbb{N}}$, and if A is infinite then A_m is not monochromatic for large enough m so F stabilises with value 0 over the sequence. Thus we have an $F \in \mathrm{AS}$ such that $|A| > F(A)$ for no monochromatic finite set A, contradicting FIPP_2.

   To formalise the proof in RCA_0, we do the following.

   (a) We replace $(A_m)_{m\in\mathbb{N}}$ by $(A_{l_m})_{m\in\mathbb{N}}$ (with $l_m \in \mathrm{Seq}$), and A by A_l (with $l \in \mathrm{Seq}$).

(b) The formula $|A_l| \leq n \vee \exists x, y \in A_l\,(f(x) \neq f(y))$ is equivalent in $\mathsf{RCA_0}$ to a Σ^0_0 formula because the quantifications can be bounded (by a term depending on l) and the extraction of the values $f(x)$ and $f(y)$ from the graph of f uses bounded (by n) quantifiers. So we can define the extensional function $F\colon \mathbb{N} \to \mathbb{N}$ by primitive recursion by

$$F(l) := \begin{cases} \mu m\,.\,|A_l| \leq m \vee \exists x, y \in A_l\,(f(x) \neq f(y)) & \text{if } l \in \mathrm{Seq} \\ 0 & \text{otherwise} \end{cases}.$$

(c) Let us prove that if the union A of the nested sequence $(A_{l_m})_{m\in\mathbb{N}}$ is finite, then $(A_{l_m})_{m\in\mathbb{N}}$ stabilises. We assume that A is finite, that is $\exists y\,\forall x \in A\,(x \leq y)$. By the strong Σ^0_1 bounding schema $\forall y\,\exists z\,\forall x \leq y\,(\exists m\,B(x,m) \to \exists m \leq z\,B(x,m))$ where $B(x,m)$ is Σ^0_1 (which is provable in $\mathsf{RCA_0}$ [64, exercise II.3.14]) we have $\exists z\,\forall x \leq y\,(\exists m\,(x \in A_{l_m}) \to \exists m \leq z\,(x \in A_{l_m}))$. So $A = A_{l_z}$, therefore $(A_{l_m})_{m\in\mathbb{N}}$ is constant for $m \geq z$.

2. This point is proved analogously to the the previous point, except that here we argue $F \in \mathrm{ASNIS}$. Consider an arbitrary sequence $(A_m)_{m\in\mathbb{N}}$ weakly converging to an infinite set A. So A is not monochromatic, that is there exists $x, y \in A$ such that $f(x) \neq f(y)$. Let $z := \max(x,y) + 1$ and note $x, y \in z$. Since $(A_m)_{m\in\mathbb{N}}$ weakly converges to A, then we have $A_m \cap z = A \cap z$ for large enough m, thus $x, y \in A_m$. So A_m is not monochromatic for large enough m, therefore F stabilises over $(A_m)_{m\in\mathbb{N}}$ with value 0.

15.43 Theorem.

1. $\mathsf{RCA_0} \vdash \mathsf{WKL_0} \leftrightarrow \Sigma^0_0\text{-}\mathsf{CUB}$ [49] [21, theorem 22.1].

2. $\mathsf{RCA_0} \vdash \mathsf{ACA_0} \leftrightarrow \Pi^0_1\text{-}\mathsf{CUB}$ [49] [21, theorem 22.2].

15.44 Proof.

1. $\underline{\to}$ First, let us remark informally that if $A(f, \underline{x})$ (where $f\colon \mathbb{N} \to n$ is a function) is Σ^0_0, then there are no quantifiers in it making the variables running through infinite many values, so f is only instantiated at finitely many points, thus f can be replaced by $\bar{f}m$ for m large enough. This being said, we can write $A(f, \underline{x})$ as $\forall m\,B(\bar{f}m, \underline{x})$ where $B(f, \underline{x})$ is a Σ^0_0 formula that expresses "if m is large enough, then $A(\bar{f}m, \underline{x})$" [21, lemma 13.3]. Formally, we take $B(s, \underline{x}) :\equiv m \geq t \to A'(s, \underline{x})$, where $A'(s, \underline{x})$ (with $s \in \mathrm{Seq}$) is obtained from $A(f, \underline{x})$ by replacing each instance of $r \in f$ (where r is term) by $\exists x, y \leq r\,(r = (x,y) \wedge s(x) = y)$, and the term t is such that $\mathsf{RCA_0} \vdash \forall f\colon \mathbb{N} \to n\,\big(m \geq t \to (A(f, \underline{x}) \leftrightarrow A'(\bar{f}m, \underline{x}))\big)$. Let us prove the existence of t by induction on the structure of A.

   $\underline{A_{\mathrm{at}}}$ If $f \notin \mathrm{FV}(A_{\mathrm{at}}(f, \underline{x}))$, then the result is trivial, so let us assume $f \in \mathrm{FV}(A_{\mathrm{at}}(f, \underline{x}))$. Then $A_{\mathrm{at}}(f, \underline{x}) \equiv r \in f$, that is $\exists x, y \leq r\,(r = (x,y) \wedge f(x) = y)$, for some term r. So $t := r + 1$ works.

$\underline{\wedge}$ For $A \wedge B$ we take for t the sum of the ts that work for A and B. Analogously for $\neg$, $\vee$, $\to$ and $\leftrightarrow$.

$\underline{\forall<}$ By induction hypothesis we have $t(w)$ working for $A(f, \underline{x}, w)$, so $t(r)$ works for $\forall w < r\, A(f, \underline{x}, w)$ (using $w \le r \to t(w) \le t(r)$, which is provable by induction on the structure of t). Analogously for $\exists<$.

Let us assume $\mathsf{WKL_0}$ and prove Σ^0_0-CUB by contraposition. We assume $(*)\ \forall k\, \exists f\colon \mathbb{N} \to n\, \forall \underline{x} \le k\, A(f, \underline{x})$ where A is Σ^0_0. Consider the bounded tree $T := \{\tau\colon l \to n \mid \forall \underline{x}, m \le l\, B(\bar{\tau}m, \underline{x})\}$. Then T is infinite because for any l there is a $\tau\colon l \to n$ in T: taking $k = l$ in $(*)$ we get an $f\colon \mathbb{N} \to n$ such that $\forall \underline{x} \le k\, A(f, \underline{x})$, that is $\forall \underline{x} \le k\, \forall m\, B(\bar{f}m, \underline{x})$, so $\tau := \bar{f}l \in T$. By $\mathsf{WKL_0}$ (actually by the equivalent bounded weak Kőnig's lemma [64, lemma IV.1.4]) we have an infinite path $f\colon \mathbb{N} \to n$ through T, that is for all l we have $\forall \underline{x}, m \le l\, B(\bar{\bar{f}l}m, \underline{x})$ where $\bar{\bar{f}l}m = \bar{f}m$. Then given any $\underline{x}, m$, taking $l = \max(\underline{x}, m)$ we get $B(\bar{f}m, \underline{x})$. So we have $\forall \underline{x}, m\, B(\bar{f}m, \underline{x})$, that is $\forall \underline{x}\, A(f, \underline{x})$. We conclude $\exists f\colon \mathbb{N} \to n\, \forall \underline{x}\, A(f, \underline{x})$.

$\underline{\leftarrow}$ First let us prove that all Σ^0_0 formulas $A(f, \underline{x})$ are continuous by showing $\mathsf{RCA_0} \vdash \forall z\, \exists y\, \forall f, g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to \forall \underline{x} \le z\, (A(f) \leftrightarrow A(g))\big)$ by induction on the structure of A [21, lemma 13.1].

$\underline{A_{\mathrm{at}}}$ If $f \notin \mathrm{FV}(A_{\mathrm{at}})$ then the result is trivial, so let us assume $x \in \mathrm{FV}(A_{\mathrm{at}})$. Then $A_{\mathrm{at}} \equiv t \in f$, that is $\exists x, y \le t\, (t = (x, y) \wedge f(x) = y)$, for some term t. We can prove $\exists w\, \forall \underline{x} \le t\, (t \le w)$ by induction on the structure of t. So $y := w + 1$ works.

$\underline{\wedge}$ For $A \wedge B$ we take for y the sum of the ys that work for A and B. Analogously for $\neg$, $\vee$, $\to$ and $\leftrightarrow$.

$\underline{\forall<}$ We want to construct a y for $\forall w < t\, A$. By induction hypothesis we have $\forall z'\, \exists y\, \forall f, g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to \forall \underline{x}, w \le z'\, (A(f) \leftrightarrow A(g))\big)$. Taking $z' := \max(z, t)$ we get $\forall z\, \exists y\, \forall f, g\colon \mathbb{N} \to n\, \big(\bar{f}y = \bar{g}y \to \forall \underline{x} \le z\, (\forall w < t\, A(f) \leftrightarrow \forall w < t\, A(g))\big)$. Analogously for $\exists<$.

Now let us prove the implication of the theorem. We assume Σ^0_0-CUB and, by contradiction, the negation of the weak Kőnig's lemma. Then there exists an infinite binary tree T with no infinite path, that is $\forall f\colon \mathbb{N} \to 2\, \exists x\, (\bar{f}x \notin T)$. By Σ^0_0-CUB (actually by Σ^0_1-CUB that results from Σ^0_0-CUB knowing that any Σ^0_0 formula is continuous) applied to the Δ^0_1 formula $\bar{f}x \notin T$ we get a k such that $\forall f\colon \mathbb{N} \to 2\, \exists x \le k\, (\bar{f}x \notin T)$. So every branch in T has length bounded by k, therefore T is actually finite, a contradiction.

2. We show that Π^0_1-CUB is equivalent to Π^0_1 comprehension, which is equivalent to Σ^0_1 comprehension, which in turn is equivalent to $\mathsf{ACA_0}$ [64, lemma III.1.3].

$\underline{\to}$ Let $\forall \underline{m}\, A(x, \underline{m})$ be a Π^0_1 formula where A is a Σ^0_0 formula. By contradiction, we assume that there is no set X such that $\forall x\, (x \in X \leftrightarrow \forall \underline{m}\, A(x, \underline{m}))$, that is there is no characteristic function $f\colon \mathbb{N} \to 2$ of a set X such that $\forall x\, (f(x) = 1 \leftrightarrow \forall \underline{m}\, A(x, \underline{m}))$. So

$$\forall f\colon \mathbb{N} \to 2\, \exists x, \underline{m}\, \forall \underline{m}'\, \neg\big((f(x) = 1 \to A(x, \underline{m})) \wedge (A(x, \underline{m}') \to f(x) = 1)\big).$$

Note that subformula starting with $\forall \underline{m}'$ is Π^0_1 and is continuous with respect to $f, x, \underline{m}$, so by Π^0_1-CUB we get a bound z on $x, \underline{m}$ uniformly on f, that is

$$\forall f \colon \mathbb{N} \to 2\, \exists x \le z$$
$$\neg\big((f(x) = 1 \to \forall \underline{m} \le z\, A(x, \underline{m})) \land (\forall \underline{m}'\, A(x, \underline{m}') \to f(x) = 1)\big).$$

But this is contradicted by the function $f \colon \mathbb{N} \to 2$ defined by

$$f(x) := \begin{cases} 1 & \text{if } \forall \underline{m} \le z\, A(x, \underline{m}) \\ 0 & \text{otherwise} \end{cases}.$$

$\underline{\leftarrow}$ Let us assume the premises $\operatorname{cont}(A(f, \underline{x}))$ and $\forall f \colon \mathbb{N} \to n\, \exists \underline{x}\, A(f, \underline{x})$ of Π^0_1-CUB, where A is a Π^0_1 formula. So we can consider the total and continuous functional $\phi \colon n^{\mathbb{N}} \to \mathbb{N}$ defined by $\phi(f) := \mu z \,.\, \exists \underline{x} \le z\, A(f, \underline{x})$. Since the Cantor space $n^{\mathbb{N}}$ is compact [64, example III.2.6], then ϕ is bounded by some z on $n^{\mathbb{N}}$. We conclude $\forall f \colon \mathbb{N} \to n\, \exists \underline{x} \le z\, A(f, \underline{x})$.

To formalise the proof in $\mathsf{RCA_0}$ we show that the third order object ϕ (that does not fit the language of $\mathsf{RCA_0}$) has a code in $\mathsf{ACA_0}$ as a continuous functional [64, definition II.6.1]. It suffices to show that ϕ has an associate $\alpha \colon \mathbb{N} \to \mathbb{N}$ [47, proposition 4.4] [21, lemma 11], namely

$$\alpha(l) := \begin{cases} \mu z \,.\, \exists \underline{x} \le z\, A(l^\frown 0, \underline{x}) + 1 & \text{if } l \in \mathrm{Seq}_{\le n} \land \forall l' \in \mathrm{Seq}_{\le n}\, (l \subseteq l' \to \\ & \quad \mu z \,.\, \exists \underline{x} \le z\, A(l^\frown 0, \underline{x}) = \\ & \quad \mu z \,.\, \exists \underline{x} \le z\, A(l'^\frown 0, \underline{x})) \\ 0 & \text{otherwise} \end{cases}$$

where $l^\frown 0 \colon \mathbb{N} \to \mathbb{N}$ is the extension of l by zeros, $l \in \mathrm{Seq}_{\le n} :\equiv l \in \mathrm{Seq} \land l^\frown 0 \colon \mathbb{N} \to n$ and $l \subseteq l'$ means that l is an initial segment of l'. Let us prove that α is an associate of ϕ, that is α satisfies the following two conditions.

$\underline{\forall \beta \colon \mathbb{N} \to n\, \exists m\, (\alpha(\bar{\beta} m) > 0)}$ Taking $f = \beta$ and $z = \mu z \,.\, \exists \underline{x} \le z\, A(\beta, \underline{x})$ in $\operatorname{cont}(A(f, \underline{x}))$ (using the notation of point 1 of definition 15.38) we get a y such that $\forall g \colon \mathbb{N} \to n\, \big(\bar{\beta} y = \bar{g} y \to \forall \underline{x} \le \mu z \,.\, \exists \underline{x} \le z\, A(\beta, \underline{x})\, (A(\beta, \underline{x}) \leftrightarrow A(g, \underline{x}))\big)$, so

$$\begin{aligned} &\forall g, g' \colon \mathbb{N} \to n\, (\bar{\beta} y = \bar{g} y = \bar{g}' y \to \mu z \,.\, \exists \underline{x} \le z\, A(\beta, \underline{x}) = \\ &\qquad \mu z \,.\, \exists \underline{x} \le z\, A(g, \underline{x}) = \mu z \,.\, \exists \underline{x} \le z\, A(g', \underline{x})). \end{aligned} \tag{15.17}$$

Let $m := y$. Taking $g = \bar{\beta} m^\frown 0$ and $g' = l'^\frown 0$ (where $l' \in \mathrm{Seq}_{\le n}$) in (15.17) we get $\bar{\beta} m \subseteq l' \to \mu z \,.\, \exists \underline{x} \le z\, A(\bar{\beta} m^\frown 0, \underline{x}) = \mu z \,.\, \exists \underline{x} \le z\, A(l'^\frown 0, \underline{x})$, so $\alpha(\bar{\beta} m) = \mu(\bar{\beta} m) + 1 > 0$.

$\underline{\forall \beta \colon \mathbb{N} \to n\, \big(\alpha(\mu m \,.\, \alpha(\bar{\beta} m) > 0) = \phi(\beta) + 1\big)}$ Let $m := \mu m \,.\, \alpha(\bar{\beta} m) > 0$ and $m' := \max(m, y)$. Using the definition of α in the first two equalities, and (15.17) with $g = \bar{\beta} m'^\frown 0$ in the third equality, we get $\alpha(\bar{\beta} m) = \mu z \,.\, \exists \underline{x} \le z\, A(\bar{\beta} m^\frown 0, \underline{x}) + 1 = \mu z \,.\, \exists \underline{x} \le z\, A(\bar{\beta} m'^\frown 0, \underline{x}) + 1 = \mu z \,.\, \exists \underline{x} \le z\, A(\beta, \underline{x}) + 1 = \phi(\beta) + 1$.

15.45 Theorem.

1. $\mathsf{RCA_0} \vdash \Sigma^0_0\text{-}\mathsf{CUB} \to (\mathsf{IPP} \to \mathsf{FIPP_2})$ [49] [21, theorem 24].

2. $\mathsf{RCA_0} \vdash \Pi^0_1\text{-}\mathsf{CUB} \to (\mathsf{IPP} \to \mathsf{FIPP_3})$ [21, theorem 24].

15.46 Proof.

1. Let us assume Σ^0_0-CUB and IPP. First we prove

$$\forall n\, \forall F \in \mathrm{AS}\, \forall f \colon \mathbb{N} \to n\, \exists m\, B(f,m),$$
$$B(f,m) :\equiv \exists c \in n\, \exists A \subseteq f^{-1}(c) \cap m\, (|A| > F(A)).$$

Consider arbitrary n, $F \in \mathrm{AS}$ and $f : \mathbb{N} \to n$. By IPP, f has an infinite colour class $f^{-1}(c)$. Since $F \in \mathrm{AS}$, then F stabilises over the nested sequence $A_m := f^{-1}(c) \cap m$ with union $f^{-1}(c)$. But $|A_m| \to \infty$. So for m large enough we have $|A_m| > F(A_m)$. Take $A := A_m$ for such an m.

Note $\mathrm{cont}(B(f,m))$ and that B is a Σ^0_1 formula, so by Σ^0_0-CUB (actually by Σ^0_1-CUB that results from Σ^0_0-CUB as showed in point 1 of proof 15.44) we get a bound k on m uniformly on f. Then by restricting f to the set k, and noting that $\exists m \leq k\, B(f,m)$ implies $B(f,k)$, we conclude $\mathsf{FIPP_2}$.

To formalise this proof in $\mathsf{RCA_0}$, we replace the set quantification of A in B by a number quantification of a code of A, so that B becomes a genuine Σ^0_1 formula.

2. The proof is similar to the previous point, but taking $B(f,m) :\equiv \exists c \in n\, \forall A\, (A \cap m = f^{-1}(c) \cap m \to |A| > F(A))$, which is equivalent to a Π^0_1 formula (namely $\forall l'\, \exists c \in n\, \forall l \leq l'\, (l \in \mathrm{Seq} \wedge A_l \cap m = f^{-1}(c) \cap m \to |A_l| > F(A_l))$ by the strong Σ^0_1 bounding schema which is provable in $\mathsf{RCA_0}$ [64, exercise II.3.14]). Then once we get a bound k, the universal quantification on A in $B(f,k)$ allows us to take $A = A_k$, that is $A = (f|_k)^{-1}(c)$, and to conclude $|A| > F(A)$.

15.47. Taking a look at point 2 of proof 15.46, we see that the reason why we need Π^0_1-CUB (rather than only Σ^0_0-CUB) is because we need a universal quantification $\forall A$ in the formula $B(f,m)$ to ensure that A is not just (essentially) a subset of a colour class but it is even the full colour class. The requirement that A is maximal increases the complexity of the instance of CUB used (from Σ^0_0-CUB to Π^0_1-CUB), forcing us to upgrade from $\mathsf{WKL_0}$ to $\mathsf{ACA_0}$ (according to theorem 15.43).

15.6 Conclusion

15.48. We discussed that Tao wants to finitise statements in analysis: to assign to soft analysis statements (infinitary and qualitative) equivalent hard analysis statements (finitary and quantitative). One of his prime examples is an almost finitisation of the infinite pigeonhole principle IPP. There are three proposed finitisations of IPP:

$\underline{\mathsf{FIPP_1}}$ Tao's original finitisation, we gave a counterexample to it;

$\underline{\mathsf{FIPP}_2}$ Our proposed correction;

$\underline{\mathsf{FIPP}_3}$ Tao's proposed correction.

We investigated, in the context of reverse mathematics, which one of the two corrections is a more faithful finitisation, that is which of the equivalence $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2$ and $\mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ is provable in a weaker theory. We concluded

$$\mathsf{WKL}_0 \vdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_2,$$
$$\mathsf{ACA}_0 \vdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3.$$

This suggest that FIPP_2 is a more faithful finitisation of IPP than FIPP_3 (but without showing $\mathsf{WKL}_0 \nvdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3$ we arrive at no definitive answer).

Chapter 16

Proof mining Hillam's theorem

16.1 Introduction

16.1. The fixed point iteration is a method of computing fixed points of continuous functions $\phi\colon [0,1] \to [0,1]$ (here we restrict ourselves to $[0,1]$):

> Fix $v_0 \in [0,1]$ and define the sequence $(v_k)_{k\in\mathbb{N}}$ by $v_{k+1} := \phi(v_k)$. If $v_k \to x$, then x is a fixed point of ϕ.

What this method does not tell us is when $(v_k)_{k\in\mathbb{N}}$ converges. Hillam's theorem [37] answers this question:

> The sequence $(v_k)_{k\in\mathbb{N}}$ converges if and only if $v_{k+1} - v_k \to 0$.

The left-to-right implication is well-known; the right-to-left implication is the interesting one.

16.2. We are going to proof mine Hillam's theorem, that is to extract computational content from Hillam's proof. In more detail, we are going to give a rate of metastability (a kind of "finitary rate of convergence") of $(v_k)_{k\in\mathbb{N}}$ in terms of a rate of metastability of $(v_{k+1}-v_k)_{k\in\mathbb{N}}$ and a rate of uniform continuity of ϕ. Schematically:

$$\begin{array}{c}\text{rate of metastability}\\ \text{of } (v_k)_{k\in\mathbb{N}}\end{array} = f\left(\begin{array}{c}\text{rate of uniform}\\ \text{continuity of } \phi\end{array}, \begin{array}{c}\text{rate of metastability}\\ \text{of } (v_{k+1}-v_k)_{k\in\mathbb{N}}\end{array}\right). \tag{16.1}$$

This is done in the following way. Let

$$\begin{aligned}
A &:\equiv \text{"}\phi \text{ is continuous"},\\
B &:\equiv \text{"}v_{k+1} - v_k \to 0\text{"},\\
C &:\equiv \text{"}(v_k)_{k\in\mathbb{N}} \text{ is a Cauchy sequence"}.
\end{aligned}$$

We write the right-to-left implication of Hillam's theorem as $A,\, B \,/\, C$, and we interpret it with $\mathrm{MD} \circ \mathrm{GG}$, getting a witness $\gamma(\alpha,\beta)$ for $(C^{\mathrm{GG}})^{\mathrm{MD}}$ as a function of witnesses α for $(A^{\mathrm{GG}})^{\mathrm{MD}}$ and β for $(B^{\mathrm{GG}})^{\mathrm{MD}}$:

$$\frac{A \quad B}{C} \qquad \rightsquigarrow \qquad \frac{\alpha \text{ for } (A^{\mathrm{GG}})^{\mathrm{MD}} \quad \beta \text{ for } (B^{\mathrm{GG}})^{\mathrm{MD}}}{\gamma(\alpha,\beta) \text{ for } (C^{\mathrm{GG}})^{\mathrm{MD}}}.$$

Then α, β and γ are the rates that we talk about in (16.1):

$$\underbrace{\begin{matrix}\text{rate of metastability}\\ \text{of } (v_k)_{k\in\mathbb{N}}\end{matrix}}_{\gamma(\alpha,\beta)\text{ for }(C^{\mathrm{GG}})^{\mathrm{MD}}} = \underbrace{f}_{\gamma}\left(\underbrace{\begin{matrix}\text{rate of uniform}\\ \text{continuity of } \phi\end{matrix}}_{\alpha\text{ for }(A^{\mathrm{GG}})^{\mathrm{MD}}}, \underbrace{\begin{matrix}\text{rate of metastability}\\ \text{of } (v_{k+1}-v_k)_{k\in\mathbb{N}}\end{matrix}}_{\beta\text{ for }(B^{\mathrm{GG}})^{\mathrm{MD}}}\right).$$

16.3. We are going to present two proof mined versions of Hillam's theorem.

1. A version that uses a rate of convergence of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$.
   (a) This version is only partially proof mined (because the soundness theorem of MD predicts that we should be able to use only a rate of metastability instead of a full rate of convergence).
   (b) This version has the advantage of giving a simpler γ.
2. A version that uses a rate of metastability of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$.
   (a) This version is fully proof mined (because it only uses what the soundness theorem of MD predicts that should be used).
   (b) This version has the disadvantage of giving a more complicated γ (because it uses a complicated term witnessing the interpretation $(\mathsf{IPP}^{\mathrm{GG}})^{\mathrm{MD}}$ of the infinite pigeonhole principle IPP).

We end with numerical tests on γ to get an idea of how good or bad this rate is.

16.2 Formalising the proof

16.4. In order to proof mine the proof of Hillam's theorem, we start by the theorem with (essentially) its original proof.

16.5 Theorem (Hillam's theorem)**.** Consider a function $\phi\colon [0,1] \to [0,1]$, take an arbitrary $v_0 \in [0,1]$ and define the sequence $(v_k)_{k\in\mathbb{N}}$ by $v_{k+1} := \phi(v_k)$. If ϕ is continuous and $v_{k+1} - v_k \to 0$, then $(v_k)_{k\in\mathbb{N}}$ converges [37].

16.6 Proof. By contradiction, let us assume that $(v_k)_{k\in\mathbb{N}}$ diverges. Then there exist two subsequences converging to distinct cluster points l_1 and l_2 of $(v_k)_{k\in\mathbb{N}}$. Say $l_1 < l_2$ and let $I :=]l_1, l_2[$. We consider two cases.

$\forall x \in I\,(|\phi(x) - x| = 0)$ Since $v_{k+1} - v_k \to 0$, and l_1 and l_2 are cluster points of $(v_k)_{k\in\mathbb{N}}$, then eventually $v_k \in I$ for some $k \in \mathbb{N}$. Then $v_{k+1} = \phi(v_k) = v_k$ by $\forall x \in I\,(\phi(x) = x)$, and analogously $v_k = v_{k+1} = v_{k+2} = \cdots$. But then $(v_k)_{k\in\mathbb{N}}$ converges, contradicting the assumption that it diverges.

$\exists x \in I\,(|\phi(x) - x| > 0)$ By the continuity of ϕ, there exist $\varepsilon, \delta > 0$ such that $(*_1)$ $\forall y \in J :=]x - \varepsilon, x + \varepsilon[\subseteq I\,(|\phi(y) - y| > \delta)$. Since $v_{k+1} - v_k \to 0$, and l_1 and l_2 are cluster points of $(v_k)_{k\in\mathbb{N}}$, then eventually $v_k \in J$ and $(*_2)$ $|v_{k+1} - v_k| < \delta$, for some $k \in \mathbb{N}$. But then $|v_{k+1} - v_k| = |\phi(v_k) - v_k| > \delta$ by $(*_1)$, contradicting $(*_2)$. This is pictured in figure 16.1.

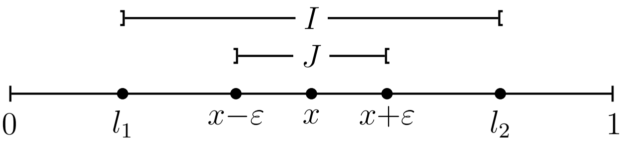

Figure 16.1: the intervals I and J.

16.7.

1. The proof 16.6 uses the Bolzano-Weierstrass theorem to produce an interval J such that $(v_k)_{k\in\mathbb{N}}$ crosses J infinitely often. The way the proof does this is by assuming that $(v_k)_{k\in\mathbb{N}}$ diverges and $v_{k+1} - v_k \to 0$, then using the Bolzano-Weierstrass theorem to get two distinct cluster points l_1 and l_2 (with $l_1 < l_2$) of $(v_k)_{k\in\mathbb{N}}$, and taking $J =]x - \varepsilon, x + \varepsilon[\subseteq]l_1, l_2[$. So $(v_k)_{k\in\mathbb{N}}$ will oscillate infinitely often between l_1 and l_1, and for large enough k will do so entering J.

   In order to give (in proof 16.15 below) a proof of theorem 16.5 in (typed) Peano arithmetic, we are going to replace the full Bolzano-Weierstrass theorem (which requires full arithmetic comprehension because it is equivalent in RCA_0 to ACA_0 [64, theorem III.2.2], similarly to example 15.30) by a "discrete" version of the Bolzano-Weierstrass theorem: the infinite pigeonhole principle IPP (which is provable in Peano arithmetic). That is we will argue the existence of such an interval J using IPP. We sketch the argument. We assume that $(v_k)_{k\in\mathbb{N}}$ diverges (that is there exists an $\varepsilon > 0$ such that $\forall n \in \mathbb{N}\, \exists k > n\, (|v_n - v_k| \geq \varepsilon)$) and $v_{k+1} - v_k \to 0$. We divide $[0, 1]$ into intervals I_i small enough (that is with length $\mathrm{lh}\, I_i < \varepsilon/3$) such that the assumption that $(v_k)_{k\in\mathbb{N}}$ diverges implies that $(v_k)_{k\in\mathbb{N}}$ will always eventually move from one I_{i_0} into a nonadjacent I_{i_1} (that is if $v_n \in I_{i_0}$, then there exists a $k > n$ such that $v_k \in I_{i_1}$ for some i_1 such that $|i_0 - i_1| \geq 2$). By IPP, $(v_k)_{k\in\mathbb{N}}$ enters infinitely often some I_{i_0}. Once inside I_{i_0}, $(v_k)_{k\in\mathbb{N}}$ will eventually jump to a nonadjacent interval. So, by IPP, $(v_k)_{k\in\mathbb{N}}$ enters infinitely often some I_{i_1} nonadjacent to I_{i_0}. Let J be an interval between I_{i_0} and I_{i_1}. So $(v_k)_{k\in\mathbb{N}}$ will oscillate infinitely often between I_{i_0} and I_{i_1}, and for k large enough it will do so entering J. This is illustrated in figure 16.2 [51].

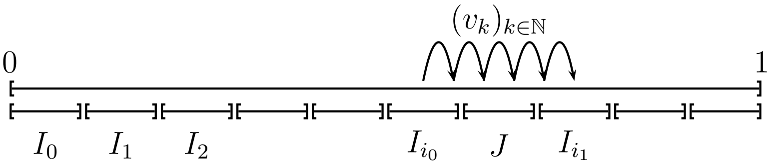

Figure 16.2: the intervals I_i and $(v_k)_{k\in\mathbb{N}}$ crossing J as it goes from I_{i_0} to I_{i_1}.

2. In proof 16.6 we considered two cases: $\forall x \in I\, (|\phi(x) - x| = 0)$ and $\exists x \in I\, (|\phi(x) - x| > 0)$. To proof mine the proof, the second case is problematic because we do not know x, so we cannot estimate the value of $|\phi(x) - x|$. Therefore in proof 16.15 we are going to replace these two cases by ε-versions of the two cases: $\forall x \in I\, (|\phi(x) - x| \leq \varepsilon)$ and $\exists x \in I\, (|\phi(x) - x| > \varepsilon)$, where ε

is a constant constructed using the parameters of the theorem (like a rate of uniform continuity of ϕ).

16.8. The proof of Hillam's theorem talks about real numbers in $[0, 1]$, the difference between two real numbers, inequality between two real numbers, and so on. So to formalise the proof in $\mathsf{WE\text{-}PA}^\omega$ we need to represent all those notions in the language of $\mathsf{WE\text{-}PA}^\omega$. Let us sketch these representations [50, chapter 4]. Two things will be particularly important later on:

1. the inequalities $r_1 < r_2$ and $r_1 \leq r_2$ between real numbers are represented by Σ^0_1 and Π^0_1 formulas, respectively;

2. there exists a term M such that the elements of $[0, 1]$ can be represented by terms bounded by M.

(We use $\mathsf{WE\text{-}PA}^\omega$ instead of PA^ω not because we noticed a need for the extensionality rule, but just to rely on literature that uses $\mathsf{WE\text{-}PA}^\omega$ [50, chapter 4].)

16.9 Definition.

$\underline{\mathbb{Q}}$ By definition 1.37, let j^{000} be a term of $\mathsf{WE\text{-}HA}^\omega$ representing a primitive recursive Cantor pairing function $j\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Every $(x, y) \in \mathbb{N} \times \mathbb{N}$ can be coded by the natural number $j(x, y)$. We represent (x, y) in $\mathsf{WE\text{-}HA}^\omega$ by the term $j^{000}x^0y^0$.

Every $q \in \mathbb{Q}$ can be written as $\frac{x/2}{y+1}$ with $x, y \in \mathbb{N}$ and x even, or as $-\frac{(x+1)/2}{y+1}$ with $x, y \in \mathbb{N}$ and x odd. We represent q in $\mathsf{WE\text{-}HA}^\omega$ by the term $\langle q\rangle :\equiv jxy$.

By definition 1.37, let e^{000} be a term representing the primitive recursive function $e\colon \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that for all $q_1, q_2 \in \mathbb{Q}$ we have: $e(\langle q_1\rangle, \langle q_2\rangle) = 0$ if and only if $q_1 = q_2$. We represent in $\mathsf{WE\text{-}HA}^\omega$ the equality between rational numbers by $e^{000}x^0y^0 =_0 0$. Analogously for $<_\mathbb{Q}$, $\geq_\mathbb{Q}$, $+_\mathbb{Q}$, $-_\mathbb{Q}$ and $|\cdot|_\mathbb{Q}$.

$\underline{\mathbb{R}}$ Every $r \in \mathbb{R}$ is the limit of some Cauchy sequence $(r_n)_{n\in\mathbb{N}}$ of rational numbers that converges fast, that is $\forall n \in \mathbb{N}\, (|r_{n+1} - r_n| < 2^{-(n+1)})$. We represent r in $\mathsf{WE\text{-}HA}^\omega$ by a term r^1 such that $\mathsf{WE\text{-}HA}^\omega \vdash R(r)$ where $R(r) :\equiv \forall n^0\, (|r(\mathrm{S}n) -_\mathbb{Q} rn|_\mathbb{Q} <_\mathbb{Q} \langle 2^{-(n+1)}\rangle)$.

At first sight, a quantification $\forall r \in \mathbb{R}\, A(r)$ would be represented in $\mathsf{WE\text{-}HA}^\omega$ by $\forall r^1\, (R(r) \to A(r))$, therefore affecting the complexity of formulas. We can avoid this by using the fast convergence: we define in $\mathsf{WE\text{-}HA}^\omega$ the functional $r^1 \mapsto \hat{r}^1$ by

$$\hat{r}n^0 := \begin{cases} rn & \text{if } \forall m \leq n\, (|r(\mathrm{S}m) - rm|_\mathbb{Q} <_\mathbb{Q} \langle 2^{-(m+1)}\rangle) \\ rk & \text{if } k = \mu m\, .\, |r(\mathrm{S}m) - rm|_\mathbb{Q} \geq_\mathbb{Q} \langle 2^{-(m+1)}\rangle \end{cases},$$

such that $\mathsf{WE\text{-}HA}^\omega \vdash \forall r^1\, R(\hat{r})$, and then $\forall r \in \mathbb{R}\, A(r)$ is represented in $\mathsf{WE\text{-}HA}^\omega$ by $\forall r^1\, A(\hat{r})$.

We represent in WE-HA$^\omega$ equality and inequality between real numbers by

$$r_1 =_{\mathbb{R}} r_2 :\equiv \forall n^0 \left(|\hat{r}_1(\mathrm{S}n) -_{\mathbb{Q}} \hat{r}_2(\mathrm{S}n)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-n} \rangle \right),$$
$$r_1 <_{\mathbb{R}} r_2 :\equiv \exists n^0 \left(\hat{r}_2(\mathrm{S}n) -_{\mathbb{Q}} \hat{r}_1(\mathrm{S}n) \geq_{\mathbb{Q}} \langle 2^{-n} \rangle \right),$$
$$r_1 \leq_{\mathbb{R}} r_2 :\equiv \forall n^0 \left(\hat{r}_1(\mathrm{S}n) -_{\mathbb{Q}} \hat{r}_2(\mathrm{S}n) <_{\mathbb{Q}} \langle 2^{-n} \rangle \right).$$

Also, we represent in WE-HA$^\omega$ the addition of real numbers by $r_1 +_{\mathbb{R}} r_2 :\equiv \lambda n \,.\, (\hat{r}_1(\mathrm{S}n) +_{\mathbb{Q}} \hat{r}_2(\mathrm{S}n))$ (the change from $\hat{r}_i(n)$ to $\hat{r}_i(\mathrm{S}n)$ ensures that $r_1 +_{\mathbb{R}} r_2$ converges fast), and analogously for $-_{\mathbb{R}}$, $\cdot_{\mathbb{R}}$ and $/_{\mathbb{R}}$.

$\underline{[0,1]}$ There exists a primitive recursive function $M \colon \mathbb{N} \to \mathbb{N}$ such that real numbers in $[0,1]$ can be represented in WE-HA$^\omega$ by $\tilde{r}^1$ such that $\tilde{r} \leq^{\mathrm{e}} M$. Let us sketch this construction.

For each $k \in \mathbb{N}$, consider the "2^{-n}-fine net of points" $N_n := \{2^{-n} i \colon i = 0, \ldots, 2^n\}$. Let $M(n) := \max_{q \in N_n} \langle q \rangle$. Since $N_0 \subseteq N_1 \subseteq \cdots$, then $M(0) \leq M(1) \leq \cdots$, thus $M \leq^{\mathrm{e}} M$.

Every $r \in [0,1]$ is within a distance $2^{-(n+1)}$ of some $q \in N_n$, so we can define the sequence $\tilde{r} = (\tilde{r}_n)_{n \in \mathbb{N}}$ by $\tilde{r}_n := \langle \mu q \in N_n \,.\, |r - q| \leq 2^{-(n+1)} \rangle$. Informally, this is a sequence, of (codes $\langle q \rangle$ of) rational numbers $q \in N_n$, converging to r. This is illustrated in figure 16.3. Since these codes are bounded $M(n)$, then the sequence is bounded by M.

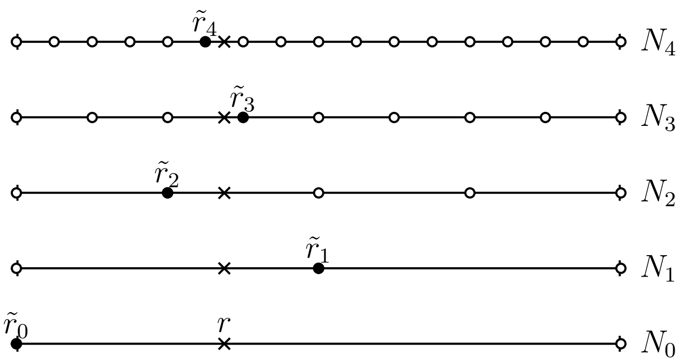

Figure 16.3: the sequence $(\tilde{r}_n)_{n \in \mathbb{N}}$ approximating r over the nets N_n.

16.10 Lemma.

1. There exists a primitive recursive and bijective pairing function $\langle \cdot, \cdot \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, the pairs $(p, q) \in \mathbb{N} \times \mathbb{N}$ with $p, q < n$ are coded by $\langle p, q \rangle < n^2$ [65, page 20].

2. We have $\forall w, x, y, z \in \mathbb{R} \, (|x - y| \geq |w - z| - |w - x| - |y - z|)$. This is pictured in figure 16.4.

3. Let $J = [a, b]$ be a interval in $\mathbb{R}$ and $\mathrm{lh}\, J := b - a$ its length. Let $x \in J$, $\varepsilon > 0$ and $I = \,]x - \varepsilon, x + \varepsilon[$. If $\mathrm{lh}\, J \geq \varepsilon$, then $\mathrm{lh}(I \cap J) \geq \varepsilon$.

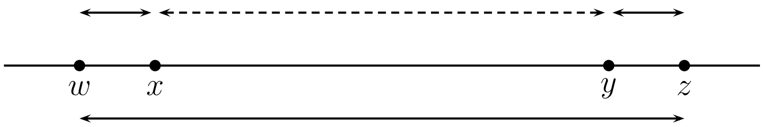

Figure 16.4: the dashed distance is greater than or equal to the big distance minus the two small distances, that is $|x-y| \geq |w-z| - |w-x| - |y-z|$.

p						
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	16	17	18	19	20	$\cdots$
3	9	10	11	12	21	$\cdots$
2	4	5	6	13	22	$\cdots$
1	1	2	7	14	23	$\cdots$
0	0	3	8	15	24	$\cdots$
	0	1	2	3	4	$\cdots$ q

Figure 16.5: a bijective pairing $\langle\cdot,\cdot\rangle$ such that $\langle p, q\rangle < n^2$ whenever $p, q < n$.

16.11 Proof.

1. The pairing function is illustrated in figure 16.5.
2. Using twice the triangular inequality we have $|w-z| = |(w-x)+(x-y)+(y-z)| \leq |w-x|+|x-y|+|y-z|$.
3. We consider the two cases $x-\varepsilon < a$ and $x-\varepsilon \geq a$. For each one of these cases, we consider the two subcases $x+\varepsilon < b$ and $x+\varepsilon \geq b$. For each case and subcase, we present in table 16.1 an interval with length at least ε that is contained in $I \cap J$. So $\mathrm{lh}(I \cap J) \geq \varepsilon$, as we wanted.

	$x+\varepsilon < b$	$x+\varepsilon \geq b$
$x-\varepsilon < a$	$[x, x+\varepsilon[\, \subseteq I \cap J$	$J \subseteq I \cap J$
$x-\varepsilon \geq a$	$I \subseteq I \cap J$	$]x-\varepsilon, x] \subseteq I \cap J$

Table 16.1: intervals, with length greater than or equal to ε, contained in $I \cap J$.

16.12. Let Π^0_1-$\mathsf{AC}^{0,0}$ be AC restricted to Π^0_1 formulas and variables of type 0 (that is $\forall x^0\, \exists y^0\, A(x,y) \to \exists Y\, \forall x\, A(x, Yx)$ where $A(x,y)$ is Π^0_1), and analogously we define Σ^0_1-$\mathsf{AC}^{0,0}$ and QF-$\mathsf{AC}^{0,0}$. In the next theorem we show that there are proofs of theorem 16.5 in WE-$\mathsf{PA}^\omega + \Pi^0_1$-$\mathsf{AC}^{0,0}$ and even in WE-$\mathsf{PA}^\omega + \mathsf{QF}$-$\mathsf{AC}^{0,0}$.

1. The proof in WE-$\mathsf{PA}^\omega + \Pi^0_1$-$\mathsf{AC}^{0,0}$ splits the interval $[0,1]$ into closed intervals $I_i := \left[\frac{i}{n}, \frac{i+1}{n}\right]$, and has simpler computations but at the expense of using Π^0_1-$\mathsf{AC}^{0,0}$ (because the formula $x \in I_i$ is Π^0_1).

   It is this proof, with its simpler computations, that we are actually going to proof mine.

For the expert, we remark that when proof mining this proof, it could happen that the terms extracted use bar recursion because of Π^0_1-$\mathsf{AC}^{0,0}$. This will not happen because the use of Π^0_1-$\mathsf{AC}^{0,0}$ is unessential. Indeed, taking a close look at the proof, we see that there are two places where we use Π^0_1-$\mathsf{AC}^{0,0}$:

(a) in the first place we can replace Π^0_1-$\mathsf{AC}^{0,0}$ by QF-$\mathsf{AC}^{0,0}$ (from which we get Σ^0_1-$\mathsf{AC}^{0,0}$) by replacing the Π^0_1 formula $|v_i - v_j| \geq 2^{-f}$ by the Σ^0_1 formula $|v_i - v_j| > 2^{-(f+1)}$ [52];

(b) in the second place we can replace Π^0_1-$\mathsf{AC}^{0,0}$ by the (very particular) bounded rule of choice $\forall x^0 \, \exists y \leq_0 Zx \, A(x, y) \, / \, \exists Y \leq Z \, \forall x \, A(x, Yx)$ [50, page 142] where $A(x, y)$ is Π^0_1, and then eliminate the rule by adding the premise (of the instance of the rule that we use) to Γ and the conclusion to Γ' in the soundness theorem of MD [52].

Nevertheless, to keep the proof simple, we prefer not to do this two changes to the proof, and instead to do the proof with Π^0_1-$\mathsf{AC}^{0,0}$.

2. The proof in WE-$\mathsf{PA}^\omega + \mathsf{QF}$-$\mathsf{AC}^{0,0}$ splits the interval $[0, 1]$ into open intervals $I'_i := \left]\frac{i}{n} - \frac{1/3}{n}, \frac{i+1}{n} + \frac{1/3}{n}\right[$, and uses QF-$\mathsf{AC}^{0,0}$ (because the formula $x \in I'_i$ is Σ^0_1, and Σ^0_1-$\mathsf{AC}^{0,0}$ follows from QF-$\mathsf{AC}^{0,0}$) but at the expense of more complicated computations.

   We use this proof only to give a theoretical guarantee that there are terms witnessing the interpretation by MD $\circ$ GG of theorem 16.5.

16.13. Below we formulate the Cauchy property of $(v_k)_{k\in\mathbb{N}}$ in the slightly convoluted form (16.4) instead of the more usual form $\forall f \in \mathbb{N} \, \exists g \in \mathbb{N} \, \forall i, j \geq g \, (|v_i - v_j| < 2^{-f})$. The reason for this is that the interpretation by MD $\circ$ GG of the former form gives us (roughly speaking) an interval where the terms of the sequence are close to each other, while the interpretation of the latter form gives us only a pair of points close to each other.

16.14 Theorem. The following is provable in WE-$\mathsf{PA}^\omega + \Pi^0_1$-$\mathsf{AC}^{0,0}$ and even in WE-$\mathsf{PA}^\omega + \mathsf{QF}$-$\mathsf{AC}^{0,0}$. Consider a function $\phi \colon [0, 1] \to [0, 1]$, take an arbitrary $v_0 \in [0, 1]$ and define the sequence $(v_k)_{k\in\mathbb{N}}$ by $v_{k+1} := \phi(v_k)$. If

1. the function ϕ is (uniformly) continuous, that is
$$\forall a \in \mathbb{N} \, \exists b \in \mathbb{N} \, \forall x, y \in [0, 1] \, (|x - y| < 2^{-b} \to |\phi(x) - \phi(y)| < 2^{-a}); \tag{16.2}$$

2. we have $v_{k+1} - v_k \to 0$, that is
$$\forall c \in \mathbb{N} \, \exists d \in \mathbb{N} \, \forall e \in \mathbb{N} \, (|v_{d+e+1} - v_{d+e}| < 2^{-c}); \tag{16.3}$$

then $(v_k)_{k\in\mathbb{N}}$ converges (is a Cauchy sequence), that is
$$\forall f \in \mathbb{N} \, \exists g \in \mathbb{N} \, \forall h \in \mathbb{N} \, \forall i, j \in [g; g + h] \, (|v_i - v_j| < 2^{-f}). \tag{16.4}$$

16.15 Proof. In the first point below we give a proof in $\mathsf{WE\text{-}PA}^\omega + \Pi^0_1\text{-}\mathsf{AC}^{0,0}$ in some detail. Then in the second point below we sketch how to adapt it to a proof in $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC}^{0,0}$. Through the proof, our main concern is that when we apply AC to some formula, we have to pay attention to the complexity of the formula.

1. By contradiction, we assume the negation of (16.4), that is there exists an $f \in \mathbb{N}$ such that $\forall g \in \mathbb{N}\,\exists h \in \mathbb{N}\,\exists i,j \in [g; g+h]\,(|v_i - v_j| \geq 2^{-f})$, thus $\forall g \in \mathbb{N}\,\exists h \in \mathbb{N}\,\exists h' \in]g; g+h]\,(|v_g - v_{h'}| \geq 2^{-(f+1)})$, where $|v_g - v_{h'}| \geq 2^{-(f+1)}$ is Π^0_1. By $\Pi^0_1\text{-}\mathsf{AC}^{0,0}$ we get $H, H' \colon \mathbb{N} \to \mathbb{N}$ such that $(*)$ $\forall g \in \mathbb{N}\,(H'(g) \in]g; g+H(g)] \wedge |v_g - v_{H'(g)}| \geq 2^{-(f+1)})$.

   Let $n := 3 \times 2^{f+1}$ and $[0,1] = \bigcup_{i=0}^{n-1} I_i$ where $I_i := \left[\frac{i}{n}, \frac{i+1}{n}\right]$. Then $\forall k \in \mathbb{N}\,\exists i < n\,(v_k \in I_i)$, where $v_k \in I_i$ is Π^0_1. By $\Pi^0_1\text{-}\mathsf{AC}^{0,0}$ we get an $F \colon \mathbb{N} \to n$ (where n denotes $\{0, 1, 2, \ldots, n-1\}$) such that $\forall k \in \mathbb{N}\,(v_k \in I_{F(k)})$. Define the colouring $F' \colon \mathbb{N} \to n^2$ with n^2 colours by $F'(k) := \langle F(k), F(H'(k))\rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing of point 1 of lemma 16.10.

   By IPP (which is provable in PA^ω) we get an $i < n^2$ such that $\forall k \in \mathbb{N}\,\exists p \in \mathbb{N}\,(p \geq k \wedge F'(p) = i)$. By $\mathsf{QF\text{-}AC}^{0,0}$ we get a $P \colon \mathbb{N} \to \mathbb{N}$ such that $\forall k \in \mathbb{N}\big(P(k) \geq k \wedge F'(P(k)) = i\big)$. Say $i = \langle i_0, i_1\rangle$ where $i_0, i_1 < n$. Then $F'(P(k)) = i$ is equivalent to $F(P(k)) = i_0 \wedge F\big(H'(P(k))\big) = i_1$, which implies $v_{P(k)} \in I_{i_0} \wedge v_{H'(P(k))} \in I_{i_1}$. Moreover, $P(k) < H'(P(k)) \leq P(k) + H(P(k))$ (by $(*)$) and $|i_0 - i_1| \geq 2$ (since $|v_{P(k)} - v_{H'(P(k))}| \geq 2^{-(f+1)}$ by $(*)$). In particular, $H(P(k)) > 0$, so below we can write $\frac{1}{2nH(P(d))}$.

   Consider $I_{i_0 \pm 1}$, where we choose the plus sign if $i_0 + 1 < i_1$, and the minus sign if $i_1 + 1 < i_0$, so that $I_{i_0 \pm 1}$ is between I_{i_0} and I_{i_1}. This is illustrated in figure 16.6.

   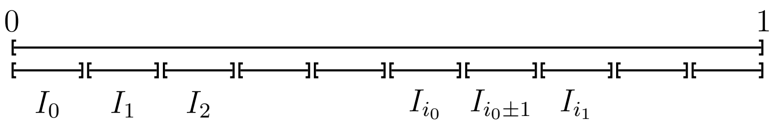

   Figure 16.6: the intervals I_i.

   Taking $c = 1 + \lceil \log_2 n \rceil$ in (16.3) we get a $d \in \mathbb{N}$ such that $\forall e \in \mathbb{N}\,(|v_{d+e+1} - v_{d+e}| < \frac{1}{2n})$. Now we consider two cases.

   $\forall x \in I_{i_0 \pm 1}\left(|\phi(x) - x| < \frac{1}{2nH(P(d))}\right)$ The sequence $(v_k)_{k \in \mathbb{N}}$ goes from $v_{P(d)} \in I_{i_0}$ to $v_{H'(P(d))} \in I_{i_1}$ in steps strictly smaller than $\frac{1}{2n}$, that is half of the length of $I_{i_0 \pm 1}$. Thus v_k enters the half of $I_{i_0 \pm 1}$ closest to I_{i_0}, for some $k \in [P(d); H'(P(d))]$. Then, to reach I_{i_1}, the sequence $(v_k)_{k \in \mathbb{N}}$

      (a) has to cover the other half of $I_{i_0 \pm 1}$;

      (b) in at most $H'(P(d)) - P(d) \leq H(P(d))$ steps;

      (c) and inside $I_{i_0 \pm 1}$ (that is for all $k \in \mathbb{N}$ such that $v_k \in I_{i_0 \pm 1}$) each step has length $|v_{k+1} - v_k| = |\phi(v_k) - v_k| \leq \frac{1}{2nH(P(d))}$.

But this is impossible because in at most $H(P(d))$ steps strictly smaller than $\frac{1}{2nH(P(d))}$ the sequence $(v_k)_{k\in\mathbb{N}}$ covers a distance strictly smaller than $\frac{1}{2n}$, that is strictly smaller than the length of half of $I_{i_0\pm1}$.

$\underline{\exists x \in I_{i_0\pm1}\left(|\phi(x)-x| \geq \frac{1}{2nH(P(d))}\right)}$ Taking $a := \left\lceil \log_2\left(6nH(P(d))\right)\right\rceil$ in (16.2) we get a $b\in\mathbb{N}$ such that (16.5), and taking $c := \max(a,b)$ in (16.3) we get a $d'\in\mathbb{N}$ such that (16.6):

$$\forall y \in [0,1]\left(|x-y| < 2^{-b} \to |\phi(x)-\phi(y)| < \frac{1}{6nH(P(d))}\right), \tag{16.5}$$

$$\forall e \in \mathbb{N}\left(|v_{d'+e+1} - v_{d'+e}| < 2^{-c} \leq \frac{1}{6nH(P(d))}, 2^{-b}\right). \tag{16.6}$$

Let $J :=]x-2^{-c}, x+2^{-c}[$. By point 2 of lemma 16.10 we have

$$\forall y \in J\Bigg(|\phi(y)-y| \geq \underbrace{|\phi(x)-x|}_{\geq\frac{1}{2nH(P(d))}} - \underbrace{|\phi(x)-\phi(y)|}_{<\frac{1}{6nH(P(d))}} - \underbrace{|y-x|}_{<\frac{1}{6nH(P(d))}} > \frac{1}{6nH(P(d))}\Bigg). \tag{16.7}$$

Since $2^{-c} \leq \operatorname{lh} I_{i_0\pm1}$ (because $2^{-c} \leq \frac{1}{6nH(P(d))}$ and $\operatorname{lh} I_{i_0\pm1} = 1/n$), then by point 3 of lemma 16.10 we have $2^{-c} \leq \operatorname{lh}(I_{i_0\pm1}\cap J)$. So, by (16.6), as $(v_k)_{k\in\mathbb{N}}$ goes from $v_{P(d')} \in I_{i_0}$ to $v_{H'(P(d'))} \in I_{i_1}$ it enters J for some $k \in [P(d'); H'(P(d'))]$. But then $|v_{k+1}-v_k| = |\phi(v_k)-v_k| > \frac{1}{6nH(P(d))}$ by (16.7), contradicting (16.6).

2. As before, we assume the negation of (16.4), getting $f\in\mathbb{N}$. Let $f' := 2f$. Then $\forall g \in \mathbb{N}\,\exists h \in \mathbb{N}\,\exists h' \in]h; g+h]\,(|v_g - v_{h'}| > 2^{-(f'+1)})$ where $|v_g - v_{h'}| > 2^{-(f'+1)}$ is Σ^0_1. By $\mathsf{QF\text{-}AC}^{0,0}$ (that implies $\Sigma^0_1\text{-}\mathsf{AC}^{0,0}$) we get h and h' as respectively functions H and H' of g.

   Let $n := 3\times 2^{f'+1}$ and $[0,1] \subseteq \bigcup_{i=0}^{n-1} I'_i$ where $I'_i := \left]\frac{i}{n} - \frac{1/3}{n}, \frac{i+1}{n} + \frac{1/3}{n}\right[$. As before we get a P such that $v_{P(k)} \in I_{i_0} \wedge v_{H'(P(k))} \in I_{i_1}$.

   Let $I := I_{i_0\pm1} \setminus (I_{i_0}\cup I_{i_1})$, which has length $\operatorname{lh} I \geq \frac{1}{3n}$. This is pictured in figure 16.7.

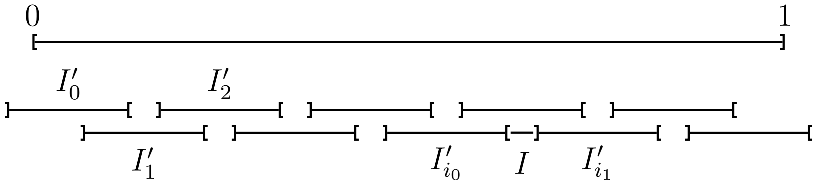

Figure 16.7: the intervals I'_i.

Taking $c = \lceil \log_2(6n)\rceil$ in (16.3) we get a $d\in\mathbb{N}$ such that $\forall e \in \mathbb{N}\left(|v_{d+e+1} - v_{d+e}| < \frac{1}{6n}\right)$. We consider two cases.

$\underline{\forall x \in I \left(|\phi(x) - x| < \frac{1}{6nH(P(d))}\right)}$ For some $k \in [P(d); H'(P(d))]$, the sequence $(v_k)_{k\in\mathbb{N}}$ enters the half of I closest to I_{i_0}. Then $(v_k)_{k\in\mathbb{N}}$ cannot cover the other half, which has length greater than or equal to $\frac{1}{6n}$, in at most $H(P(d))$ steps of length strictly smaller than $\frac{1}{6nH(P(d))}$.

$\underline{\exists x \in I \left(|\phi(x) - x| \geq \frac{1}{6nH(P(d))}\right)}$ Taking $a := \left\lceil \log_2 \left(18nH(P(d))\right)\right\rceil$ in (16.2) we get a $b \in \mathbb{N}$ such that (16.8), and taking $c := \max(a, b)$ in (16.3) we get a $d' \in \mathbb{N}$ such that (16.9):

$$\forall y \in [0,1] \left(|x - y| < 2^{-b} \to |\phi(x) - \phi(y)| < \frac{1}{18nH(P(d))}\right), \tag{16.8}$$

$$\forall e \in \mathbb{N} \left(|v_{d'+e+1} - v_{d'+e}| < 2^{-c} \leq \frac{1}{18nH(P(d))}, 2^{-b}\right). \tag{16.9}$$

Let $J :=]x - 2^{-c}, x + 2^{-c}[$. We have

$$\forall y \in J \Bigg(|\phi(y) - y| \geq \underbrace{|\phi(x) - x|}_{\geq \frac{1}{6nH(P(d))}} - \underbrace{|\phi(x) - \phi(y)|}_{< \frac{1}{18nH(P(d))}} - \underbrace{|y - x|}_{< \frac{1}{18nH(P(d))}} > \frac{1}{18nH(P(d))}\Bigg). \tag{16.10}$$

As $(v_k)_{k\in\mathbb{N}}$ goes from $v_{P(d')} \in I_{i_0}$ to $v_{H'(P(d'))} \in I_{i_1}$ it enters J for some $k \in [P(d'); H'(P(d'))]$. But then $|v_{k+1} - v_k| = |\phi(v_k) - v_k| > \frac{1}{18nH(P(d))}$ by (16.10), contradicting (16.9).

16.3 Rates of uniform continuity, convergence and metastability

16.16. In the next definition we are going to define the notions of rate of uniform continuity, rate of convergence and rate of metastability. Let us motivate this definitions. The motivation takes place at a mathematical level, not at a logic level, so we are not going to concern ourselves much with the amounts of logic and of axiom of choice used. Let $\phi\colon [0,1] \to [0,1]$ be a function, $(v_k)_{k\in\mathbb{N}}$ be a sequence of real numbers, and $l \in \mathbb{R}$.

<u>Rate of uniform continuity of ϕ</u> By definition, ϕ is uniformly continuous if and only if (16.11) below holds true. Equivalently, we can restrict this formula to ε and δ of the form 2^{-a} and 2^{-b} (with $a, b \in \mathbb{N}$) respectively, getting (16.12). Taking b as a function of a (by AC) we get an $\alpha\colon \mathbb{N} \to \mathbb{N}$ satisfying (16.13). We call *rate of uniform continuity* of ϕ to such an α.

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in [0,1]\, (|x - y| < \delta \to |\phi(x) - \phi(y)| < \varepsilon) \Leftrightarrow \tag{16.11}$$

$$\begin{gathered}\forall a \in \mathbb{N}\, \exists b \in \mathbb{N}\, \forall x, y \in [0,1] \\ (|x - y| < 2^{-b} \to |\phi(x) - \phi(y)| < 2^{-a})\end{gathered} \Leftrightarrow \tag{16.12}$$

$$\begin{gathered}\exists \alpha\colon \mathbb{N} \to \mathbb{N}\, \forall a \in \mathbb{N}\, \forall x, y \in [0,1] \\ (|x - y| < 2^{-\alpha(a)} \to |\phi(x) - \phi(y)| < 2^{-a}).\end{gathered} \tag{16.13}$$

Rate of convergence of $(v_k)_{k\in\mathbb{N}}$ (with limit l) By definition, $(v_k)_{k\in\mathbb{N}}$ converges to l if and only if the formula (16.14) below holds true. Making the change of variable $e = e' - d$ we get (16.15). Equivalently, we can restrict this formula to ε of the form 2^{-c} (with $c \in \mathbb{N}$), getting (16.16). Taking d as a function of c (by AC) we get a $\beta\colon \mathbb{N} \to \mathbb{N}$ satisfying (16.17). We call *rate of convergence* of $(v_k)_{k\in\mathbb{N}}$ (with limit l) to such a β.

$$\forall\varepsilon > 0\, \exists d \in \mathbb{N}\, \forall e' \geq d\, (|v_{e'} - l| < \varepsilon) \;\Leftrightarrow \tag{16.14}$$
$$\forall\varepsilon > 0\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < \varepsilon) \;\Leftrightarrow \tag{16.15}$$
$$\forall c \in \mathbb{N}\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < 2^{-c}) \;\Leftrightarrow \tag{16.16}$$
$$\exists\beta\colon \mathbb{N} \to \mathbb{N}\, \forall c, e \in \mathbb{N}\, (|v_{\beta(c)+e} - l| < 2^{-c}). \tag{16.17}$$

Rate of metastability of $(v_k)_{k\in\mathbb{N}}$ (with limit l) By definition, $(v_k)_{k\in\mathbb{N}}$ converges to l if and only if the formula (16.18) below holds true. Making the change of variable $e = e' - d$ we get (16.19). Equivalently, we can restrict this formula to ε of the form 2^{-c} (with $c \in \mathbb{N}$), getting (16.20). Adding a double negation and moving one negation inside we get formulas (16.21) and (16.22). Taking e as a function of d (by AC, but could be done by $\mathsf{QF\text{-}AC}$ by adapting the derivation so that where is $|v_{d+e} - l| \geq 2^{-c}$ would be $|v_{d+e} - l| > 2^{-(c+1)}$, to be in line with the characterisation theorem $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC} \vdash A \leftrightarrow (A^{\mathrm{N}})^{\mathrm{D}}$ of D after $\mathrm{N} \in \{\mathrm{GG}, \mathrm{Ko}, \mathrm{Kr}, \mathrm{Ku}\}$ [55, section 5.1] [50, proposition 10.13], like the characterisation theorem of S) we get (16.23). Moving the remaining negation inside we get (16.24). Finally, bounding d as a function of c and E we get a $\beta\colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying (16.25). We call *rate of metastability* of $(v_k)_{k\in\mathbb{N}}$ (with limit l) to such a β.

$$\forall\varepsilon > 0\, \exists d \in \mathbb{N}\, \forall e' \geq d\, (|v_{e'} - l| < \varepsilon) \;\Leftrightarrow \tag{16.18}$$
$$\forall\varepsilon > 0\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < \varepsilon) \;\Leftrightarrow \tag{16.19}$$
$$\forall c \in \mathbb{N}\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < 2^{-c}) \;\Leftrightarrow \tag{16.20}$$
$$\neg\neg\forall c \in \mathbb{N}\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < 2^{-c}) \;\Leftrightarrow \tag{16.21}$$
$$\neg\exists c \in \mathbb{N}\, \forall d \in \mathbb{N}\, \exists e \in \mathbb{N}\, (|v_{d+e} - l| \geq 2^{-c}) \;\Leftrightarrow \tag{16.22}$$
$$\neg\exists c \in \mathbb{N}\, \exists E\colon \mathbb{N} \to \mathbb{N}\, \forall d \in \mathbb{N}\, (|v_{d+E(d)} - l| \geq 2^{-c}) \;\Leftrightarrow \tag{16.23}$$
$$\forall c \in \mathbb{N}\, \forall E\colon \mathbb{N} \to \mathbb{N}\, \exists d \in \mathbb{N}\, (|v_{d+E(d)} - l| < 2^{-c}) \;\Leftrightarrow \tag{16.24}$$
$$\begin{gathered}\exists\beta\colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall c \in \mathbb{N}\, \forall E\colon \mathbb{N} \to \mathbb{N}\, \exists d \leq \beta(c, E)\\ (|v_{d+E(d)} - l| < 2^{-c}).\end{gathered} \tag{16.25}$$

Alternatively, negating both sides of the axiom of choice $\forall d \in \mathbb{N}\, \exists e \in \mathbb{N}\, \neg A(e) \leftrightarrow \exists E\colon \mathbb{N} \to \mathbb{N}\, \forall d \in \mathbb{N}\, \neg A(E(d))$ we get $\exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, A(e) \leftrightarrow \forall E\colon \mathbb{N} \to \mathbb{N}\, \exists d \in \mathbb{N}\, A(E(d))$, and use this to get (16.26) below.

$$\forall\varepsilon > 0\, \exists d \in \mathbb{N}\, \forall e' \geq d\, (|v_{e'} - l| < \varepsilon) \;\Leftrightarrow$$
$$\forall\varepsilon > 0\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < \varepsilon) \;\Leftrightarrow$$
$$\forall c \in \mathbb{N}\, \exists d \in \mathbb{N}\, \forall e \in \mathbb{N}\, (|v_{d+e} - l| < 2^{-c}) \;\Leftrightarrow$$
$$\forall c \in \mathbb{N}\, \forall E\colon \mathbb{N} \to \mathbb{N}\, \exists d \in \mathbb{N}\, (|v_{d+E(d)} - l| < 2^{-c}) \;\Leftrightarrow \tag{16.26}$$

$$\exists \beta \colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall c \in \mathbb{N}\, \forall E \colon \mathbb{N} \to \mathbb{N}\, \exists d \leq \beta(c, E)$$
$$(|v_{d+E(d)} - l| < 2^{-c}).$$

Formula (16.25) gives us a single index $d + E(d)$ such that $v_{d+E(d)}$ is close to l. We can actually upgrade this to an entire interval $[d; d + H(d)] = \{d, d + 1, \ldots, d + H(d)\}$ (where $H \colon \mathbb{N} \to \mathbb{N}$ is an arbitrary function) such that for all $k \in [d; d + H(d)]$ we have that v_k is close to l:

$$\begin{gathered} \exists \beta' \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall c \in \mathbb{N}\, \forall H \colon \mathbb{N} \to \mathbb{N}\, \exists d \leq \beta'(c, H) \\ \forall k \in [d; d + H(d)]\, (|v_k - l| < 2^{-c}). \end{gathered} \tag{16.27}$$

Indeed, let us define $E_H \colon \mathbb{N} \to \mathbb{N}$ and $\beta' \colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by

$$\begin{aligned} E_H(d) &:= \mu n \leq H(d) \,.\, \forall k \in [d; d + H(d)]\, (|v_k - l| \leq |v_{n+d} - l|), \\ \beta'(c, H) &:= \beta(c, E_H) \end{aligned}$$

(note that E_H is well defined because when n runs from 0 to $H(d)$, the value $n+d$ runs through $[d; d+H(d)]$, so the k that maximises $|v_k-l|$ over $[d; d+H(d)]$ will be met by $n + d$). Taking $E = E_H$ in (16.25) we get a $d \leq \beta(c, E_H)$ such that $|v_{d+E_H(d)} - l| < 2^{-c}$. But $\forall k \in [d; d + H(d)]\, (|v_k - l| \leq |v_{d+E_H(d)} - l|)$ by definition of E_H. From these two formulas we conclude $\forall k \in [d; d+H(d)]\, (|v_k - l| < 2^{-c})$, proving (16.27), as we wanted.

Rate of metastability of $(v_k)_{k\in\mathbb{N}}$ (without mentioning a limit) By definition, $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence if and only if the formula (16.28) below holds true. This formula is equivalent to (16.29). Equivalently, we can restrict (16.29) to ε of the form 2^{-f} (with $f \in \mathbb{N}$), getting (16.30). Adding a double negation and moving one negation inside we get formulas (16.31) and (16.32). Taking h as a function of g (by AC, but could be done by $\mathsf{QF\text{-}AC}$ as before) we get (16.33). Moving the remaining negation inside we get (16.34). Finally, bounding g as a function of f and H we get a $\gamma \colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying (16.35). We call *rate of metastability* of $(v_k)_{k\in\mathbb{N}}$ (without mentioning a limit) to such a γ.

$$\forall \varepsilon > 0\, \exists g \in \mathbb{N}\, \forall i, j \geq g\, (|v_i - v_j| < \varepsilon) \;\Leftrightarrow \tag{16.28}$$
$$\forall \varepsilon > 0\, \exists g \in \mathbb{N}\, \forall h \in \mathbb{N}\, \forall i, j \in [g; g + h]\, (|v_i - v_j| < \varepsilon) \;\Leftrightarrow \tag{16.29}$$
$$\forall f \in \mathbb{N}\, \exists g \in \mathbb{N}\, \forall h \in \mathbb{N}\, \forall i, j \in [g; g + h]\, (|v_i - v_j| < 2^{-f}) \;\Leftrightarrow \tag{16.30}$$
$$\neg\neg \forall f \in \mathbb{N}\, \exists g \in \mathbb{N}\, \forall h \in \mathbb{N}\, \forall i, j \in [g; g + h]\, (|v_i - v_j| < 2^{-f}) \;\Leftrightarrow \tag{16.31}$$
$$\neg \exists f \in \mathbb{N}\, \forall g \in \mathbb{N}\, \exists h \in \mathbb{N}\, \exists i, j \in [g; g + h]\, (|v_i - v_j| \geq 2^{-f}) \;\Leftrightarrow \tag{16.32}$$
$$\begin{gathered} \neg \exists f \in \mathbb{N}\, \exists H \colon \mathbb{N} \to \mathbb{N}\, \forall g \in \mathbb{N} \\ \exists i, j \in [g; g + H(g)]\, (|v_i - v_j| \geq 2^{-f}) \end{gathered} \;\Leftrightarrow \tag{16.33}$$
$$\forall f \in \mathbb{N}\, \forall H \colon \mathbb{N} \to \mathbb{N}\, \exists g \in \mathbb{N}\, \forall i, j \in [g; g + H(g)]\, (|v_i - v_j| < 2^{-f}) \;\Leftrightarrow \tag{16.34}$$
$$\begin{gathered} \exists \gamma \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall f \in \mathbb{N}\, \forall H \colon \mathbb{N} \to \mathbb{N}\, \exists g \leq \gamma(c, H) \\ \forall i, j \in [g; g + H(g)]\, (|v_i - v_j| < 2^{-f}). \end{gathered} \tag{16.35}$$

Alternatively, negating both sides of the axiom of choice as before, we get (16.36) below.

$$\begin{aligned}
\forall \varepsilon > 0\, \exists g \in \mathbb{N}\, \forall i, j \geq g\, (|v_i - v_j| < \varepsilon) &\Leftrightarrow \\
\forall \varepsilon > 0\, \exists g \in \mathbb{N}\, \forall h \in \mathbb{N}\, \forall i, j \in [g; g+h]\, (|v_i - v_j| < \varepsilon) &\Leftrightarrow \\
\forall f \in \mathbb{N}\, \exists g \in \mathbb{N}\, \forall h \in \mathbb{N}\, \forall i, j \in [g; g+h]\, (|v_i - v_j| < 2^{-f}) &\Leftrightarrow \\
\forall f \in \mathbb{N}\, \forall H \colon \mathbb{N} \to \mathbb{N}\, \exists g \in \mathbb{N}\, \forall i, j \in [g; g+H(g)]\, (|v_i - v_j| < 2^{-f}) &\Leftrightarrow \quad (16.36) \\
\exists \gamma \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall f \in \mathbb{N}\, \forall H \colon \mathbb{N} \to \mathbb{N}\, \exists g \leq \gamma(c, H)& \\
\forall i, j \in [g; g+H(g)]\, (|v_i - v_j| < 2^{-f}).&
\end{aligned}$$

16.17 Definition. Let $\phi \colon [0,1] \to [0,1]$ be a function, $(v_k)_{k\in\mathbb{N}}$ be a sequence of real numbers, and $l \in \mathbb{R}$.

1. We say that $\alpha \in \mathbb{N} \to \mathbb{N}$ is a *rate of uniform continuity* of ϕ if and only if
$$\forall a \in \mathbb{N}\, \forall x, y \in [0,1]\, (|x - y| < 2^{-\alpha(a)} \to |\phi(x) - \phi(y)| < 2^{-a}).$$
2. We say that $\beta \colon \mathbb{N} \to \mathbb{N}$ is a *rate of convergence* of $(v_k)_{k\in\mathbb{N}}$ (with limit l) if and only if
$$\forall c, e \in \mathbb{N}\, (|v_{\beta(e)+e} - l| < 2^{-c}).$$
3. We say that $\beta \colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a *rate of metastability* of $(v_k)_{k\in\mathbb{N}}$ (with limit l) if and only if
$$\forall c \in \mathbb{N}\, \forall E \colon \mathbb{N} \to \mathbb{N}\, \exists d \leq \beta(c, E)\, (|v_{d+E(d)} - l| < 2^{-c}).$$
4. We say that $\gamma \colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a *rate of metastability* of $(v_k)_{k\in\mathbb{N}}$ (without mentioning a limit) if and only if
$$\forall f \in \mathbb{N}\, \forall H \colon \mathbb{N} \to \mathbb{N}\, \exists g \leq \gamma(f, H)\, \forall i, j \in [g; g+H(g)]\, (|v_i - v_j| < 2^{-f}).$$

16.18. In general, a functional f has no majorant, that is there is no functional f^{M} such that $f \leq^{\mathrm{e}} f^{\mathrm{M}}$ [50, proposition 3.70.2]. But if f is a function $f \colon \mathbb{N}^n \to \mathbb{N}$ (or in other words, f has type $0 \cdots 0$) we can construct a majorant f^{M} (we can even construct the functional $f \mapsto f^{\mathrm{M}}$ in HA^ω). This will be needed later on and so we do it in the next definition.

16.19 Definition. For each function $f \colon \mathbb{N}^n \to \mathbb{N}$, we define the function $f^{\mathrm{M}} \colon \mathbb{N}^n \to \mathbb{N}$ [50, definition 3.65] by

$$f^{\mathrm{M}}(k_1, \ldots, k_n) := \max\{f(l_1, \ldots, l_n) : 0 \leq l_1 \leq k_1 \wedge \cdots \wedge 0 \leq l_n \leq k_n\}.$$

16.20. Theorem 16.14 is of the form $T :\equiv A \wedge B \to C$ where

$$\begin{aligned}
&A :\equiv \text{“}\phi \text{ is continuous”}, \\
&B :\equiv \text{“}v_{k+1} - v_k \to 0\text{”}, \\
&C :\equiv \text{“}(v_k)_{k\in\mathbb{N}} \text{ is a Cauchy sequence”}.
\end{aligned}$$

(Actually, to represent the theorem in $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC}$ we could need to use the hypothesis "ϕ is a function", that is $\forall x, y \in [0,1]\,(x = y \to \phi x = \phi y)$, but it turns out that its interpretation by MD $\circ$ GG is unnecessary because it essentially duplicates the interpretation of A.) Since $\mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC} \vdash T$ by theorem 16.14, then $\mathsf{WE\text{-}HA}^\omega + \mathsf{QF\text{-}AC} \vdash T^{\mathrm{GG}}$ (where $T^{\mathrm{GG}} \equiv A^{\mathrm{GG}} \wedge B^{\mathrm{GG}} \to C^{\mathrm{GG}}$) by the soundness of GG, thus there are (closed monotone) terms $\underline{q}$ of $\mathsf{WE\text{-}HA}^\omega$ such that $(*)$ $\mathsf{WE\text{-}HA}^\omega + \mathsf{QF\text{-}AC} \vdash \exists \underline{x} \leq^{\mathrm{e}} \underline{q}\, \forall \underline{y}\, (T^{\mathrm{GG}})_{\mathrm{D}}(\underline{x}\underline{\ell}; \underline{y})$ (where $\mathrm{FV}(T) = \{\underline{\ell}\}$) by the soundness theorem of MD. However, instead of extracting terms $\underline{q}$ such that $(*)$, it is simpler to extract terms $\underline{t}(\underline{r}, \underline{s})$ such that holds the rule

$$\frac{\exists \underline{a} \leq^{\mathrm{e}} \underline{r}\, \forall \underline{b}\, (A^{\mathrm{GG}})_{\mathrm{D}}(\underline{a}\underline{\ell}; \underline{b}) \quad \exists \underline{c} \leq^{\mathrm{e}} \underline{s}\, \forall \underline{d}\, (B^{\mathrm{GG}})_{\mathrm{D}}(\underline{c}\underline{\ell}; \underline{d})}{\exists \underline{e} \leq^{\mathrm{e}} \underline{t}(\underline{r}, \underline{s})\, \forall \underline{f}\, (C^{\mathrm{GG}})_{\mathrm{D}}(\underline{e}\underline{\ell}; \underline{f})}$$

(the terms $\underline{t}(\underline{r}, \underline{s})$ exist because if the premises of the rule are provable in $\mathsf{WE\text{-}HA}^\omega + \mathsf{QF\text{-}AC}$, then we have terms working for $A \wedge B$, and so by $(*)$ and the fact that MD interprets modus ponens we get terms working for C, that is we get the conclusion of the rule). So now we compute the interpretation of A, B and C by MD $\circ$ GG. The formulas A, B and C are (essentially) Π_3 formulas, so it saves us some work to compute in general the interpretation by MD $\circ$ GG of a Π_3 formula. That is what we do in the next proposition.

16.21 Proposition. Let each variable in $\underline{\ell}, \underline{w}, \underline{y}, \underline{z}$ have type 0 or 00, $\underline{x}$ have type 0,

$$(A(\underline{w}, \underline{x}, \underline{y})^{\mathrm{GG}})^{\mathrm{D}} \equiv \exists \underline{z}\, A_{\mathrm{qf}}(\underline{w}, \underline{x}, \underline{y}, \underline{z}),$$
$$\mathrm{FV}(\forall \underline{w}\, \exists \underline{x}\, \forall \underline{y}\, A(\underline{w}, \underline{x}, \underline{y})) = \{\underline{\ell}\},$$
$$\big((\forall \underline{w}\, \exists \underline{x}\, \forall \underline{y}\, A(\underline{w}, \underline{x}, \underline{y}))^{\mathrm{GG}}\big)^{\mathrm{MD}} \equiv \tilde{\exists} \underline{a}\, \exists \underline{B} \leq^{\mathrm{e}} \underline{a}\, \forall \underline{\ell}, \underline{c}\, ((\forall \underline{w}\, \exists \underline{x}\, \forall \underline{y}\, A(\underline{w}, \underline{x}, \underline{y}))^{\mathrm{GG}})_{\mathrm{D}}(\underline{B}\underline{\ell}; \underline{c}).$$

1. We have
$$\mathsf{WE\text{-}PA}^\omega + \mathsf{AC} \vdash \tilde{\exists} \underline{a}\, \exists \underline{B} \leq^{\mathrm{e}} \underline{a}\, \forall \underline{\ell}, \underline{c}\, \big((\forall \underline{w}\, \exists \underline{x}\, \forall \underline{y}\, A(\underline{w}, \underline{x}, \underline{y}))^{\mathrm{GG}}\big)_{\mathrm{D}}(\underline{B}\underline{\ell}; \underline{c}) \to$$
$$\tilde{\exists} \underline{\mathrm{X}}\, \forall \underline{\ell}, \underline{w}, \underline{Y}\, \exists \underline{x} \leq^{\mathrm{e}} \underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{Y}\, A(\underline{w}, \underline{x}, \underline{Y}\underline{x}).$$
2. From closed monotone terms $\underline{t}_{\underline{a}}$ witnessing $\underline{a}$ we can construct closed monotone terms $\underline{t}_{\underline{\mathrm{X}}}$ witnessing $\underline{\mathrm{X}}$:
$$\mathsf{WE\text{-}PA}^\omega + \mathsf{AC} \vdash \exists \underline{B} \leq^{\mathrm{e}} \underline{t}_{\underline{a}}\, \forall \underline{\ell}, \underline{c}\, \big((\forall \underline{w}\, \exists \underline{x}\, \forall \underline{y}\, A(\underline{w}, \underline{x}, \underline{y}))^{\mathrm{GG}}\big)_{\mathrm{D}}(\underline{B}\underline{\ell}; \underline{c}) \to$$
$$\forall \underline{\ell}, \underline{w}, \underline{Y}\, \exists \underline{x} \leq^{\mathrm{e}} \underline{t}_{\underline{\mathrm{X}}}\underline{\ell}\underline{w}\underline{Y}\, A(\underline{w}, \underline{x}, \underline{Y}\underline{x}).$$
3. If some variables in $\underline{w}, \underline{y}$ range in $[0,1]$, then we can assume that the bounds $\underline{\mathrm{X}}$ and the terms $\underline{t}_{\underline{\mathrm{X}}}$ are independent of those variables.

16.22 Proof.

1. Below, in step (16.37) we compute the translation by GG (simplified using the intuitionistic $\neg \forall x_1 \neg \cdots \neg \forall x_n \neg B \leftrightarrow \neg \forall x_1, \ldots, x_n \neg B$). In step (16.38) we compute the interpretation by D. In step (16.39) we compute the interpretation by MD. In implication (16.40) we remove the double negation.

In implication (16.41) we take $\underline{\mathrm{Y}} = \lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}$. In implication (16.42) we take $\underline{z} = \underline{\mathrm{Z}}\underline{\ell}\underline{w}(\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x})(\underline{Y}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}))$. In implication (16.43) we take $\underline{x} = \underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}$ (note that $\underline{\ell}^{\mathrm{M}}, \underline{w}^{\mathrm{M}}, \underline{Y}^{\mathrm{M}}\underline{x}$ are defined because of the restrictions on the types of $\underline{\ell}, \underline{w}, \underline{y}$, and that $\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x} \leq^{\mathrm{e}} \lambda\underline{x}, \underline{Z} \,.\, \underline{Y}^{\mathrm{M}}\underline{x}$). In implication (16.44) we take $\underline{\mathrm{X}} = \lambda\underline{\ell}, \underline{w}, \underline{Y} \,.\, \underline{\mathrm{X}}'\underline{\ell}^{\mathrm{M}}\underline{w}^{\mathrm{M}}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}^{\mathrm{M}}\underline{x}$. In implication (16.45) we use that $(A(\underline{w}, \underline{x}, \underline{Y}\underline{x})^{\mathrm{GG}})^{\mathrm{D}} \equiv \exists\underline{z}\, A_{\mathrm{qf}}(\underline{w}, \underline{x}, \underline{Y}\underline{x}, \underline{z})$ is equivalent to $A(\underline{w}, \underline{x}, \underline{Y}\underline{x})$ by the characterisation theorems of D and GG.

$$\forall\underline{w}\,\exists\underline{x}\,\forall\underline{y}\, A(\underline{w}, \underline{x}, \underline{y}) \rightsquigarrow \tag{16.37}$$

$$\forall\underline{w}\,\neg\forall\underline{x}\,\neg\forall\underline{y}\, A^{\mathrm{GG}}(\underline{w}, \underline{x}, \underline{y}) \rightsquigarrow \tag{16.38}$$

$$\begin{gathered}\exists\underline{\mathrm{X}}, \underline{\mathrm{Z}}\,\forall\underline{w}, \underline{\mathrm{Y}} \\ \neg\neg A_{\mathrm{qf}}\big(\underline{w}, \underline{\mathrm{X}}\underline{w}\underline{\mathrm{Y}}, \underline{\mathrm{Y}}(\underline{\mathrm{X}}\underline{w}\underline{\mathrm{Y}})(\underline{\mathrm{Z}}\underline{w}\underline{\mathrm{Y}}), \underline{\mathrm{Z}}\underline{w}\underline{\mathrm{Y}}(\underline{\mathrm{Y}}(\underline{\mathrm{X}}\underline{w}\underline{\mathrm{Y}})(\underline{\mathrm{Z}}\underline{w}\underline{\mathrm{Y}}))\big)\end{gathered} \rightsquigarrow \tag{16.39}$$

$$\begin{gathered}\tilde{\exists}\underline{\mathrm{X}}', \underline{\mathrm{Z}}'\,\exists\underline{\mathrm{X}}, \underline{\mathrm{Z}} \leq^{\mathrm{e}} \underline{\mathrm{X}}', \underline{\mathrm{Z}}'\,\forall\underline{\ell}, \underline{w}, \underline{\mathrm{Y}} \\ \neg\neg A_{\mathrm{qf}}\big(\underline{w}, \underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}, \underline{\mathrm{Y}}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{\mathrm{Y}})(\underline{\mathrm{Z}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}), \underline{\mathrm{Z}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}(\underline{\mathrm{Y}}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{\mathrm{Y}})(\underline{\mathrm{Z}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}))\big)\end{gathered} \to \tag{16.40}$$

$$\begin{gathered}\tilde{\exists}\underline{\mathrm{X}}', \underline{\mathrm{Z}}'\,\exists\underline{\mathrm{X}}, \underline{\mathrm{Z}} \leq^{\mathrm{e}} \underline{\mathrm{X}}', \underline{\mathrm{Z}}'\,\forall\underline{\ell}, \underline{w}, \underline{\mathrm{Y}} \\ A_{\mathrm{qf}}\big(\underline{w}, \underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}, \underline{\mathrm{Y}}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{\mathrm{Y}})(\underline{\mathrm{Z}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}), \underline{\mathrm{Z}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}(\underline{\mathrm{Y}}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{\mathrm{Y}})(\underline{\mathrm{Z}}\underline{\ell}\underline{w}\underline{\mathrm{Y}}))\big)\end{gathered} \to \tag{16.41}$$

$$\begin{gathered}\tilde{\exists}\underline{\mathrm{X}}', \underline{\mathrm{Z}}'\,\exists\underline{\mathrm{X}}, \underline{\mathrm{Z}} \leq^{\mathrm{e}} \underline{\mathrm{X}}', \underline{\mathrm{Z}}'\,\forall\underline{\ell}, \underline{w}, \underline{Y} \\ A_{\mathrm{qf}}\big(\underline{w}, \underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}, \underline{Y}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}), \\ \underline{\mathrm{Z}}\underline{\ell}\underline{w}(\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x})(\underline{Y}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}))\big)\end{gathered} \to \tag{16.42}$$

$$\begin{gathered}\tilde{\exists}\underline{\mathrm{X}}'\,\exists\underline{\mathrm{X}} \leq^{\mathrm{e}} \underline{\mathrm{X}}'\,\forall\underline{\ell}, \underline{w}, \underline{Y}\,\exists\underline{z} \\ A_{\mathrm{qf}}(\underline{w}, \underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}, \underline{Y}(\underline{\mathrm{X}}\underline{\ell}\underline{w}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}\underline{x}), \underline{z})\end{gathered} \to \tag{16.43}$$

$$\tilde{\exists}\underline{\mathrm{X}}'\,\forall\underline{\ell}, \underline{w}, \underline{Y}\,\exists\underline{x} \leq^{\mathrm{e}} \underline{\mathrm{X}}'\underline{\ell}^{\mathrm{M}}\underline{w}^{\mathrm{M}}\lambda\underline{x}, \underline{Z} \,.\, \underline{Y}^{\mathrm{M}}\underline{x}\,\exists\underline{z}\, A_{\mathrm{qf}}(\underline{w}, \underline{x}, \underline{Y}\underline{x}, \underline{z}) \to \tag{16.44}$$

$$\tilde{\exists}\underline{\mathrm{X}}\,\forall\underline{\ell}, \underline{w}, \underline{Y}\,\exists\underline{x} \leq^{\mathrm{e}} \underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{Y}\,\exists\underline{z}\, A_{\mathrm{qf}}(\underline{w}, \underline{x}, \underline{Y}\underline{x}, \underline{z}) \to \tag{16.45}$$

$$\tilde{\exists}\underline{\mathrm{X}}\,\forall\underline{\ell}, \underline{w}, \underline{Y}\,\exists\underline{x} \leq^{\mathrm{e}} \underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{Y}\, A(\underline{w}, \underline{x}, \underline{Y}\underline{x}).$$

2. The terms $\underline{t}_{\underline{a}}$ contain witnesses for $\underline{\mathrm{X}}'$, and the previous point describes how to construct $\underline{\mathrm{X}}$ from $\underline{\mathrm{X}}'$.

3. If, for example, $\underline{y}$ is intended to represent real numbers in $[0, 1]$, then we can assume $\underline{y} \leq^{\mathrm{e}} M$, so $\underline{Y} \leq^{\mathrm{e}} \lambda\underline{x} \,.\, M$, therefore we can replace the bound $\underline{\mathrm{X}}\underline{\ell}\underline{w}\underline{Y}$ by the greater than or equal to bound $\underline{\mathrm{X}}\underline{\ell}\underline{w}(\lambda\underline{x} \,.\, M)$, which is independent of $\underline{Y}$.

16.23. Now let us return to the question of computing the interpretation of A, B and C by MD $\circ$ GG. In rigour, we are not going to compute the exact complicated interpretations by MD $\circ$ GG, but the simpler formulas given by proposition 16.21.

ϕ is continuous We rewrite this statement, that is (16.46) below, as (16.47), where now we prefer to have the inequality $\leq_{\mathbb{R}}$ in the premise so that the interpretation of $(|\tilde{x} - \tilde{y}| \leq 2^{-b} \to |\phi\tilde{x} - \phi\tilde{y}| < 2^{-a})$ by MD $\circ$ GG is an existential formula as necessary to use the proposition 16.21. By points 1 and 3 of proposition 16.21 we get (16.48). In equivalence (16.48) we use that quantifying over

y or Y makes no difference, and that we can replace $\leq$ by $<$ and vice-versa (adjusting b and B as necessary).

$$\begin{array}{c} \forall\varepsilon > 0\,\forall x \in [0,1]\,\exists\delta > 0\,\forall y \in [0,1] \\ (|x-y| < \delta \to |\phi x - \phi y| \leq \varepsilon) \end{array} \leftrightarrow \tag{16.46}$$

$$\forall a\,\forall x\,\exists b\,\forall y\,(|\tilde{x}-\tilde{y}| \leq 2^{-b} \to |\phi\tilde{x} - \phi\tilde{y}| < 2^{-a}) \rightsquigarrow \tag{16.47}$$

$$\tilde{\exists} B\,\forall a, x, Y\,\exists b \leq Ba\,(|\tilde{x}-\tilde{y}| \leq 2^{-b} \to |\phi\tilde{x} - \phi\widetilde{Yb}| < 2^{-a}) \leftrightarrow \tag{16.48}$$

$$\tilde{\exists} B\,\forall a\,\forall x, y \in [0,1]\,\exists b \leq Ba\,(|x-y| < 2^{-b} \to |\phi x - \phi y| < 2^{-a}) \leftrightarrow \tag{16.49}$$

$$\tilde{\exists} B\,\forall a\,\forall x, y \in [0,1]\,(|x-y| < 2^{-Ba} \to |\phi x - \phi y| < 2^{-a}). \tag{16.50}$$

Formula (16.50) says that B is a monotone rate of uniform continuity of ϕ.

$\underline{v_{k+1} - v_k \to 0}$ We rewrite this statement, that is (16.51) below, as (16.52). By point 1 of proposition 16.21 we get (16.53).

$$\forall\varepsilon > 0\,\exists d \in \mathbb{N}\,\forall e \geq d\,(|v_{e+1} - v_e| < \varepsilon) \leftrightarrow \tag{16.51}$$

$$\forall c\,\exists d\,\forall e\,(|v_{d+e+1} - v_{d+e} < 2^{-c}) \rightsquigarrow \tag{16.52}$$

$$\tilde{\exists} D\,\forall c, E\,\exists d \leq^{\mathrm{e}} DcE\,(|v_{d+Ed+1} - v_{d+Ed} < 2^{-c}). \tag{16.53}$$

Formula (16.53) says that D is a monotone rate of metastability of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$ (with limit 0).

$\underline{(v_k)_{k\in\mathbb{N}} \text{ is a Cauchy sequence}}$ We rewrite this statement, that is (16.54) below, as (16.56). By points 1 and 3 of proposition 16.21 we get (16.56).

$$\forall\varepsilon > 0\,\exists g \in \mathbb{N}\,\forall i, j \geq g\,(|v_i - v_j| < \varepsilon) \leftrightarrow \tag{16.54}$$

$$\forall f\,\exists g\,\forall h\,\forall i, j \in [g; g+h]\,(|v_i - v_j| < 2^{-f}) \rightsquigarrow \tag{16.55}$$

$$\begin{array}{c} \tilde{\exists} G\,\forall f, H, I, J\,\exists g \leq GfH \\ (Ig, Jg \in [g; g+Hg] \to |v_{Ig} - v_{Jg}| < 2^{-f}) \end{array} \leftrightarrow \tag{16.56}$$

$$\tilde{\exists} G\,\forall f, H\,\exists g \leq GfH\,\forall i, j \in [g; g+Hg]\,(|v_i - v_j| < 2^{-f}). \tag{16.57}$$

Formula (16.57) says that G is a monotone rate of metastability of $(v_k)_{k\in\mathbb{N}}$ (with mentioning a limit).

16.24. Let us put together the picture that developed in this section. To keep the picture simple, we leave rigour aside for a moment.

1. Theorem 16.14 is of the form $A \wedge B \to C$ where

$$\begin{aligned} A &:\equiv \text{“}\phi \text{ is continuous”}, \\ B &:\equiv \text{“}v_{k+1} - v_k \to 0\text{”}, \\ C &:\equiv \text{“}(v_k)_{k\in\mathbb{N}} \text{ is a Cauchy sequence”}. \end{aligned}$$

2. The soundness theorem of MD (composed with GG) predicts that we have $(C^{\mathrm{GG}})^{\mathrm{MD}}$ as a function f of $(A^{\mathrm{GG}})^{\mathrm{MD}}$ and $(B^{\mathrm{GG}})^{\mathrm{MD}}$:

$$(C^{\mathrm{GG}})^{\mathrm{MD}} = f\big((A^{\mathrm{GG}})^{\mathrm{MD}}, (B^{\mathrm{GG}})^{\mathrm{MD}}\big).$$

3. We computed:

$$\begin{aligned}(A^{\mathrm{GG}})^{\mathrm{MD}} &\equiv \text{“rate of uniform continuity of } \phi\text{”},\\ (B^{\mathrm{GG}})^{\mathrm{MD}} &\equiv \text{“rate of metastability of } (v_{k+1}-v_k)_{k\in\mathbb{N}}\text{”},\\ (C^{\mathrm{GG}})^{\mathrm{MD}} &\equiv \text{“rate of metastability of } (v_k)_{k\in\mathbb{N}}\text{”}.\end{aligned}$$

4. Putting all together, MD $\circ$ GG predicts

$$\begin{matrix}\text{rate of metastability}\\ \text{of } (v_k)_{k\in\mathbb{N}}\end{matrix} = f\left(\begin{matrix}\text{rate of uniform}\\ \text{continuity of } \phi\end{matrix}, \begin{matrix}\text{rate of metastability}\\ \text{of } (v_{k+1}-v_k)_{k\in\mathbb{N}}\end{matrix}\right).$$

Now our task is to find f.

16.4 Partial proof mining

16.25. In the next theorem we present a partially proof mined version of theorem 16.5. The reason why it is only partially proof mined is because we use the hypothesis “β is a rate of convergence of $(v_{k+1}-v_k)_{k\in\mathbb{N}}$ (with limit 0)” instead of “β is a rate of metastability of $(v_{k+1}-v_k)_{k\in\mathbb{N}}$ (with limit 0)”. The former one is stronger, and the latter one is the what gives us the interpretation of $v_{k+1}-v_k \to 0$ by MD $\circ$ GG.

16.26. At some point in proof 16.28 we have a number $l \in \mathbb{N}$, a function $L\colon \mathbb{N} \to \mathbb{N}$ and a colouring $f\colon \mathbb{N} \to n$, and want to get two points p and q such that p and q have the same colour i (that is $f(p) = f(q) = i$), p occurs after l (that is $l \leq p$) and q occurs after $L(p)$ (that is $L(p) \leq q$).

1. One way of doing this is to use IPP to get a colour i that occurs infinitely often. Then we take p to be some occurrence of i after l, and q to be some occurrence of i after $L(p)$.

2. Another way of doing this is to consider a strictly monotone bound L' on L, in the sense of $\forall n \in \mathbb{N}\,(L'(n) < L'(n+1))$ and $\forall n \in \mathbb{N}\,(L(n) \leq L'(n))$. For example, we can define L' by $L'(0) := L(0)$ and $L'(n+1) := \max(L'(n)+1, L(n+1))$. Then we consider the finite sequence $L'^0(l) < \cdots < L'^n(l)$ composed of $n+1$ distinct terms, where $L'^0(l) := l$, $L'^1(l) := L'(l)$, $L'^2(l) := L'(L'(l))$, and so on. By the (finite) pigeonhole principle there exists a colour $i \in n$ and indices $u, v \leq n$, with $u < v$, such that $L'^u(l)$ and $L'^v(l)$ have colour i. Then we take $p := L'^u(k)$ and $q := L'^v(l)$.

In the next theorem, β will be a rate of convergence and we will be using the latter way of getting p and q with $l = \beta(c)$ and $L = \beta \circ C$ (for the moment let us not mind about what c and C are).

16.27 Theorem. Consider a function $\phi\colon [0,1] \to [0,1]$, take an arbitrary $v_0 \in [0,1]$ and define the sequence $(v_k)_{k\in\mathbb{N}}$ by $v_{k+1} := \phi(v_k)$. If

1. the function ϕ is uniformly continuous and $\alpha\colon \mathbb{N} \to \mathbb{N}$ is a modulus of uniform continuity of ϕ, that is

$$\forall a \in \mathbb{N}\, \forall x, y \in [0,1]\, (|x-y| < 2^{-\alpha(a)} \to |\phi(x) - \phi(y)| < 2^{-a}); \tag{16.58}$$

2. we have $v_{k+1} - v_k \to 0$ and $\beta\colon \mathbb{N} \to \mathbb{N}$ is a rate of convergence of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$ (with limit 0), that is

$$\forall c, e \in \mathbb{N}\, (|v_{\beta(c)+e+1} - v_{\beta(c)+e}| < 2^{-c}); \tag{16.59}$$

then $\Phi(\alpha, \beta, \cdot, \cdot) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a rate of metastability of $(v_k)_{k\in\mathbb{N}}$ (without mentioning a limit), that is

$$\forall f \in \mathbb{N}\, \forall H\colon \mathbb{N} \to \mathbb{N}\, \exists g \leq \Phi(\alpha, \beta, f, H)\, \forall i, j \in [g; g+H(g)]\, (|v_i - v_j| < 2^{-f}), \tag{16.60}$$

where we defined

1. $n := 3 \times 2^{f+1}$;
2. $c := 1 + \lceil \log_2 n \rceil$;
3. $A\colon \mathbb{N} \to \mathbb{N}$ by $A(k) := \lceil \log_2 \max(6nH(k), 1) \rceil$;
4. $C\colon \mathbb{N} \to \mathbb{N}$ by $C(k) := \max\big(A(k), \alpha(A(k))\big)$;
5. $(u_k)_{k\in\mathbb{N}}$ by $u_0 := \beta(c)$ and $u_{k+1} := \max\big(u_k + 1, \beta(C(u_k))\big)$;
6. $\Phi \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by $\Phi(\alpha, \beta, f, H) := u_{n^2}$.

16.28 Proof. First, let us note $\forall k \in \mathbb{N}\, (u_k < u_{k+1})$, $\forall k \in \mathbb{N}\, (\beta(c) \leq u_k)$, and $\forall k \in \mathbb{N}\, \big(\beta(C(u_k)) \leq u_{k+1}\big)$.

By contradiction, we assume the negation of (16.60), that is there exist $f \in \mathbb{N}$ and $H\colon \in \mathbb{N} \to \mathbb{N}$ such that $\forall g \leq \Phi(\alpha, \beta, f, H)\, \exists i, j \in [g; g+H(g)]\, (|v_i - v_j| \geq 2^{-f})$, thus $\forall g \leq \Phi(\alpha, \beta, f, H)\, \exists h' \in\,]g; g+H(g)]\, (|v_g - v_{h'}| \geq 2^{-(f+1)})$. So define $H'\colon \mathbb{N} \to \mathbb{N}$ by

$$H'(g) := \begin{cases} \mu h' \in\,]g; g+H(g)]\, .\, |v_g - v_{h'}| \geq 2^{-(f+1)} & \text{if } g \leq \Phi(\alpha, \beta, f, H) \\ 0 & \text{otherwise} \end{cases}.$$

We have $[0,1] = \bigcup_{i=0}^{n-1} I_i$ where $I_i := \big[\frac{i}{n}, \frac{i+1}{n}\big]$. So define $F \in \mathbb{N} \to n$ by $F(k) := \mu m < n\, .\, v_k \in I_m$. Define the colouring $F'\colon \mathbb{N} \to n^2$ with n^2 colours by $F'(k) := \langle F(k), F(H'(k))\rangle$, where $\langle\cdot,\cdot\rangle$ is the pairing of point 1 of lemma 16.10.

By the (finite) pigeonhole principle (applied to the list of n^2+1 distinct numbers $u_0, \ldots, u_{n^2}$ coloured by F' with n^2 colours) there exist $i < n^2$ and $j_0, j_1 \leq n^2$, with $j_0 < j_1$, such that $F'(u_{j_0}) = F'(u_{j_1}) = i$. Say $i = \langle i_0, i_1\rangle$ where $i_0, i_1 < n$. Then $F'(u_{j_0}) = F'(u_{j_1}) = i$ is equivalent to $F(u_{j_0}) = F(u_{j_1}) = i_0 \wedge F(H'(u_{j_0})) = F(H'(u_{j_1})) = i_1$, which implies $v_{u_{j_0}}, v_{u_{j_1}} \in I_{i_0} \wedge v_{H'(u_{j_0})}, v_{H'(u_{j_1})} \in I_{i_1}$. Moreover, $u_{j_0} < H'(u_{j_0}) \leq u_{j_0} + H(u_{j_0})$ and $u_{j_1} < H'(u_{j_1}) \leq u_{j_1} + H(u_{j_1})$ (by definition of H', since $u_{j_0}, u_{j_1} \leq u_{n^2} = \Phi(\alpha, \beta, f, H)$) and $|i_0 - i_1| \geq 2$ (since $|v_{u_{j_0}} - v_{H'(u_{j_0})}| \geq 2^{-(f+1)}$ by definition of H'). In particular, $H(u_{j_0}) > 0$, so below we can write $\frac{1}{2nH(u_{j_0})}$. This is pictured in figure 16.8.

Consider $I_{i_0\pm 1}$, where we choose the plus sign if $i_0 + 1 < i_1$, and the minus sign if $i_1 + 1 < i_0$, so that $I_{i_0\pm 1}$ is between I_{i_0} and I_{i_1}. Now we consider two cases.

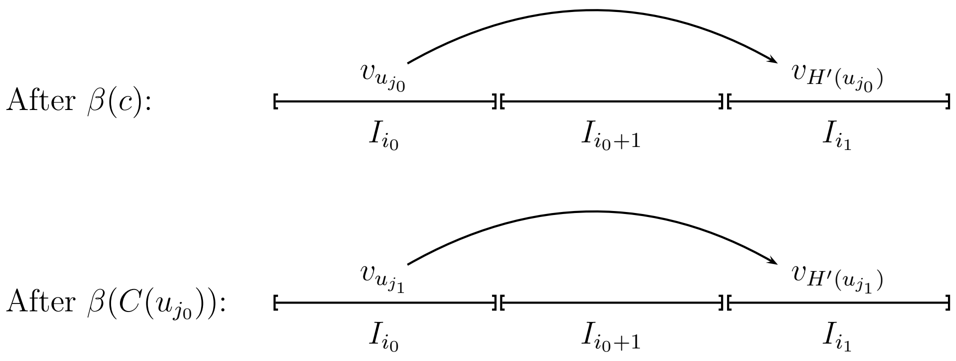

Figure 16.8: the sequence $(v_k)_{k\in\mathbb{N}}$ jumping from I_{i_0} into I_{i_1}.

<u>$\forall x \in I_{i_0\pm1}\left(|\phi(x)-x| < \frac{1}{2nH(u_{j_0})}\right)$</u> By (16.59) we have $\forall e \in \mathbb{N}\left(|v_{\beta(c)+e+1} - v_{\beta(c)+e}| < \frac{1}{2n}\right)$. So $(v_k)_{k\in\mathbb{N}}$ goes from $v_{u_{j_0}} \in I_{i_0}$ to $v_{H'(u_{j_0})} \in I_{i_1}$ in steps strictly smaller than $\frac{1}{2n}$, that is half of the length of $I_{i_0\pm1}$. Thus v_k enters the half of $I_{i_0\pm1}$ closest to I_{i_0}, for some $k \in [u_{j_0}; H'(u_{j_0})]$. Then, to reach I_{i_1}, the sequence $(v_k)_{k\in\mathbb{N}}$

1. has to cover the other half of $I_{i_0\pm1}$;
2. in at most $H'(u_{j_0}) - u_{j_0} \leq H(u_{j_0})$ steps;
3. and inside $I_{i_0\pm1}$ (that is for all $k \in \mathbb{N}$ such that $v_k \in I_{i_0\pm1}$) each step has length $|v_{k+1} - v_k| = |\phi(v_k) - v_k| \leq \frac{1}{2nH(u_{j_0})}$.

But this is impossible because in at most $H(u_{j_0})$ steps strictly smaller than $\frac{1}{2nH(u_{j_0})}$ the sequence $(v_k)_{k\in\mathbb{N}}$ covers a distance strictly smaller than $\frac{1}{2n}$, that is strictly smaller than the length of half of $I_{i_0\pm1}$.

<u>$\exists x \in I_{i_0\pm1}\left(|\phi(x)-x| \geq \frac{1}{2nH(u_{j_0})}\right)$</u> By (16.58) and (16.59) we have

$$\forall y \in [0,1]\left(|x-y| < 2^{-\alpha(A(u_{j_0}))} \to |\phi(x)-\phi(y)| < \frac{1}{6nH(u_{j_0})}\right),$$

$$\forall e \in \mathbb{N}\left(|v_{\beta(C(u_{j_0}))+e+1} - v_{\beta(C(u_{j_0}))+e}| < 2^{-C(u_{j_0})} \leq \frac{1}{6nH(u_{j_0})}, 2^{-\alpha(A(u_{j_0}))}\right). \tag{16.61}$$

Let $J :=]x - 2^{-C(u_{j_0})}, x + 2^{-C(u_{j_0})}[$. By point 2 of lemma 16.10 we have

$$\forall y \in J\left(|\phi(y)-y| \geq \underbrace{|\phi(x)-x|}_{\geq\frac{1}{2nH(u_{j_0})}} - \underbrace{|\phi(x)-\phi(y)|}_{<\frac{1}{6nH(u_{j_0})}} - \underbrace{|y-x|}_{<\frac{1}{6nH(u_{j_0})}} > \frac{1}{6nH(u_{j_0})}\right). \tag{16.62}$$

Since $2^{-C(u_{j_0})} \leq \mathrm{lh}\, I_{i_0\pm1}$ (because $2^{-C(u_{j_0})} \leq \frac{1}{6nH(u_{j_0})}$ and $\mathrm{lh}\, I_{i_0\pm1} = 1/n$), then by point 3 of lemma 16.10 we have $2^{-C(u_{j_0})} \leq \mathrm{lh}(I_{i_0\pm1} \cap J)$. Since $j_0 < j_1$, then $\beta(C(u_{j_0})) \leq u_{j_1}$. So, by (16.61), as $(v_k)_{k\in\mathbb{N}}$ goes from $v_{u_{j_1}} \in I_{i_0}$ to $v_{H'(u_{j_1})} \in I_{i_1}$ it enters J for some $k \in [u_{j_1}; H'(u_{j_1})]$. But then $|v_{k+1} - v_k| = |\phi(v_k) - v_k| > \frac{1}{6nH(u_{j_0})}$ by (16.62), contradicting (16.61).

16.29. The bound Φ presented in the previous theorem has low complexity: it is primitive recursive on α, β and H (the use of the full β is essential because β is iterated a variable number of times).

The bound Φ is independent of v_0 and ϕ. Let us explain this.

$\underline{v_0}$ The independence from $v_0 \in [0,1]$ can be explained because $(*)$ real numbers in $[0,1]$ can be represented by $\tilde{r}^1$ such that $\tilde{r} \leq^{\mathrm{e}} M$ (see definition 16.9).

$\underline{\phi}$ The independence from $\phi\colon [0,1] \to [0,1]$ can be explained in the following way: given a modulus of uniform continuity α of ϕ, we can restrict ourselves to $\phi\colon [0,1] \cap \mathbb{Q} \to [0,1]$, and consider $\phi\colon \mathbb{N} \to [0,1]$ (by identifying $\mathbb{Q} \cap [0,1]$ with a enumeration of it), and so $\phi \leq^{\mathrm{e}} \lambda x^0 . M$ by $(*)$ [52].

16.5 Full proof mining

16.30. In theorem 16.27 we avoided dealing with IPP by replacing it by the (finite) pigeonhole principle. The price to pay is that we need the stronger hypothesis "β is a rate of convergence of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$ (with limit 0)" instead of the weaker "β is a rate of metastability of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$ (with limit 0)". The theorem is not fully proof mined because MD $\circ$ GG gives us the weaker hypothesis, not the stronger hypothesis.

So, to "officially" follow MD $\circ$ GG, in the next theorem we use only the weaker hypothesis, getting a fully proof mined theorem. But now the price to pay is that we have to deal with IPP. This will take the form of a term γ witnessing $(\mathsf{IPP}^{\mathrm{GG}})^{\mathrm{MD}}$ and constructing a bound that uses γ. In the full proof mining we never look into what exactly γ is, but rather treat it as an "oracle", because it is difficult to write down γ. However, in the next remark we sketch a description of γ.

16.31 Remark. In order to appreciate the complexity of the term γ witnessing $(\mathsf{IPP}^{\mathrm{GG}})^{\mathrm{MD}}$, let us sketch it. In (16.63) below we write IPP. In equivalence (16.63) we replace $\forall f \colon \mathbb{N} \to n\, A(f)$ by its official meaning $\forall f^1\, A(f_n)$ where $f_n(m) := \min(f(m), k)$ (to be really formal we should write $\forall f^1\, A(tnf)$ where t^{010} is a term such that $\mathsf{WE\text{-}HA}^\omega \vdash tnfm =_0 \min(f(m), n)$). In step (16.64) we compute the translation by GG. In step (16.65) we compute the interpretation by D. Finally, in equivalence (16.66) we use $\mathsf{QF\text{-}AC}$ to get the more readable (16.67).

$$\forall n^0\, \forall f \colon \mathbb{N} \to n\, \exists i \leq_0 n\, \forall k^0\, \exists m \geq_0 k\, (fm =_0 i) \leftrightarrow \tag{16.63}$$

$$\forall n, f^1\, \exists i\, \big(i \leq n \wedge \forall k\, \exists m\, (m \geq k \wedge \min(fm, n) = i)\big) \rightsquigarrow \tag{16.64}$$

$$\forall n, f^1\, \neg \forall i\, \neg \big(i \leq n \wedge \forall k\, \neg \forall m\, \neg (m \geq k \wedge \min(fm, n) = i)\big) \rightsquigarrow \tag{16.65}$$

$$\begin{gathered} \exists \mathrm{I}, \mathrm{M}\, \forall n, f, \mathrm{K}\, \Big(\underbrace{\mathrm{I}nf\mathrm{K}}_{\equiv:\alpha} \leq n \wedge \\ \underbrace{\mathrm{M}nf\mathrm{K}(\overbrace{\mathrm{K}\alpha(\mathrm{M}nf\mathrm{K})}^{\beta:\equiv})}_{\equiv:\delta} \geq \beta \wedge \min(f\delta, n) = \alpha\Big) \end{gathered} \leftrightarrow \tag{16.66}$$

$$\forall n\, \forall f \colon \mathbb{N} \to n\, \forall \mathrm{K}\, \exists i, M\, \big(i \leq n \wedge M(\mathrm{K}iM) \geq \mathrm{K}iM \wedge f(M(\mathrm{K}iM)) = i\big). \tag{16.67}$$

Let us fix some bijective coding $\langle \cdot \rangle \colon \bigcup_{n\in\mathbb{N}} \mathbb{N}^n \to \mathbb{N}$ of tuples of natural numbers, and let us denote by $(s)_i$ the i-th component of the tuple coded by $s \in \mathbb{N}$, denote by $\operatorname{lh} s$ the length of that tuple, and denote by $^\frown$ the concatenation of tuples. Using mainly the recursor R_1, we can define a term B, called *finite bar recursion*, by

$$BG^{010}g^1n^0s^0 :=_0 \begin{cases} \langle\rangle & \text{if } \operatorname{lh} s > n \\ Xs^\frown BGgn(s^\frown Xs) & \text{if } \operatorname{lh} s \leq n \end{cases},$$

$$g_s := \lambda x \,.\, f(s^\frown\langle x\rangle^\frown BGgn(s^\frown\langle x\rangle)), \qquad Xs := G(\operatorname{lh} s)g_s.$$

Informally (and dropping G, g and n in $BFfns$ to keep the notation simple), B defines a backward recursion: we start with the value $B\langle s_0,\ldots,s_n\rangle = \langle\rangle$, then compute $B\langle s_0,\ldots,s_{n-1}\rangle = \langle x_n\rangle$ having access to the function $x \mapsto B\langle s_0,\ldots,s_{n-1},x\rangle$ (that is to all values $B\langle s_0,\ldots,s_{n-1},x\rangle$ with x running through $\mathbb{N}$), and then compute $B\langle s_0,\ldots,s_{n-2}\rangle = \langle x_{n-1},x_n\rangle$ having access to $x \mapsto B\langle s_0,\ldots,s_{n-2},x\rangle$, and so on, until we achieve a final result $B\langle\rangle = \langle x_0,\ldots,x_n\rangle$.

Taking $G = \mathrm{K}$ and g defined by $gs = \max((s)_0,\ldots,(s)_{\operatorname{lh} s-1})$ in $BGgns$, we get $\langle x_0,\ldots,x_n\rangle := BGgn\langle\rangle$. Then we define $M_k := g_{\langle x_0,\ldots,x_k\rangle}$ for $k = 0,\ldots,n$, and $i := f(M_0(\mathrm{K}0M_0)) \leq n$ (these i and M depend on n, f and K). We can prove that i and $M := M_i$ witness (16.67). So $\mathrm{M} := \lambda n, f, \mathrm{K} \,.\, M$ and $\mathrm{I} := \lambda n, f, \mathrm{K} \,.\, i$ witness the (16.66) [50, pages 213–214] [61].

Finally, we take γ as being a term majorising M. Officially, we should also give term majorising I, but this is trivial since $i \leq n$: take $\lambda n, f, \mathrm{K} \,.\, n$.

Let us remark that we can assume that γ does not take f as an input. Since $f_n \leq^{\mathrm{e}} n^1$ where $n^1 := \lambda k^0 \,.\, n$, then $\gamma \leq^{\mathrm{e}} \gamma'$ where $\gamma' := \lambda n, \mathrm{K} \,.\, (\gamma n n^1 \mathrm{K})$, so we can replace the bound γ by γ' that does not take f as an input.

16.32. Analogously to paragraph 16.26, at some point in proof 16.34 we have a number $l \in \mathbb{N}$, a function $L\colon \mathbb{N} \to \mathbb{N}$ and a colouring $f\colon \mathbb{N} \to n$, and want to get two points p and q such that p and q have the same colour i (that is $f(p) = f(q) = i$), p occurs after l (that is $l \leq p$) and q occurs after $L(p)$ (that is $L(p) \leq q$). But, at first sight, (16.67) seems to only gives us p: taking $\mathrm{K}iM := l$ we get i and M such that $p := M(\mathrm{K}iM) \geq \mathrm{K}iM = l$ and $fp = i$. So the problem is to choose a K so good that (16.67) gives us both p and q. The solution [56] is to take

$$\mathrm{K}iM = \begin{cases} l & \text{if } Ml < l \vee f(Ml) \neq i \\ L(Ml) & \text{otherwise} \end{cases}.$$

Indeed, for this K the formula (16.67) gives us i and M such that $(*_1)$ $M(\mathrm{K}iM) \geq \mathrm{K}iM$ and $(*_2)$ $f(M(\mathrm{K}iM)) = i$, and then we define the following p and q.

$\underline{p}$ Let $p := Ml$. Let us argue $p \geq l$ and $fp = i$.

$\underline{p \geq l}$ If $p < l$, then $\mathrm{K}iM = l$, thus $(*_1)$ means $p \geq l$, and we get a contradiction.

$\underline{fp = i}$ If $fp \neq i$, then $\mathrm{K}iM = l$, thus $(*_2)$ means $fp = i$, and we get a contradiction.

$\underline{q}$ Let $q := M(L(Ml))$. Let us argue $q \geq Lp$ and $fq = i$. Since we already proved $p \geq l \wedge fp = i$, that is $Ml \geq l \wedge f(Ml) = i$, we have $\mathrm{K}iM = L(Ml) = Lp$.

$\underline{q \geq Lp}$ The formula $(*_1)$ means $q \geq Lp$.

$\underline{fq = i}$ The formula $(*_2)$ means $fq = i$.

The definition of K is reminiscent of the way that D interprets the contraction axiom $A \to A \wedge A$: its interpretation (essentially) asks for terms $\underline{t}$ such that

$$A_{\mathrm{D}}(\underline{a}; \underline{t}) \to A_{\mathrm{D}}(\underline{a}; \underline{d}) \wedge A_{\mathrm{D}}(\underline{a}; \underline{f})$$

like

$$\underline{t} := \begin{cases} \underline{f} & \text{if} \quad A_{\mathrm{D}}(\underline{a}; \underline{d}) \\ \underline{d} & \text{if} \ \neg A_{\mathrm{D}}(\underline{a}; \underline{d}) \end{cases}$$

(the exact details are given in proof 5.8).

16.33 Theorem. Consider a function $\phi\colon [0,1] \to [0,1]$, take an arbitrary $v_0 \in [0,1]$ and define the sequence $(v_k)_{k\in\mathbb{N}}$ by $v_{k+1} := \phi(v_k)$. If

1. the function ϕ is continuous and $\alpha\colon \mathbb{N} \to \mathbb{N}$ is a modulus of uniform continuity of ϕ, that is

$$\forall a \in \mathbb{N}\, \forall x, y \in [0,1]\, (|x - y| < 2^{-\alpha(a)} \to |\phi(x) - \phi(y)| < 2^{-a}); \tag{16.68}$$

2. we have $v_{k+1} - v_k \to 0$ and $\beta\colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a monotone rate of metastability of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$, so $\beta \leq^{\mathrm{e}} \beta$ and

$$\begin{gathered} \forall c \in \mathbb{N}\, \forall E'\colon \mathbb{N} \to \mathbb{N}\, \forall E \leq^{\mathrm{e}} E'\, \exists d \leq \beta(c, E') \\ (|v_{d+E(d)+1} - v_{d+E(d)}| < 2^{-c}); \end{gathered} \tag{16.69}$$

3. the functional $\gamma\colon \mathbb{N} \times \mathbb{N}^{(\mathbb{N}\times\mathbb{N}^{\mathbb{N}})} \to \mathbb{N}^{\mathbb{N}}$ witnesses $(\mathsf{IPP}^{\mathrm{GG}})^{\mathrm{MD}}$, so $\gamma \leq^{\mathrm{e}} \gamma$ and

$$\begin{gathered} \forall n \in \mathbb{N}\, \forall F'\colon \mathbb{N} \to n\, \forall K'\colon n \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall K \leq^{\mathrm{e}} K'\, \exists i < n \\ \exists L \leq^{\mathrm{e}} \gamma(n, K')\, \big(L(K(i, L)) \geq K(i, L) \wedge F'\big(L(K(i, L))\big) = i\big); \end{gathered} \tag{16.70}$$

then $\Psi(\alpha, \beta, \gamma, \cdot, \cdot)\colon \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a rate of metastability of $(v_n)_{k\in\mathbb{N}}$, that is

$$\begin{gathered} \forall f \in \mathbb{N}\, \forall H\colon \mathbb{N} \to \mathbb{N}\, \exists g \leq \Psi(\alpha, \beta, \gamma, f, H) \\ \forall i, j \in [g; g + H(g)]\, (|v_i - v_j| < 2^{-f}), \end{gathered} \tag{16.71}$$

where we defined

1. $n := 3 \times 2^{f+1}$;
2. $c := 1 + \lceil \log_2 n \rceil$;
3. $E'\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ by $E'(L, d) := (H + \mathrm{id})^{\mathrm{M}}\big(\max(L^{\mathrm{M}}(d), d)\big)$;
4. $D'\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ by $D'(L, c') := \beta(c', E'(L^{\mathrm{M}}, \cdot))$;
5. $A\colon \mathbb{N} \to \mathbb{N}$ by $A(k) := \lceil \log_2 \max(6nH(k), 1) \rceil$;

6. $C\colon \mathbb{N}\to\mathbb{N}$ by $C(k) := \max\big(A(k), \alpha(A(k))\big)$;
7. $K'\colon \mathbb{N}\times\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by $K'(i, L) := \max\Big(D'(L^{\mathrm{M}}, c), D'\big(L^{\mathrm{M}}, C^{\mathrm{M}}\big(L^{\mathrm{M}}(D'(L^{\mathrm{M}}, c))\big)\big)\Big)$;
8. $\Psi\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} \times (\mathbb{N}^{\mathbb{N}})^{(\mathbb{N}\times\mathbb{N}^{(\mathbb{N}\times\mathbb{N}^{\mathbb{N}})})} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by $\Psi(\alpha,\beta,\gamma,f,H) := \gamma(n^2, K')\big(K'(0, \gamma(n^2, K'))\big)$.

16.34 Proof.

1. Let us define
   (a) $I_{Ld} := \big[\max(L(d), d); (H+\mathrm{id})\big(\max(L(d), d)\big)\big[$;
   (b) $E\colon \mathbb{N}^{\mathbb{N}}\times\mathbb{N}\to\mathbb{N}$ by $E(L, d) := \mu k\,.\, d+k \in I_{Ld} \wedge \forall m \in I_{Ld}\,(|v_{m+1} - v_m| \leq |v_{d+k+1} - v_{d+k}|)$;
   (c) $D\colon \mathbb{N}^{\mathbb{N}}\times\mathbb{N}\to\mathbb{N}$ by $D(L, c') := \mu d \leq \beta(c', E'(L^{\mathrm{M}}, \cdot))\,.\,\forall m \in I_{Ld}\,(|v_{m+1} - v_m| < 2^{-c'})$;
   (d) $K_{F'}\colon \mathbb{N}\times\mathbb{N}^{\mathbb{N}}\to\mathbb{N}$ (depending on an $F'\colon \mathbb{N}\to n^2$) by
   $$K_{F'}(i, L) := \begin{cases} D(L, c) & \text{if } L(D(L, c)) < D(L, c) \vee \\ & \quad F'\big(L(D(L, c))\big) \neq i \\ D\big(L, C\big(L(D(L, c))\big)\big) & \text{otherwise}\end{cases}.$$

   Now let us remark that A, D and E are well-defined.

   $\underline{A}$ In A, the logarithm is applied to $\max(6nH(k), 1) \geq 1$.

   $\underline{D}$ Later we are going to prove $E \leq^{\mathrm{e}} E'$, so $E(L,\cdot) \leq^{\mathrm{e}} E'(L^{\mathrm{M}},\cdot)$. By (16.69) we have $\forall c' \in \mathbb{N}\,\forall L\colon \mathbb{N}\to\mathbb{N}\,\exists d \leq \beta(c', E'(L^{\mathrm{M}},\cdot))\,\forall m \in I_{Ld}\,(|v_{m+1} - v_m| \leq |v_{d+E(L,d)+1} - v_{d+E(L,d)}| < 2^{-c'})$, so D is defined everywhere.

   $\underline{E}$ When k runs through $\mathbb{N}$, $d+k$ runs through $[d; +\infty[$ which contains I_{Ld}. So, for some k, the number $d+k$ will eventually take the value $m \in I_{Ld}$ that maximises $|v_{m+1} - v_m|$ on I_{Ld}.

2. Now let us prove
   $$\begin{gathered}\forall L \leq^{\mathrm{e}} \gamma(n^2, K') \\ \Big(L(D(L, c)), L\Big(D\big(L, C\big(L(D(L, c))\big)\big)\Big) \leq^{\mathrm{e}}_0 \Psi(\alpha,\beta,\gamma,f,H)\Big).\end{gathered} \tag{16.72}$$

   To do so, we start by proving $E \leq^{\mathrm{e}} E'$, $D \leq^{\mathrm{e}} D'$ and $K_{F'} \leq^{\mathrm{e}} K'$.

   $\underline{E \leq^{\mathrm{e}} E'}$ We take arbitrary $L \leq^{\mathrm{e}} L'$ and $d \leq^{\mathrm{e}} d'$, and prove $E(L, d) \leq^{\mathrm{e}}_0 E'(L', d')$ and $E'(L, d) \leq^{\mathrm{e}}_0 E'(L', d')$:
   $$\begin{aligned} E(L, d) &\leq \\ (H+\mathrm{id})\big(\max(L(d), d)\big) &\leq && (H+\mathrm{id} \leq^{\mathrm{e}} (H+\mathrm{id})^{\mathrm{M}},\ L \leq^{\mathrm{e}} L^{\mathrm{M}}) \\ (H+\mathrm{id})^{\mathrm{M}}\big(\max(L^{\mathrm{M}}(d), d)\big) &= \\ E'(L, d) &\leq && ((H+\mathrm{id})^{\mathrm{M}} \leq^{\mathrm{e}} (H+\mathrm{id})^{\mathrm{M}},\ L^{\mathrm{M}} \leq^{\mathrm{e}} L'^{\mathrm{M}}) \\ E'(L', d'). \end{aligned}$$

$\underline{D \leq^{\mathrm{e}} D'}$ We take arbitrary $L \leq^{\mathrm{e}} L'$ and $c' \leq^{\mathrm{e}} c''$, and prove $D(L, c') \leq^{\mathrm{e}}_0 D'(L', c'')$ and $D'(L, c') \leq^{\mathrm{e}}_0 D'(L', c'')$:

$$\begin{aligned} D(L, c') &\leq \\ \beta(c', E'(L^{\mathrm{M}}, \cdot)) &= \\ D'(L, c') &\leq && (\beta \leq^{\mathrm{e}} \beta,\ E' \leq^{\mathrm{e}} E',\ L^{\mathrm{M}} \leq^{\mathrm{e}} L'^{\mathrm{M}}) \\ D'(L', c''). \end{aligned}$$

$\underline{K_{F'} \leq^{\mathrm{e}} K'}$ We take arbitrary $i \leq^{\mathrm{e}} i'$ and $L \leq^{\mathrm{e}} L'$, and prove $K_{F'}(i, L) \leq^{\mathrm{e}}_0 K'(i', L')$ and $K'(i, L) \leq^{\mathrm{e}}_0 K'(i', L')$:

$$\begin{aligned} K_{F'}(i, L) &\leq \\ \max\Big(D(L, c), D\big(L, C\big(L(D(L, c))\big)\big)\Big) &\leq && (D \leq^{\mathrm{e}} D',\ L \leq^{\mathrm{e}} L^{\mathrm{M}},\ C \leq^{\mathrm{e}} C^{\mathrm{M}}) \\ \max\Big(D'(L^{\mathrm{M}}, c), D'\big(L^{\mathrm{M}}, C^{\mathrm{M}}\big(L^{\mathrm{M}}(D'(L^{\mathrm{M}}, c))\big)\big)\Big) &= \\ K'(i, L) &\leq && (D' \leq^{\mathrm{e}} D',\ L^{\mathrm{M}} \leq^{\mathrm{e}} L'^{\mathrm{M}},\ C^{\mathrm{M}} \leq^{\mathrm{e}} C^{\mathrm{M}}) \\ K'(i', L'). \end{aligned}$$

$\underline{\text{Proof of (16.72)}}$ We take arbitrary $L \leq^{\mathrm{e}} \gamma(n^2, K')$. First, we compute

$$\begin{aligned} D(L, c), D\big(L, C\big(L(D(L, c))\big)\big) &\leq \\ \max\Big(D(L, c), D\big(L, C\big(L(D(L, c))\big)\big)\Big) &\leq && \text{(previous point)} \\ K'(i, L) &= && (K' \text{ does not depend on } i) \\ K'(0, L) &\leq && (K' \leq^{\mathrm{e}} K',\ L \leq^{\mathrm{e}} \gamma(n^2, K')) \\ K'(0, \gamma(n^2, K')). \end{aligned}$$

Since we just proved $D(L, c), D\big(L, C\big(L(D(L, c))\big)\big) \leq^{\mathrm{e}} K'(0, \gamma(n^2, K'))$, then applying the left side to L and the right side to $\gamma(n^2, K')$ we get (16.72).

3. By contradiction, we assume the negation of (16.71), that is there exist $f \in \mathbb{N}$ and $H\colon \mathbb{N} \to \mathbb{N}$ such that $\forall g \leq \Psi(\alpha, \beta, \gamma, f, H)\, \exists i, j \in [g; g + H(g)]\, (|v_i - v_j| \geq 2^{-f})$, thus $\forall g \leq \Psi(\alpha, \beta, \gamma, f, H)\, \exists h' \in\,]g; g + H(g)]\, (|v_g - v_{h'}| \geq 2^{-(f+1)})$. So define $H'\colon \mathbb{N} \to \mathbb{N}$ by

$$H'(g) := \begin{cases} \mu h' \in\,]g; g + H(g)]\,.\, |v_g - v_{h'}| \geq 2^{-(f+1)} & \text{if } g \leq \Psi(\alpha, \beta, \gamma, f, H) \\ 0 & \text{otherwise} \end{cases}.$$

We have $[0, 1] = \bigcup_{i=0}^{n-1} I_i$ where $I_i := [\frac{i}{n}, \frac{i+1}{n}]$ (not to be confused with I_{Ld} defined above). So define $F\colon \mathbb{N} \to n$ by $F(k) := \mu m < n\,.\, v_k \in I_m$. Define the colouring $F' \colon \mathbb{N} \to n^2$ with n^2 colours by $F'(k) := \langle F(k), F(H'(k))\rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing of point 1 of lemma 16.10.

By (16.70) there exist $i < n^2$ and $L \leq^{\mathrm{e}} \gamma(n^2, K')$ such that $(*_1)$ $L(K_{F'}(i, L)) \geq K_{F'}(i, L)$ and $(*_2)$ $F'\big(L(K_{F'}(i, L))\big) = i$. Let

$$d_0 := D(L, c), \qquad j_0 := L(\overbrace{D(L, c)}^{=d_0}),$$
$$d_1 := D\big(L, C\big(\underbrace{L(D(L, c))}_{=j_0}\big)\big), \qquad j_1 := L\Big(\underbrace{D\big(L, C\big(L(D(L, c))\big)\big)}_{=d_1}\Big).$$

Say $i = \langle i_0, i_1 \rangle$ with $i_0, i_1 < n$. Let us prove some statements.

$\underline{j_0 \geq d_0}$ If $j_0 < d_0$, then $K_{F'}(i, L) = d_0$, thus $(*_1)$ is $j_0 \geq d_0$, and we arrive at a contradiction.

$\underline{v_{j_0} \in I_{i_0} \text{ and } v_{H'(j_0)} \in I_{i_1}}$ By definition of F and F', it suffices to show $F'(j_0) = i$. If $F'(j_0) \neq i$, then $K_{F'}(i, L) = d_0$, so $(*_2)$ is $F'(j_0) = i$, and we arrive at a contradiction.

$\underline{j_1 \geq d_1}$ By the first previous point and the proof of the second previous point, we have $\neg(j_0 < d_0 \vee F'(j_0) \neq i)$, so $K_{F'}(i, L) = d_1$, thus $(*_1)$ is $j_1 \geq d_1$.

$\underline{v_{j_1} \in I_{i_0} \text{ and } v_{H'(j_1)} \in I_{i_1}}$ By definition of F and F', it suffices to show $F'(j_1) = i$. We already know $K_{F'}(i, L) = d_1$, so $(*_2)$ is $F'(j_1) = i$.

$\underline{j_0, j_1 \leq \Psi(\alpha, \beta, \gamma, f, H)}$ It follows from (16.72) since $L \leq^{\mathrm{e}} \gamma(n^2, K')$.

$\underline{|i_0 - i_1| \geq 2}$ It follows from $|v_{j_0} - v_{H'(j_0)}| \geq 2^{-(f+1)}$ (by $j_0 \leq \Psi(\alpha, \beta, \gamma, f, H)$ and definition of H'), and $v_{j_0} \in I_{i_0}$ and $v_{H'(j_0)} \in I_{i_1}$.

$\underline{j_0 < H'(j_0) \text{ and } j_1 < H'(j_1)}$ It follows from $j_0, j_1 \leq \Psi(\alpha, \beta, \gamma, f, H)$ and the definition of H'.

Below we can write $\frac{1}{2nH(j_0)}$ since $H(j_0) \neq 0$ because $j_0 < H'(j_0) \leq j_0 + H(j_0)$. Consider $I_{i_0 \pm 1}$, where we choose the plus sign if $i_0 + 1 < i_1$ and the minus sign if $i_1 + 1 < i_0$, so that $I_{i_0 \pm 1}$ is between I_{i_0} and I_{i_1}. We consider two cases.

$\underline{\forall x \in I_{i_0 \pm 1} \big(|\phi(x) - x| < \frac{1}{2nH(j_0)}\big)}$ By (16.69) and the definition of E, we have $\forall e \in I_{Ld_0} \, (|v_{e+1} - v_e| < \frac{1}{2n})$, where $I_{Ld_0} = [j_0; (H + \mathrm{id})(j_0)[\, \supseteq [j_0; H'(j_0)[$ (because $L(d_0) = j_0 \geq d_0$ and $H'(j_0) \leq (H + \mathrm{id})(j_0)$). So $(v_k)_{k \in \mathbb{N}}$ goes from $v_{j_0} \in I_{i_0}$ to $v_{H'(j_0)} \in I_{i_1}$ in steps strictly smaller than $\frac{1}{2n}$, that is half of the length of $I_{i_0 \pm 1}$. Thus v_k enters the half of $I_{i_0 \pm 1}$ closest to I_{i_0}, for some $k \in [j_0; H'(j_0)]$. Then, to reach I_{i_1}, the sequence $(v_k)_{k \in \mathbb{N}}$

(a) has to cover the other half of $I_{i_0 \pm 1}$;
(b) in at most $H'(j_0) - j_0 \leq H(j_0)$ steps;
(c) and inside $I_{i_0 \pm 1}$ (that is for all $k \in \mathbb{N}$ such that $v_k \in I_{i_0 \pm 1}$) each step has length $|v_{k+1} - v_k| = |\phi(v_k) - v_k| < \frac{1}{2nH(j_0)}$.

But this is impossible because in at most $H(j_0)$ steps strictly smaller than $\frac{1}{2nH(j_0)}$ the sequence $(v_k)_{k \in \mathbb{N}}$ covers a distance strictly smaller than $\frac{1}{2n}$, that is strictly smaller than the length of half of $I_{i_0 \pm 1}$.

$\underline{\exists x \in I_{i_0\pm 1}\left(|\phi(x)-x| \geq \frac{1}{2nH(j_0)}\right)}$ By (16.68), (16.69) and the definitions of d_0, D and C, we have

$$\forall y \in [0,1]\left(|x-y| < 2^{-\alpha(A(j_0))} \to |\phi(x)-\phi(y)| < \frac{1}{6nH(j_0)}\right),$$

$$\forall e \in I_{Ld_1} \supseteq [j_1; H'(j_1)[\left(|v_{e+1}-v_e| < 2^{-C(j_0)} \leq \frac{1}{6nH(j_0)}, 2^{-\alpha(A(j_0))}\right) \tag{16.73}$$

where $I_{Ld_1} \supseteq [j_1; H'(j_1)[$ (because $L(d_1) = j_1 \geq d_1$ and $H'(j_1) \leq (H + \mathrm{id})(j_1)$). Let $J :=]x - 2^{-C(j_0)}, x + 2^{-C(j_0)}[$. By point 2 of lemma 16.10 we have

$$\forall y \in J\left(|\phi(y)-y| \geq \underbrace{|\phi(x)-x|}_{\geq \frac{1}{2nH(j_0)}} - \underbrace{|\phi(x)-\phi(y)|}_{< \frac{1}{6nH(j_0)}} - \underbrace{|y-x|}_{< \frac{1}{6nH(j_0)}} > \frac{1}{6nH(j_0)}\right). \tag{16.74}$$

Since $2^{-C(j_0)} \leq \mathrm{lh}\, I_{i_0\pm 1}$ (because $2^{-C(j_0)} \leq \frac{1}{6nH(j_0)}$ and $\mathrm{lh}\, I_{i_0\pm 1} = 1/n$), then by point 3 of lemma 16.10 we have $2^{-C(j_0)} \leq \mathrm{lh}(I_{i_0\pm 1} \cap J)$. So, by (16.73), as $(v_k)_{k\in\mathbb{N}}$ goes from $v_{j_1} \in I_{i_0}$ to $v_{H'(j_1)} \in I_{i_1}$ it enters J for some $k \in [j_1; H'(j_1)[$. But then $|v_{k+1}-v_k| = |\phi(v_k)-v_k| > \frac{1}{6nH(j_0)}$ by (16.74), contradicting (16.73).

16.35. The bound Ψ presented in the previous theorem has low complexity: it is primitive recursive (almost only uses addition, multiplication and exponentiation) on α, β, γ and H. We can say that all the complexity of the bound is contained in γ arising from $(\mathsf{IPP}^{\mathrm{GG}})^{\mathrm{MD}}$. The bound is also uniform on v_0 and ϕ (analogously to paragraph 16.29).

16.6 Computer testing

16.36. In theorems 16.27 and 16.33 we gave bounds $\Omega(\alpha, \beta, f, H)$ on g. It is natural to ask if this bounds are good or bad, that is if their value is close to g or not, or in other words if Ω/g is close to 1 or if Ω/g is very large. In this section we are going to experimentally answer this question: we are going to choose some ϕ, v_0, f and H, find rates α and β, and compute the value of Ω/g.

Since the βs that we are going to find are not just rates of metastability but even rates of convergence, we take Ω as being the simpler bound Φ given in theorem 16.27, which makes use of rates of convergence. This also saves us from dealing with the term γ witnessing $(\mathsf{IPP}^{\mathrm{GG}})^{\mathrm{MD}}$ which, as explained in remark 16.31, is difficult to write down.

16.37 Proposition. In table 16.2 we list, for several functions $\phi\colon [0,1] \to [0,1]$ and initial points $v_0 \in [0,1]$:

1. rates of uniformity continuity $\alpha\colon \mathbb{N} \to \mathbb{N}$ of ϕ;

2. rates of convergence $\beta\colon \mathbb{N} \to \mathbb{N}$ of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$ (with limit 0), where $(v_k)_{k\in\mathbb{N}}$ is defined by $v_{k+1} := \phi(v_k)$.

$\phi(x)$	v_0	$\alpha(a)$	$\beta(c)$
$1 - x/2$	1	$\max(0, a-1)$	c
$\frac{1}{1+x}$	1	a	c
$\cos x$	1	a	$5\max(0, c-1)$
$(x - 1/2)^3 + 1/2$	0	a	$3\max(0, c-1)$

Table 16.2: rates of uniformity continuity α of ϕ and rates of convergence β of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$ (with limit 0).

16.38 Proof. First, let us make two remarks.

1. If $\phi\colon [a, b] \to [a, b]$ (with $a, b \in \mathbb{R}$) is continuously differentiable, then ϕ is Lipschitz continuous with Lipschitz constant $c := \max_{x\in[a,b]}|f'(x)|$, that is $\forall x, y \in [a, b]\,(|\phi(x) - \phi(y)| \le c|x - y|)$.

   Let us sketch the proof: by the fundamental theorem of calculus and the monotonicity of the integral we have $|\phi(x)-\phi(y)| = \left|\int_x^y \phi'(z)\,\mathrm{d}z\right| \le \left|\int_x^y |\phi'(z)|\,\mathrm{d}z\right| \le \left|\int_x^y c\,\mathrm{d}z\right| = c|x - y|$.

2. If $\phi\colon [a, b] \to [a, b]$ (with $a, b \in \mathbb{R}$) is Lipschitz continuous with Lipschitz constant c and $v_0 \in [a, b]$, then the sequence $(v_k)_{k\in\mathbb{N}}$ defined by $v_{k+1} := \phi(v_k)$ satisfies $\forall k \in \mathbb{N}\,(|v_{k+1} - v_k| \le c^k|v_1 - v_0|)$ (with $c \neq 0$ for c^0 to be defined).

   Let us sketch the proof by induction on k: in the induction step we assume $|v_{k+1} - v_k| \le c^k|v_1 - v_0|$ by induction hypothesis, and so $|v_{k+2} - v_{k+1}| = |\phi(v_{k+1}) - \phi(v_k)| \le c|v_{k+1} - v_k| \le c \cdot c^k|v_1 - v_0| = c^{k+1}|v_1 - v_0|$, as we wanted.

Let us prove that the αs and βs in table 16.2 are correct rates. In the i-th item below we take care of the α and β in the $(i + 1)$-th line of table 16.2: the α is taken care in the first subitem and the β in the second subitem.

1. (a) We take arbitrary $x, y \in [0, 1]$. By remark 1 we have $|\phi(x) - \phi(y)| \le \frac{1}{2}|x - y|$. So, if $|x - y| < 2^{-\alpha(a)} = 2^{-\max(0,a-1)}$, then $|\phi(x) - \phi(y)| \le \frac{1}{2}|x - y| < \frac{1}{2} \cdot 2^{-\max(0,a-1)} \le 2^{-a}$, as we wanted.

   (b) By remark 2 we have $\forall k \in \mathbb{N}\,(|v_{k+1} - v_k| \le (1/2)^k|1/2 - 1| < 2^{-k})$, so $\forall c, e \in \mathbb{N}\,(|v_{\beta(c)+e+1} - v_{\beta(c)+e}| < 2^{-(\beta(c)+e)} \le 2^{-c})$, as we wanted.

2. (a) We take arbitrary $x, y \in [0, 1]$. By remark 1 we have $|\phi(x) - \phi(y)| \le 1|x - y|$. So, if $|x - y| < 2^{-\alpha(a)} = 2^{-a}$, then $|\phi(x) - \phi(y)| \le |x - y| < 2^{-a}$, as we wanted.

   (b) By remark 2, applied to ϕ restricted to $[a, b] = [1/2, 1]$, we have $\forall k \in \mathbb{N}\,(|v_{k+1} - v_k| \le (4/9)^k|1/2 - 1| < 2^{-k})$, so $\forall c, e \in \mathbb{N}\,(|v_{\beta(c)+e+1} - v_{\beta(c)+e}| < 2^{-(\beta(c)+e)} \le 2^{-c})$, as we wanted.

3. (a) We take arbitrary $x, y \in [0,1]$. By remark 1 we have $|\phi(x) - \phi(y)| \leq (\sin 1)|x-y|$. So, if $|x-y| < 2^{-\alpha(a)} = 2^{-a}$, then $|\phi(x)-\phi(y)| \leq (\sin 1)|x-y| \leq |x-y| < 2^{-a}$, as we wanted.
   (b) By remark 2 we have $\forall k \in \mathbb{N}\,(|v_{k+1} - v_k| \leq (\sin 1)^k |\cos 1 - 1| < 2^{-(k/5+1)})$ (because $\sin 1 < 2^{-1/5}$ and $|\cos 1 - 1| < 2^{-1}$), thus $\forall c, e \in \mathbb{N}\,(|v_{\beta(c)+e+1} - v_{\beta(c)+e}| < 2^{-((\beta(c)+e)/5+1)} \leq 2^{-(\beta(c)/5+1)} \leq 2^{-c})$, as we wanted.

4. (a) We take arbitrary $x, y \in [0,1]$. By remark 1 we have $|\phi(x) - \phi(y)| \leq \frac{3}{4}|x-y|$. So, if $|x-y| < 2^{-\alpha(a)} = 2^{-a}$, then $|\phi(x) - \phi(y)| \leq \frac{3}{4}|x-y| \leq |x-y| < 2^{-a}$, as we wanted.
   (b) By remark 2 we have $\forall k \in \mathbb{N}\,(|v_{k+1} - v_k| \leq (3/4)^k |3/8 - 0| < 2^{-(k/3+1)})$ (because $3/4 < 2^{-1/3}$ and $|3/8 - 0| < 2^{-1}$), thus $\forall c, e \in \mathbb{N}\,(|v_{\beta(c)+e+1} - v_{\beta(c)+e}| < 2^{-((\beta(c)+e)/3+1)} \leq 2^{-(\beta(c)/3+1)} \leq 2^{-c})$, as we wanted.

16.39 Remark.

1. The unique fixed point of $\phi(x) = 1 - x/2$ is $2/3$.

2. The unique fixed point (in $[0,1]$) of $\phi(x) = \frac{1}{1+x}$ is $\frac{\sqrt{5}-1}{2} = 0.618033\ldots$ and is equal to both $1/\varphi$ and $\varphi - 1$ where $\varphi = \frac{\sqrt{5}+1}{2} = 1.618033\ldots$ is the golden ratio.

3. The unique fixed point of cos is $0.739085\ldots$ and is called *Dottie number*. It is named after the professor of French that noted that inserting any number in a calculator and pressing repeatedly the cos button always produces $0.739085\ldots$ [42]. We can prove by contradiction that the Dottie number is transcendental using this result: if $x \neq 0$ is an algebraic number, then $\cos x$ is transcendental [60, theorem 9.11].

4. The unique fixed point (in $[0,1]$) of $\phi(x) = (x - 1/2)^3 + 1/2$ is $1/2$.

These fixed points are illustrated in figure 16.9.

16.40 Program. Below we present a program, written in the numerically oriented programming language of the numerical computational software Scilab [7]. For better readability, the program is divided into three listings.

In lines 1 to 6 of listing 16.1 we define the function ϕ. Since we are going do numerically test the bound for the four functions ϕ in table 16.2, there are four possible definitions listed in lines 2 to 5. The definition in use is the one not commented out by "//". The remaining lines define v_0, $(v_k)_{k\in\mathbb{N}}$, α, β, f and H.

```
function y = phi(x)
    y = 1 - x / 2
  // y = 1 / (1 + x)
  // y = cos(x)
  // y = (x - 1 / 2)^3 + 1 / 2
endfunction
```

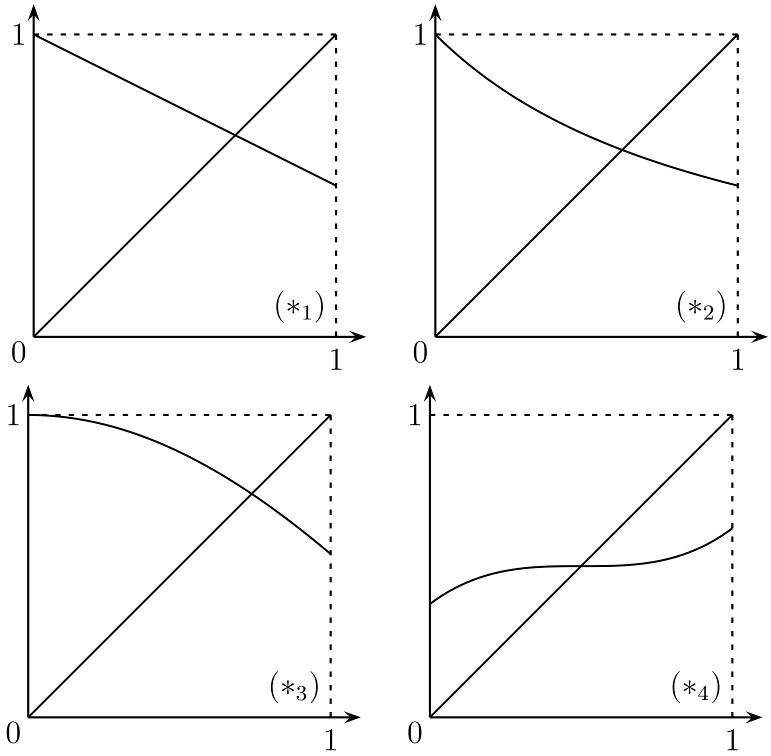

Figure 16.9: the fixed points of $(*_1)$ $\phi(x) = 1 - x/2$, $(*_2)$ $\phi(x) = \frac{1}{1+x}$, $(*_3)$ $\phi(x) = \cos x$ and $(*_4)$ $\phi(x) = (x - 1/2)^3 + 1/2$.

```
     v0 = 1
//  v0 = 1
//  v0 = 1
//  v0 = 0

function y = v(k)
   if k == 0 then
      y = v0
   else
      y = phi(v(k - 1))
   end
endfunction

function y = alp(a)
       y = max([0  a - 1])
   // y = a
   // y = a
   // y = a
endfunction

function y = bet(c)
       y = c
```

```
  // y = c
  // y = 5 * max([0 c - 1])
  // y = 3 * max([0 c - 1])
endfunction

   f = 2
// f = 3
// f = 4
// f = 5

function y = H(k)
      y = 1
  // y = modulo(k,3) + 1
  // y = k + 1
  // y = k^2 + 1
endfunction
```

Listing 16.1: definitions of ϕ, $(v_k)_{k\in\mathbb{N}}$, α, β, f and H.

In listing 16.2 we compute the bound $\Phi(\alpha,\beta,f,H)$ of theorem 16.27. To do so, in this listing we define the numbers n and c, the functions A and C, and the sequence $(u_k)_{k\in\mathbb{N}}$ by an iterative method (to avoid memory limitations associated to recursive methods).

```
n = 3 * 2^(f + 1)

c = 1 + ceil(log2(n))

function y = A(k)
  y = ceil(log2(max([6 * n * H(k) 1])))
endfunction

function y = C(k)
  y = max([A(k) alp(A(k))])
endfunction

function y = u(k)
  previous = bet(c)
  for i = 1 : k
    next = max([previous + 1 bet(C(previous))])
    previous = next
  end
  y = previous
endfunction

function y = Phi
  y = u(n^2)
```

```
endfunction
```

Listing 16.2: computation of $\Phi(\alpha, \beta, f, H)$.

Finally, in listing 16.3 we compute the least $g \in \mathbb{N}$ such that $\forall i, j \in [g; g + H(g)]\,(|v_i - v_j| < 2^{-f})$. To do so, we start with $g = 0$ (resulting from combining lines 2 and 5), and we keep increasing the value of g by 1 as long as $\neg\forall i, j \in [g; g + H(g)]\,|v_i - v_j| < 2^{-f}$.

```
function findLeastg
  g = -1
  found = %f
  while ~found
    g = g + 1
    found = %t
    for i = g : g + H(g)
      for j = g : g + H(g)
        if abs(v(i) - v(j)) >= 2^-f then
          found = %f
        end
      end
    end
  end
  disp(g)
endfunction
```

Listing 16.3: computation of the least $g \in \mathbb{N}$ such that $\forall i, j \in [g; g+H(g)]\,(|v_i - v_j| < 2^{-f})$.

In table 16.3 we list in the first six columns the inputs of the program, in the seventh and eighth columns the outputs of the program, and in last column the quotient Φ/g rounded (where g is the least $g \in \mathbb{N}$ such that $\forall i, j \in [g; g+H(g)]\,(|v_i - v_j| < 2^{-f})$). We can see that, in average, $\Phi/g \approx 5000$.

$\phi(x)$	v_0	$\alpha(a)$	$\beta(c)$	f	$H(k)$	Φ	g	Φ/g
$1 - \frac{x}{2}$	1	$\max(0, a-1)$	c	2	1	583	2	292
$\frac{1}{1+x}$	1	a	c	3	$k \bmod 3 + 1$	2313	2	1157
$\cos x$	1	a	$5\max(0, c-1)$	4	$k+1$	9289	6	1548
$(x - \frac{1}{2})^3 + \frac{1}{2}$	0	a	$3\max(0, c-1)$	5	k^2+1	36927	2	18464

Table 16.3: values of $\Phi(\alpha, \beta, f, H)$, the least $g \in \mathbb{N}$ such that $\forall i, j \in [g; g+H(g)]\,(|v_i - v_j| < 2^{-f})$, and $\Phi(\alpha, \beta, f, H)/g$ rounded.

16.41. From table 16.3, the conclusion that we reach on how the bound Φ compares with the least g is $\Phi/g \approx 5000$.

16.7 Conclusion

16.42. We considered Hillam's theorem characterising the convergence of a fixed point iteration $v_{k+1} := \phi(v_k)$ of a continuous function $\phi\colon [0,1] \to [0,1]$: the sequence $(v_k)_{k\in\mathbb{N}}$ converges if and only if $v_{k+1} - v_k \to 0$. We extract computational content from Hillam's theorem. This was done in three steps.

1. We showed that Hillam's theorem is provable $\mathsf{WE\text{-}HA}^\omega + \mathsf{QF\text{-}AC}$, so the soundness theorem of MD (composed with GG) predicts that we can extract computational content.

2. We computed what form the computational should take:

$$\begin{array}{c}\text{rate of metastability}\\ \text{of } (v_k)_{k\in\mathbb{N}}\end{array} = f\left(\begin{array}{c}\text{rate of uniform}\\ \text{continuity of } \phi\end{array}, \begin{array}{c}\text{rate of metastability}\\ \text{of } (v_{k+1} - v_k)_{k\in\mathbb{N}}\end{array}\right).$$

3. We presented two proof mined versions of Hillam's theorem.

   Partial proof mining It gives a simpler rate/bound, but a weaker proof mining (which uses a full rate of convergence of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$).

   Full proof mining It gives a more complicated rate/bound, but a stronger proof mining (which uses only a rate of metastability of $(v_{k+1} - v_k)_{k\in\mathbb{N}}$).

Then we did a computer testing and conclude that our bound is about 5000 times greater than the exact value.

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