

Proof Interpretations with Truth*

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12 August 2009

Abstract

This article systematically investigates so-called “truth variants” of several functional interpretations. We start by showing a close relation between two variants of modified realizability, namely modified realizability with truth and q-modified realizability. Both variants are shown to be derived from a single “functional interpretation with truth” of intuitionistic linear logic. This analysis suggests that, contrary to what is believed, several functional interpretations have truth and q-variants. These variants, however, require a more involved modification than the ones previously considered. Following this lead we present truth and q-variants of the Diller-Nahm interpretation, the bounded modified realizability and the bounded functional interpretation.

Keywords Modified realizability with truth and q-variant, intuitionistic linear logic, Aczel and Kleene slashes

Subject classification 03F07, 03F10, 03F25

1 Introduction

Modified realizability associates to each formula A of Heyting arithmetic in all finite types a new formula “ $\mathbf{x} \text{ mr } A$ ” by induction on the structure of A as:

$$\begin{aligned}\mathbf{x}, \mathbf{y} \text{ mr } A \wedge B &:\equiv (\mathbf{x} \text{ mr } A) \wedge (\mathbf{y} \text{ mr } B) \\ z, \mathbf{x}, \mathbf{y} \text{ mr } A \vee B &:\equiv (z = 0 \rightarrow (\mathbf{x} \text{ mr } A)) \wedge (z \neq 0 \rightarrow (\mathbf{y} \text{ mr } B)) \\ f \text{ mr } A \rightarrow B &:\equiv \forall \mathbf{x}((\mathbf{x} \text{ mr } A) \rightarrow (f\mathbf{x} \text{ mr } B)) \\ f \text{ mr } \forall z A &:\equiv \forall z(fz \text{ mr } A) \\ z, \mathbf{x} \text{ mr } \exists z A &:\equiv \mathbf{x} \text{ mr } A.\end{aligned}$$

Intuitively, the fresh tuple of free-variables $\mathbf{x}$ in the formula “ $\mathbf{x} \text{ mr } A$ ” captures the existential information of the formula A , in the style of the informal BHK interpretation. The formula A is

*This is the pre-peer reviewed version of the following article: Jaime Gaspar and Paulo Oliva, “Proof interpretations with truth”, *Mathematical Logic Quarterly*, volume 56, number 6, pages 591–610, 2010, which has been published in final form at <http://onlinelibrary.wiley.com/doi/10.1002/malq.200910112/abstract>.

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reduced to $\exists x(x \text{ mr } A)$ in such a way that if A is provable (in Heyting arithmetic, for instance) then a tuple of terms t (in Gödel's T) can be extracted from this proof such that “ $t \text{ mr } A$ ”.

Note that this tuple of terms t extracted from the proof of A is a witnesses for the statement $\exists x(x \text{ mr } A)$. One may then wonder what has been gained, since ideally we would like to obtain witnesses for the original theorem A (and not necessarily for its interpretation $\exists x(x \text{ mr } A)$). As it turns out, the modified realizability of a formula A retains a relation with the original formula A for the small class of $\exists$ -free formulas (that is, formulas without disjunctions and existential quantifiers), in the sense that the *truth property*

$$(x \text{ mr } A_{\text{ef}}) \rightarrow A_{\text{ef}}$$

holds for all $\exists$ -free formulas A_{ef} . In this way, from a proof of $\exists z A_{\text{ef}}$, where A_{ef} is existential-free, one obtains a tuple of terms such that $(s, t \text{ mr } \exists z A_{\text{ef}}) \equiv (t \text{ mr } A_{\text{ef}}[s/z])$, which given the truth property

$$(t \text{ mr } A_{\text{ef}}[s/z]) \rightarrow A_{\text{ef}}[s/z]$$

implies that s is a witness for $\exists z A_{\text{ef}}$.

One desires, however, to extend this property to larger classes. This was achieved by Stephen Kleene and Peter Aczel by adapting the modified realizability clauses for disjunction, existential quantification and implication. Kleene extended the truth property to *disjunctive and existential formulas* by considering a variant mq of mr , called q -modified realizability, where one adds A and B to the disjunction and existential quantifier clauses:

$$\begin{aligned} z, x, y \text{ mq } A \vee B &::= (z = 0 \rightarrow (x \text{ mq } A) \wedge A) \wedge (z \neq 0 \rightarrow (y \text{ mq } B) \wedge B) \\ z, x \text{ mq } \exists z A &::= (x \text{ mq } A) \wedge A. \end{aligned}$$

Aczel extended the truth property to *all formulas* by considering a variant mrt of mr , called modified realizability with truth, where one only adds $A \rightarrow B$ to the implication clause:

$$f \text{ mrt } A \rightarrow B ::= \forall x((x \text{ mrt } A) \rightarrow (fx \text{ mrt } B)) \wedge (A \rightarrow B).$$

The modified realizability with truth can be quite useful, as demonstrated in [10–12], since the interpretation of A does not lose the relation to the original formula A .

Recently, Jørgensen [9] presented a q -variant of the Diller-Nahm interpretation. In the same paper, Jørgensen shows that simply adding $A \rightarrow B$ to the implication clause of the Diller-Nahm interpretation does not lead to a sound “Diller-Nahm with truth”. However, Jørgensen’s work does not rule out the possibility that a more comprehensive modification works.

In this paper we make mainly two points. (1) We first show a close relation between the q -modified realizability and the modified realizability with truth. Then we factorise the two in terms of the two standard embeddings of intuitionistic logic into linear logic composed with a single modified realizability with truth of intuitionistic linear logic. (2) The factorisation suggests that the universal quantifier clause of modified realizability with truth should be

$$f \text{ mrt } \forall z A ::= \forall z(fz \text{ mrt } A) \wedge \forall z A.$$

For modified realizability with truth the addition of $\forall z A$ is redundant, but for other proof interpretations this makes a difference. Indeed, following this clue, we show that by adding $\forall z A$ to the universal quantifier clause, we get truth variants of the bounded modified realizability, the

Diller-Nahm interpretation, and the bounded functional interpretation.

Acknowledgements. The authors would like to thank Ulrich Kohlenbach for interesting discussions regarding the topic of proof interpretations with truth. The first author would also like to thank the support given by Fundação para a Ciência e a Tecnologia, whereas the second author gratefully acknowledges support of the Royal Society (grant 516002.K501/RH/kk).

1.1 Organisation

Section 2. Modified realizability with truth, relation between q- and truth variants. *Section 3.* Truth interpretation of intuitionistic linear logic, relation to interpretations of intuitionistic logic. *Section 4.* Kleene and Aczel slash translations. *Section 5.* Bounded modified realizability with truth. *Section 6.* Diller-Nahm with truth. *Section 7.* Bounded functional interpretation with truth. *Section 8.* Conclusion.

2 Modified Realizability with Truth

Let us start with a discussion (and some new results) about the two “truth variants” of modified realizability.

2.1 Definitions

Let us use the abbreviation $A \diamond_z B := (z = 0 \rightarrow A) \wedge (z \neq 0 \rightarrow B)$.

Definition 1 (q-variant of modified realizability). Each formula A of $\mathbb{L}^\omega$ is associated to a new formula $\mathbf{x} \text{ mq } A$ inductively as follows. If A_{at} is an atomic formula, then $\mathbf{x} \text{ mq } A_{\text{at}} := A_{\text{at}}$ where $\mathbf{x}$ is the empty tuple. Assuming we already have defined $\mathbf{x} \text{ mq } A$ and $\mathbf{y} \text{ mq } B$, then

$$\begin{aligned} \mathbf{x}, \mathbf{y} \text{ mq } A \wedge B &:= (\mathbf{x} \text{ mq } A) \wedge (\mathbf{y} \text{ mq } B) \\ z, \mathbf{x}, \mathbf{y} \text{ mq } A \vee B &:= ((\mathbf{x} \text{ mq } A) \wedge A) \diamond_z ((\mathbf{y} \text{ mq } B) \wedge B) \\ f \text{ mq } A \rightarrow B &:= \forall \mathbf{x} ((\mathbf{x} \text{ mq } A) \wedge A \rightarrow (f \mathbf{x} \text{ mq } B)) \\ f \text{ mq } \forall z A &:= \forall z (f z \text{ mq } A) \\ z, \mathbf{x} \text{ mq } \exists z A &:= (\mathbf{x} \text{ mq } A) \wedge A. \end{aligned}$$

The *q-variant of modified realizability* of a formula A is then defined as $A^{\text{mq}} := \exists \mathbf{x} (\mathbf{x} \text{ mq } A)$.

Remark 2. We have $\mathbb{L}^\omega \vdash (\mathbf{x} \text{ mq } A) \rightarrow A$ for all disjunctive and existential formulas A of $\mathbb{L}^\omega$.

Definition 3 (modified realizability with truth). Each formula A of $\mathbb{L}^\omega$ is associated to a new formula $\mathbf{x} \text{ mrt } A$ inductively as follows. If A_{at} is an atomic formula, then $\mathbf{x} \text{ mrt } A_{\text{at}} := A_{\text{at}}$ where $\mathbf{x}$ is the empty tuple. Assuming we already have defined $\mathbf{x} \text{ mrt } A$ and $\mathbf{y} \text{ mrt } B$, then

$$\begin{aligned} \mathbf{x}, \mathbf{y} \text{ mrt } A \wedge B &:= (\mathbf{x} \text{ mrt } A) \wedge (\mathbf{y} \text{ mrt } B) \\ z, \mathbf{x}, \mathbf{y} \text{ mrt } A \vee B &:= (\mathbf{x} \text{ mrt } A) \diamond_z (\mathbf{y} \text{ mrt } B) \\ f \text{ mrt } A \rightarrow B &:= \forall \mathbf{x} ((\mathbf{x} \text{ mrt } A) \rightarrow (f \mathbf{x} \text{ mrt } B)) \wedge (A \rightarrow B) \\ f \text{ mrt } \forall z A &:= \forall z (f z \text{ mrt } A) \\ z, \mathbf{x} \text{ mrt } \exists z A &:= \mathbf{x} \text{ mrt } A. \end{aligned}$$

The *modified realizability with truth* of a formula A is then defined as $A^{\text{mrt}} := \exists \mathbf{x}(\mathbf{x} \text{ mrt } A)$.

Remark 4. We have $\mathbb{L}^\omega \vdash (\mathbf{x} \text{ mrt } A) \rightarrow A$ for all formulas A of $\mathbb{L}^\omega$.

2.2 Relation between the q- and the truth variants of modified realizability

Intuitively, the truth-variant is stronger than the q-variant, since the truth property of mrt holds for all formulas A , whereas for the q-variant it only holds for disjunctive and existential formulas. Our next (apparently new) theorem gives a rigorous mathematical meaning to this intuition.

Theorem 5. For all formulas A of $\mathbb{L}^\omega$ we have

$$\mathbb{L}^\omega \vdash (\mathbf{x} \text{ mrt } A) \leftrightarrow (\mathbf{x} \text{ mq } A) \wedge A.$$

Proof. By an easy induction on the structure of the formulas.

$A \rightarrow B$.

$$\begin{aligned} f \text{ mrt } A \rightarrow B &\equiv \forall \mathbf{x}((\mathbf{x} \text{ mrt } A) \rightarrow (f\mathbf{x} \text{ mrt } B)) \wedge (A \rightarrow B) \\ &\stackrel{(\text{IH})}{\leftrightarrow} \forall \mathbf{x}((\mathbf{x} \text{ mq } A) \wedge A \rightarrow (f\mathbf{x} \text{ mq } B) \wedge B) \wedge (A \rightarrow B) \\ &\leftrightarrow \forall \mathbf{x}((\mathbf{x} \text{ mq } A) \wedge A \rightarrow (f\mathbf{x} \text{ mq } B)) \wedge (A \rightarrow B) \\ &\equiv (f \text{ mq } A \rightarrow B) \wedge (A \rightarrow B). \end{aligned}$$

$\forall z A$.

$$\begin{aligned} f \text{ mrt } \forall z A &\equiv \forall z(fz \text{ mrt } A) \\ &\stackrel{(\text{IH})}{\leftrightarrow} \forall z((fz \text{ mq } A) \wedge A) \\ &\leftrightarrow \forall z(fz \text{ mq } A) \wedge \forall z A \\ &\equiv (f \text{ mq } \forall z A) \wedge \forall z A. \end{aligned}$$

The other cases are treated similarly. □

2.3 Characterisation

Although we will show that it is possible to produce truth variants for almost any given functional interpretation, it seems that only realizability interpretations have a “nice” characterisation, in the sense that the truth variant of the interpretation can be characterised by the same principles as the version “without truth”. For instance, let $\mathbb{L}^\#$ be the theory $\mathbb{L}^\omega$ extended with the characterisation principles for the standard modified realizability, namely

$$\begin{aligned} \text{AC} : \quad &\forall \mathbf{x} \exists y A(\mathbf{x}, y) \rightarrow \exists f \forall \mathbf{x} A(\mathbf{x}, f\mathbf{x}) \\ \text{IP}_{\text{ef}} : \quad &(A_{\text{ef}} \rightarrow \exists \mathbf{x} B) \rightarrow \exists \mathbf{x} (A_{\text{ef}} \rightarrow B), \end{aligned}$$

where A_{ef} is an existential-free formula.

Theorem 6. For all formulas A of IL^ω we have

- (i) $\text{IL}^\# \vdash \exists x(x \text{ mq } A) \leftrightarrow A$
- (ii) $\text{IL}^\# \vdash \exists x(x \text{ mrt } A) \leftrightarrow A$.

Proof. By the characterisation of modified realizability we have

$$\text{IL}^\# \vdash \exists x(x \text{ mr } A) \leftrightarrow A,$$

for any formula A in the language of IL^ω . In order to prove (i), one can easily show by induction on A that

$$\text{IL}^\# \vdash (x \text{ mq } A) \leftrightarrow (x \text{ mr } A),$$

since $\text{IL}^\# \vdash (x \text{ mr } A) \rightarrow A$. Point (ii) follows from (i) by Theorem 5. $\square$

3 A Truth Interpretation of Linear Logic

The second author [14] has shown that modified realizability of IL^ω can be decomposed into a modified realizability of *classical* linear logic together with Girard's embedding of IL^ω into linear logic. Recently [5], this has been turned into a finer decomposition in terms of *intuitionistic* linear logic ILL^ω . In this section we present a truth variant for this functional interpretation of ILL^ω . We will then show how the q- and truth-variants of the modified realizability correspond to the truth interpretation of linear logic via two of Girard's embeddings of IL^ω into ILL^ω .

3.1 Intuitionistic logic and intuitionistic linear logic

Our languages for intuitionistic logic IL^ω and intuitionistic linear logic ILL^ω are typed over the ground base types $\mathbf{b}$ (booleans) and $\mathbf{i}$ (integers) and have two ground constants $\mathbf{t}^{\mathbf{b}}$ (true) and $\mathbf{f}^{\mathbf{b}}$ (false) and a ground constant of type $\mathbf{0}$ (to ensure that for every type ρ there is a closed term $\mathbf{0}^\rho$). Also they have term application $(t^{\rho \rightarrow \sigma} q^\rho)^\sigma$, λ -abstraction $(\lambda x^\rho . t^\sigma)^{\rho \rightarrow \sigma}$ and definition of terms by cases $(b)(\gamma_0, \gamma_1)$, with axioms

$$\begin{array}{ll} A_{\text{at}}[(\lambda x . t[x])q] \multimap A_{\text{at}}[t[q]] & A_{\text{at}}[(\mathbf{t})(c_0, c_1)] \multimap A_{\text{at}}[c_0] \\ A_{\text{at}}[\lambda x . tx] \multimap A_{\text{at}}[t] & A_{\text{at}}[(\mathbf{f})(c_0, c_1)] \multimap A_{\text{at}}[c_1] \end{array}$$

(restricted to atomic formulas A_{at} , but can be generalised to arbitrary formulas). We assume only equality between boolean, for which we require the following axioms:

$$\begin{array}{ll} \vdash b = b & !(b_0 = b_1) \vdash b_1 = b_0 \\ !(b_0 = b_1), !(b_1 = b_2) \vdash b_0 = b_2 & !(b_0 = b_1), A[b_0] \vdash A[b_1] \\ \Gamma, !(t = f) \vdash 0 & \vdash !(b = \mathbf{t}) \oplus !(b = \mathbf{f}). \end{array}$$

For IL^ω we drop the bangs and replace $\oplus$ by $\vee$ and $\multimap$ by $\leftrightarrow$ in the axioms above.

IL^ω is based on $\wedge, \vee, \rightarrow, \forall, \exists, \perp$. Its pure logical axioms are the standard ones (cf. [1], section 2.1). It has an if-then-else logical constructor $A \diamond_b B$ defined as

$$A \diamond_b B \equiv (b = \mathbf{t} \rightarrow A) \& (b = \mathbf{f} \rightarrow B).$$

ILL^ω is based on $\otimes, \&, \oplus, \multimap, \forall, \exists, !, 0, \top$. We define $1 \equiv !\top$. Its pure logical axioms are shown in Table 1. It also has an if-then-else defined as

$$A \diamond_b B \equiv (! (b = t) \multimap A) \& (! (b = f) \multimap B).$$

One can prove that the if-then-else is well-behaved, in the sense that the rules shown in table 2 hold in ILL^ω .

IL^ω can be embedded into ILL^ω via any of the two Girard translations $(\cdot)^*$ and $(\cdot)^\circ$.

Definition 7 ([6]). Let us define two embeddings of IL^ω into ILL^ω . Each associates for any formula A of intuitionistic logic its linear translation A^* and A° inductively as

$$\begin{array}{ll} A_{\text{at}}^* \equiv A_{\text{at}}, & \text{if } A_{\text{at}} \neq \perp \\ \perp^* \equiv 0 & \\ (A \wedge B)^* \equiv A^* \& B^* & \\ (A \vee B)^* \equiv !A^* \oplus !B^* & \\ (A \rightarrow B)^* \equiv !A^* \multimap B^* & \\ (\forall x A)^* \equiv \forall x A^* & \\ (\exists x A)^* \equiv \exists x !A^* & \\ A_{\text{at}}^\circ \equiv !A_{\text{at}}, & \text{if } A_{\text{at}} \neq \perp \\ \perp^\circ \equiv 0 & \\ (A \wedge B)^\circ \equiv A^\circ \otimes B^\circ & \\ (A \vee B)^\circ \equiv A^\circ \oplus B^\circ & \\ (A \rightarrow B)^\circ \equiv !(A^\circ \multimap B^\circ) & \\ (\forall x A)^\circ \equiv !\forall x A^\circ & \\ (\exists x A)^\circ \equiv \exists x A^\circ. & \end{array}$$

The translations are sound in the following sense: if $\text{IL}^\omega \vdash A$, then $\text{ILL}^\omega \vdash A^*$ and $\text{ILL}^\omega \vdash A^\circ$ (soundness for the translation $(\cdot)^\circ$ is proved by noticing that each A° is equivalent to $!A^*$, as the following lemma shows). The two embeddings are related to each other as follows:

Lemma 8. For any formula A of IL^ω we have $\text{ILL}^\omega \vdash A^\circ \multimap !A^*$.

Proof. By a simple structural induction on the formula A noting that the equivalences

$$\begin{array}{ll} 0 \multimap !0 & !A \otimes !B \multimap !(A \& B) \\ !A \oplus !B \multimap !(A \oplus !B) & !(A \multimap !B) \multimap !(A \multimap B) \\ !\forall x !A \multimap !\forall x A & \exists x !A \multimap !\exists x !A \end{array}$$

hold in ILL^ω . □

3.2 Definition

The following definition presents (in a uniform way) both the modified realizability interpretation of intuitionistic linear logic [5] and its truth variant.

Definition 9 (Modified realizability of ILL^ω [5] and its truth variant). Let F be a mapping of formulas of ILL^ω such that

- ($\star_1$) $F(A)[t/z] \equiv F(A[t/z])$
- ($\star_2$) if $!\Gamma \vdash A$ then $!F(\Gamma) \vdash F(A)$.

Each formula A of ILL^ω is associated to a new formula $\{A\}_y^x$ inductively as follows. If A_{at} is an atomic formula, then $\{A\}_y^x \equiv A_{\text{at}}$ where x and y are the empty tuple. Assuming we already have defined $\{A\}_y^x$ and $\{B\}_w^v$, then

$$\begin{aligned} \{A \otimes B\}_{y,w}^{x,v} &:= \{A\}_y^x \otimes \{B\}_w^v \\ \{A \& B\}_{y,w,b}^{x,v} &:= \{A\}_y^x \diamond_b \{B\}_w^v \\ \{A \oplus B\}_{y,w}^{x,v,b} &:= \{A\}_y^x \diamond_b \{B\}_w^v \\ \{A \multimap B\}_{x,w}^{f,g} &:= \{A\}_{fxw}^x \multimap \{B\}_w^{gx} \\ \{\forall z A(z)\}_{y,z}^f &:= \{A(z)\}_y^{fz} \\ \{\exists z A(z)\}_y^{x,z} &:= \{A(z)\}_y^x \\ \{!A\}^x &:= !\forall y \{A\}_y^x \otimes !F(A). \end{aligned}$$

If $F(A) \equiv 1$, then $\{\cdot\}$ is called *modified realizability* and the “ $\otimes !F(A)$ ” in the clause of bang becomes superfluous and so may be omitted. If $F(A) \equiv A$, then $\{\cdot\}$ is called *modified realizability with truth*.

Remark 10. We have $\text{ILL}^\omega \vdash \{!A\}^x \multimap !F(A)$ for all formulas A of ILL^ω .

In the case $F(A) \equiv A$, the idea is that in a functional interpretation with truth we want Eloise to win the game $!A$ only when A is “true”. We will show that these variants of the modified realizability of ILL^ω are also sound, and that moreover they refine both the truth and q-variants of modified realizability (cf. Proposition 13).

3.3 Soundness theorem

Theorem 11 (Soundness for modified realizability of ILL^ω). Let $A_0, \dots, A_n$ and B be arbitrary formulas of ILL^ω and let z be all their free variables. From any proof of $A_0(z), \dots, A_n(z) \vdash B(z)$ in ILL^ω we can effectively extract terms $\mathbf{a}_0, \dots, \mathbf{a}_n$ and $\mathbf{b}$ of ILL^ω such that $\{A_0(z)\}_{\mathbf{a}_0}^{x_0}, \dots, \{A_n(z)\}_{\mathbf{a}_n}^{x_n} \vdash \{B(z)\}_w^b$ is provable in ILL^ω , where $\text{FV}(\mathbf{a}_i) \subseteq \{x_0, \dots, x_n, z, w\}$ and $\text{FV}(\mathbf{b}) \subseteq \{x_0, \dots, x_n, z\}$.

Proof. By induction on the derivation of $A_0(z), \dots, A_n(z) \vdash B(z)$. Since the truth interpretation coincides with the standard interpretation for pure ILL^ω , we only need to check the rules for the modality $!A$ (see [5] for the soundness proof for the pure fragment of ILL^ω).

(con).

$$\begin{aligned} & \frac{\{\Gamma\}_{\gamma[x_0, x_1]}^u, \{!A\}^{x_0}, \{!A\}^{x_1} \vdash \{B\}_w^{b[x_0, x_1]}}{\{\Gamma\}_{\gamma[x_0, x_1]}^u, !\forall y \{A\}_y^{x_0}, !F(A), !\forall y \{A\}_y^{x_1}, !F(A) \vdash \{B\}_w^{b[x_0, x_1]}} \text{ (D9)} \\ & \frac{\{\Gamma\}_{\gamma[x, x]}^u, !\forall y \{A\}_y^x, !F(A), !\forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^{b[x, x]}}{\{\Gamma\}_{\gamma[x, x]}^u, !\forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^{b[x, x]}} \text{ (con)} \\ & \frac{\{\Gamma\}_{\gamma[x, x]}^u, !\forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^{b[x, x]}}{\{\Gamma\}_{\gamma[x, x]}^u, \{!A\}^x \vdash \{B\}_w^{b[x, x]}} \text{ (\otimes L, D9)} \end{aligned}$$

(wkn).

$$\begin{aligned} & \frac{\{\Gamma\}_{\gamma}^u \vdash \{B\}_w^b}{\{\Gamma\}_{\gamma}^u, !\forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^b} \text{ (wkn)} \\ & \frac{\{\Gamma\}_{\gamma}^u, !\forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^b}{\{\Gamma\}_{\gamma}^u, \{!A\}^x \vdash \{B\}_w^b} \text{ (\otimes L, D9)} \end{aligned}$$

(!R).

$$\begin{array}{c}
\frac{\{\Gamma\}^u \vdash \{A\}_y^a}{! \forall v \{\Gamma\}_v^u, !F(\Gamma) \vdash \{A\}_y^a} \text{ (D9)} \\
\frac{\frac{\frac{\{\Gamma\}_v^u, !F(\Gamma) \vdash \{A\}_y^a}{! \forall v \{\Gamma\}_v^u, !F(\Gamma) \vdash \forall y \{A\}_y^a} (\forall R)}{! \forall v \{\Gamma\}_v^u, !F(\Gamma) \vdash ! \forall y \{A\}_y^a} (!R) \quad \frac{\frac{\Gamma \vdash A}{!F(\Gamma) \vdash F(A)} (\star_2)}{!F(\Gamma) \vdash !F(A)} (!R) \\
\hline
\frac{! \forall v \{\Gamma\}_v^u, !F(\Gamma) \vdash ! \forall y \{A\}_y^a \otimes !F(A)}{\{\Gamma\}^u \vdash \{A\}^a} (\text{con}, \otimes R) \\
\hline
\{\Gamma\}^u \vdash \{A\}^a \quad (\otimes L, D9)
\end{array}$$

(!L).

$$\begin{array}{c}
\frac{\{\Gamma\}_\gamma^v, \{A\}_a^x \vdash \{B\}_w^b}{\{\Gamma\}_\gamma^v, \forall y \{A\}_y^x \vdash \{B\}_w^b} (\forall L) \\
\frac{\{\Gamma\}_\gamma^v, \forall y \{A\}_y^x \vdash \{B\}_w^b}{\{\Gamma\}_\gamma^v, ! \forall y \{A\}_y^x \vdash \{B\}_w^b} (!L) \\
\frac{\{\Gamma\}_\gamma^v, ! \forall y \{A\}_y^x \vdash \{B\}_w^b}{\{\Gamma\}_\gamma^v, ! \forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^b} (\text{wkn}) \\
\frac{\{\Gamma\}_\gamma^v, ! \forall y \{A\}_y^x, !F(A) \vdash \{B\}_w^b}{\{\Gamma\}_\gamma^v, \{A\}^x \vdash \{B\}_w^b} (\otimes R, D9)
\end{array}$$

$\vdash !(b = t) \oplus !(b = f)$.

$$\begin{array}{c}
\frac{\vdash t = t}{\vdash t = t} (\star_2) \\
\frac{\vdash t = t}{\vdash !(t = t)} (!R) \quad \frac{\vdash F(t = t)}{\vdash !F(t = t)} (!R) \\
\hline
\vdash !(t = t) \otimes !F(t = t) \quad (\otimes R) \\
\hline
\vdash !(t = t) \otimes !F(t = t) \quad (\star_1) \\
\hline
\vdash !(b = t) \otimes !F(b = t) \quad \vdots \\
\hline
\vdash !(b = t) \multimap (! (b = t) \otimes !F(b = t)) \quad \vdash !(b = f) \multimap (! (b = f) \otimes !F(b = f)) \\
\hline
\vdash (! (b = t) \otimes !F(b = t)) \diamond_b (! (b = f) \otimes !F(b = f)) \quad (\& R) \\
\hline
\vdash !(b = t) \oplus !(b = f) \quad (D9)
\end{array}$$

This concludes the proof. □

Corollary 12. ILL^ω has the following disjunction and existence properties:

- (i) If $\text{ILL}^\omega \vdash \exists x !A$, then there is a term t such that $\text{ILL}^\omega \vdash !A[t/x]$.
- (ii) If $\text{ILL}^\omega \vdash !A \oplus !B$, then $\text{ILL}^\omega \vdash !A$ or $\text{ILL}^\omega \vdash !B$.

We have discussed two possible choices of F which satisfy conditions $(\star_1)$ and $(\star_2)$, namely

- (1) $F(A) :\equiv 1$
- (2) $F(A) :\equiv A$.

As it turns out, the following other choices also satisfy the necessary condition, and hence lead to different variants of modified realizability:

- (3) $F(A) :\equiv A^*$, where $(\cdot)^*$ is any proof translation satisfying $(\star_1)$ and $(\star_2)$
- (4) $F(A) :\equiv !B \multimap A$, for a fixed formula B .

3.4 Relation to interpretations of intuitionistic logic

We now show that the two correspondences of Lemma 8 and Theorem 5 are deeply related. More precisely, the q-variant of modified realizability corresponds to Girard's $(\cdot)^*$ embedding, whereas the truth-variant corresponds to the $(\cdot)^\circ$ embedding.

Proposition 13. For all formulas A of ILL^ω we have

- (i) $\text{ILL}^\omega \vdash \{A^\circ\}^x \multimap (x \text{ mrt } A)^\circ$
- (ii) $\text{ILL}^\omega \vdash !\forall y \{A^*\}_y^x \multimap !(x \text{ mq } A)^*$.

Proof. The proof is by induction on the structure of formulas. We consider a few cases. For point (i):

$A \vee B$. We can prove (a) $\text{ILL}^\omega \vdash !A \diamond_b !B \multimap !(A \diamond_b B)$. Also we can prove by induction on the structure of formulas (b) $\text{ILL}^\omega \vdash \{A^\circ\}^x \multimap !\{A^\circ\}^x$ (or, equivalently but easier, that for all formula A of ILL^ω there exists a formula B of ILL^ω such that $\text{ILL}^\omega \vdash \{A^\circ\}^x \multimap !B$).

$$\begin{aligned}
 \{(A \vee B)^\circ\}^{x,y,b} &\equiv \{A^\circ\}^x \diamond_b \{B^\circ\}^y \\
 &\stackrel{(b)}{\multimap} !\{A^\circ\}^x \diamond_b !\{B^\circ\}^y \\
 &\stackrel{(a)}{\multimap} !(\{A^\circ\}^x \diamond_b !\{B^\circ\}^y) \\
 &\stackrel{(b)}{\multimap} !(\{A^\circ\}^x \diamond_b \{B^\circ\}^y) \\
 &\stackrel{(IH)}{\multimap} !((x \text{ mrt } A)^\circ \diamond_b (y \text{ mrt } B)^\circ) \\
 &\equiv !((b = t) \multimap (x \text{ mrt } A)^\circ) \& (! (b = f) \multimap (y \text{ mrt } B)^\circ) \\
 &\multimap !((b = t) \multimap (x \text{ mrt } A)^\circ) \otimes !((b = f) \multimap (y \text{ mrt } B)^\circ) \\
 &\equiv (b, x, y \text{ mrt } (A \vee B))^\circ.
 \end{aligned}$$

$A \rightarrow B$.

$$\begin{aligned}
 \{(A \rightarrow B)^\circ\}^f &\equiv !\forall x (\{A^\circ\}^x \multimap \{B^\circ\}^{fx}) \otimes (A \rightarrow B)^\circ \\
 &\stackrel{(IH)}{\multimap} !\forall x ((x \text{ mrt } A)^\circ \multimap (fx \text{ mrt } B)^\circ) \otimes (A \rightarrow B)^\circ \\
 &\multimap !\forall x !((x \text{ mrt } A)^\circ \multimap (fx \text{ mrt } B)^\circ) \otimes (A \rightarrow B)^\circ \\
 &\equiv (f \text{ mrt } (A \rightarrow B))^\circ.
 \end{aligned}$$

$\forall z A$. We can prove that (a) if $\text{ILL}^\omega \vdash A \leftrightarrow B$ then $\text{ILL}^\omega \vdash A^\circ \multimap B^\circ$, by applying the soundness theorem of $(\cdot)^\circ$ to $\text{ILL}^\omega \vdash A \rightarrow B$ and $\text{ILL}^\omega \vdash B \rightarrow A$. We can also prove that (b) $\text{ILL}^\omega \vdash \forall z (fz \text{ mrt } A) \leftrightarrow \forall z (fz \text{ mrt } A) \wedge \forall z A$, using the instance $\text{ILL}^\omega \vdash fz \text{ mrt } A \rightarrow A$ of the truth property of mrt.

$$\begin{aligned}
 \{(\forall z A)^\circ\}^f &\equiv !\forall z \{A^\circ\}^{fz} \otimes !\forall z A^\circ \\
 &\stackrel{(IH)}{\multimap} !\forall z (fz \text{ mrt } A)^\circ \otimes !\forall z A^\circ \\
 &\equiv (\forall z (fz \text{ mrt } A) \wedge \forall z A)^\circ \\
 &\stackrel{(a,b)}{\multimap} (\forall z (fz \text{ mrt } A))^\circ \\
 &\equiv (f \text{ mrt } \forall z A)^\circ.
 \end{aligned}$$

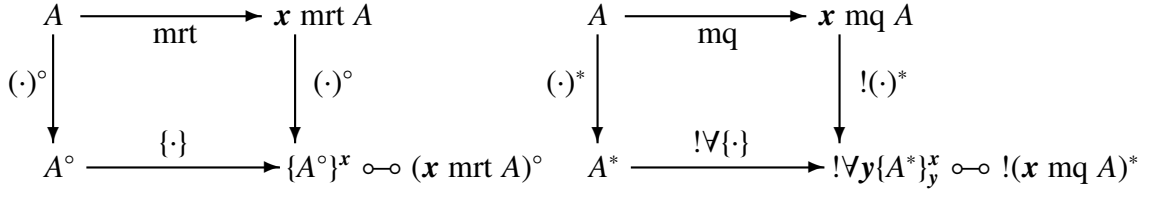


Figure 1: Diagrams of Proposition 13

For point (ii):

$A \wedge B$.

$$\begin{aligned}
! \forall y, w, b \{ (A \wedge B)^* \}_{y,w,b}^{x,v} &\equiv ! \forall y, w, b \{ A^*_y^x \diamond_b B^*_w^v \} \\
&\circ \circ ! \forall y, w \{ A^*_y^x \& B^*_w^v \} \\
&\circ \circ ! (\forall y \{ A^*_y^x \} \& \forall w \{ B^*_w^v \}) \\
&\circ \circ ! (! \forall y \{ A^*_y^x \} \& ! \forall w \{ B^*_w^v \}) \\
&\circ \circ ! ((x \text{ mq } A)^* \& (v \text{ mq } B)^*) \\
&\stackrel{(IH)}{\circ \circ} ! ((x \text{ mq } A)^* \& (v \text{ mq } B)^*) \\
&\equiv ! (x, v \text{ mq } (A \wedge B))^*.
\end{aligned}$$

$A \rightarrow B$.

$$\begin{aligned}
! \forall x, w \{ (A \rightarrow B)^* \}_{x,w}^f &\equiv ! \forall x, w \{ ! \forall y \{ A^*_y^x \} \otimes A^* \multimap \{ B^*_w^x \}^f \} \\
&\circ \circ ! \forall x \{ ! \forall y \{ A^*_y^x \} \otimes A^* \multimap \forall w \{ B^*_w^x \}^f \} \\
&\stackrel{(IH)}{\circ \circ} ! \forall x \{ ! (x \text{ mq } A)^* \otimes A^* \multimap \forall w \{ B^*_w^x \}^f \} \\
&\circ \circ ! \forall x \{ ! ((x \text{ mq } A)^* \& A^*) \multimap \forall w \{ B^*_w^x \}^f \} \\
&\circ \circ ! \forall x \{ ! ((x \text{ mq } A)^* \& A^*) \multimap \forall w \{ B^*_w^x \}^f \} \\
&\circ \circ ! \forall x \{ ! ((x \text{ mq } A)^* \& A^*) \multimap ! \forall w \{ B^*_w^x \}^f \} \\
&\stackrel{(IH)}{\circ \circ} ! \forall x \{ ! ((x \text{ mq } A)^* \& A^*) \multimap ! (fx \text{ mq } B)^* \} \\
&\circ \circ ! \forall x \{ ! ((x \text{ mq } A)^* \& A^*) \multimap (fx \text{ mq } B)^* \} \\
&\circ \circ ! \forall x \{ ! ((x \text{ mq } A)^* \& A^*) \multimap (fx \text{ mq } B)^* \} \\
&\equiv ! (f \text{ mq } (A \rightarrow B))^*.
\end{aligned}$$

The remaining cases can be checked in a similar way. □

Remark 14. Although factorisation (ii) seems less clean than factorisation (i), due to the presence of bangs and universal quantifiers, they are actually perfectly analogous. This is because bangs and universal quantifiers are also present in factorisation (i) hidden behind the notation: A° already includes a bang as it is equivalent to a banged formula, and $\{A^\circ\}^x$ has in it quantifications $\forall y$ that are responsible for the variables y not being shown.

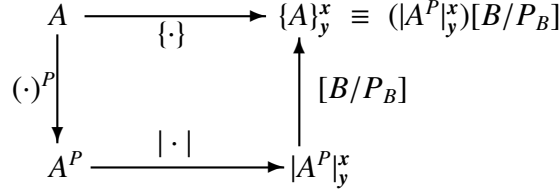


Figure 2: Alternative view on truth variants

3.5 Extrapolating on linear-intuitionistic relation

Note that we have a single soundness theorem (Theorem 11) for both the usual functional interpretation and its truth variant, with a single procedure for constructing witnesses. This shows that the witnesses extracted via a standard functional interpretation are precisely the same witnesses obtain via its truth variant. In fact, the truth variant of the interpretation is in a precise sense only enriching the verification proof $\{A_0(z)\}_{a_0}^{x_0}, \dots, \{A_n(z)\}_{a_n}^{x_n} \vdash \{B(z)\}_w^b$, maintaining some of the information which was available from the given proof $A_0, \dots, A_n \vdash B$.

We sketch an alternative way of looking at functional interpretations with truth which makes this even more explicit. Let T^ω be a given theory for which we have a functional interpretation $|\cdot| : T^\omega \rightarrow T^\omega$. In order to obtain a truth variant of the interpretation $|\cdot|$ we do the following: First, define an extension of T^ω (call it T_p^ω) by adding new atomic formulas P_A for each formula A of the original theory T^ω . Also, for each theorem $A_0, \dots, A_n \vdash B$ of T^ω we add axioms $P_{A_0} \wedge \dots \wedge P_{A_n} \rightarrow P_B$ in T_p^ω . Assuming $|\cdot|$ is trivial for Horn clauses, the original interpretation extends trivially¹ to an interpretation $|\cdot| : T_p^\omega \rightarrow T_p^\omega$. Now, define a formula translation $(\cdot)^P : T^\omega \mapsto T_p^\omega$ by simply replacing $!A$ by $!A \otimes !P_A$ inductively. It is easy to show that the truth variant $\{\cdot\}$ is related to the standard functional interpretation $|\cdot|$ as

$$\{A\}_y^x \equiv (|A^P|_y^x)[B/P_B].$$

This correspondence is illustrated in Figure 2. The same comments also hold for truth variants in the intuitionistic context. We have shown in Proposition 13 that both the q- and truth variants of modified realizability arise from a single truth interpretation of linear logic combined with the two different embeddings $(\cdot)^*$ and $(\cdot)^\circ$ of LL^ω into ILL^ω . As the truth interpretation of linear logic adds a “copy” P_A of the original formula $!A$ in parallel to interpreting $!A$ itself, this is also what is happening behind the scenes in the q- and truth-variants of modified realizability. A copy of A is created precisely at the places where the embedding $(\cdot)^*$ introduces a $!A$, i.e. in the clauses for $A \rightarrow B, A \vee B$ and $\exists zA$. Modified realizability with truth, on the other hand, introduces a copy of A in the places where the $(\cdot)^\circ$ introduces a $!A$, i.e. $A \rightarrow B$ and $\forall zA$. In the case of modified realizability, creating a copy of $\forall zA$ in the interpretation of the universal quantifier is redundant, since $\forall z(fz \text{ mrt } A)$ already implies $\forall zA$, since $(fz \text{ mrt } A) \rightarrow A$.

This leads us to a heuristic for obtaining truth and q-variants of an interpretation: For the q-variant we add a copy of the formula being interpreted in the clauses for $A \rightarrow B$ (in the premise) and $A \vee B$ and $\exists zA$; whereas for the truth variant we add a copy of A in the clauses for $A \rightarrow B$ and $\forall zA$. A second heuristic is suggested by Theorem 5: Once we got a q-variant $(\cdot)^{Iq}$ of an interpretation $(\cdot)^I$ (what can usually be done just by copying what is done for modified realizability) we may get a variant with truth $(\cdot)^{It}$ by defining it as $A^{It} \equiv A^{Iq} \wedge A$.

¹Assuming the interpretation $|\cdot|$ does not rely on particular properties of the system T^ω which fail in the extended systems T_p^ω . This is in fact why the heuristic above does not apply to Gödel’s dialectica interpretation.

In the remaining sections we show how these methodologies led us to new truth variants of bounded modified realizability, the Diller-Nahm interpretation, and the bounded functional interpretation. We start, however, by recasting Aczel's and Kleene's slash translations in terms of these two heuristics.

4 The Slash Translation and its Variants

Let a finite sequence of sentences Γ be given. To each formula A we assign a relation $\Gamma \mid A$ by induction on A as follows:

$$\begin{aligned}\Gamma \mid A_{\text{at}} &::= \Gamma \vdash A_{\text{at}} \\ \Gamma \mid A \wedge B &::= \Gamma \mid A \text{ and } \Gamma \mid B \\ \Gamma \mid A \vee B &::= \Gamma \mid A \text{ or } \Gamma \mid B \\ \Gamma \mid A \rightarrow B &::= \text{if } \Gamma \mid A \text{ then } \Gamma \mid B \\ \Gamma \mid \exists z A &::= \Gamma \mid A[n/z] \text{ for some numeral } n \\ \Gamma \mid \forall z A &::= \Gamma \mid A[n/z] \text{ for all numerals } n,\end{aligned}$$

where $\Gamma \vdash A$ denotes provability in a given fixed formal system. Intuitively, this is closely related Tarski's semantics, viewing Γ as a “structure” and $\Gamma \mid A$ as “ A is valid in Γ ”. Formally, this translation is sound in the sense that if $\Gamma \mid \Gamma$ and $\Gamma \vdash A$ then $\Gamma \mid \forall n A$, where $\text{FV}(A) = \{n\}$ and $\Gamma \mid \Gamma$ means $\Gamma \mid A$ for each $A \in \Gamma$. We demonstrate now how the slash translation above is related to Kleene's and Aczel's slash translations in the same way that modified realizability is related to its q- and truth variants (cf. Section 2).

Definition 15 (q-variant of slash translation). Kleene's slash $\Gamma \mid_k A$ is defined (cf. [17], definition 3.1.2) by induction on A as

$$\begin{aligned}\Gamma \mid_k A_{\text{at}} &::= \Gamma \vdash A_{\text{at}} \\ \Gamma \mid_k A \wedge B &::= \Gamma \mid_k A \text{ and } \Gamma \mid_k B \\ \Gamma \mid_k A \vee B &::= (\Gamma \mid_k A \text{ and } \Gamma \vdash A) \text{ or } (\Gamma \mid_k B \text{ and } \Gamma \vdash B) \\ \Gamma \mid_k A \rightarrow B &::= \text{if } (\Gamma \mid_k A \text{ and } \Gamma \vdash A) \text{ then } \Gamma \mid_k B \\ \Gamma \mid_k \forall z A &::= \Gamma \mid_k A[n/z] \text{ for all numerals } n \\ \Gamma \mid_k \exists z A &::= (\Gamma \mid_k A[n/z] \text{ and } \Gamma \vdash A[n/z]) \text{ for some numeral } n.\end{aligned}$$

Remark 16. We have that $\Gamma \mid_k A$ implies $\Gamma \vdash A$ for all disjunctive and existential formulas A .

Definition 17 (Truth variant of slash translation). Aczel's slash $\Gamma \mid_a A$ is defined (cf. [17], theorem 5.1.18) by induction on A as

$$\begin{aligned}\Gamma \mid_a A_{\text{at}} &::= \Gamma \vdash A_{\text{at}} \\ \Gamma \mid_a A \wedge B &::= \Gamma \mid_a A \text{ and } \Gamma \mid_a B \\ \Gamma \mid_a A \vee B &::= \Gamma \mid_a A \text{ or } \Gamma \mid_a B \\ \Gamma \mid_a A \rightarrow B &::= (\text{if } \Gamma \mid_a A \text{ then } \Gamma \mid_a B) \text{ and } \Gamma \vdash A \rightarrow B \\ \Gamma \mid_a \forall z A &::= \Gamma \vdash \forall z A \text{ and } \Gamma \mid_a A[n/z] \text{ for all numerals } n \\ \Gamma \mid_a \exists z A &::= \Gamma \mid_a A[n/z] \text{ for some numeral } n.\end{aligned}$$

Remark 18. We have that $\Gamma \Vdash_a A$ implies $\Gamma \vdash A$ for all formulas A .

The next proposition shows that Kleene's slash is related to Aczel's slash in the same way that the q -variant of modified realizability is related to its truth variant.

Proposition 19 (See [18], exercise 3.5.3). In both definitions above, assume $\Gamma \vdash A$ denotes provability in some extension of intuitionistic logic. For all formulas A we have $\Gamma \Vdash_a A$ if and only if $(\Gamma \Vdash_k A \text{ and } \Gamma \vdash A)$.

Proof. By an easy induction on the structure of A . □

Corollary 20 (Strengthening of [17], 5.1.17). If $\Gamma \vdash A$ then $\Gamma \Vdash_k A$ iff $\Gamma \Vdash_a A$.

5 Bounded Modified Realizability with Truth

Let us consider now the q - and truth variants of the bounded modified realizability [3]. In the following, $\text{IL}_{\leq^*}^\omega$ denotes the formal system described in [3] containing an extensional majorizability relation $\leq^*$ and primitive bounded quantifications $\forall z \leq^* tA$ and $\exists z \leq^* tA$.

5.1 Definitions

Definition 21 (q -variant of bounded modified realizability). Each formula A of $\text{IL}_{\leq^*}^\omega$ is associated to a new formula $\mathbf{x} \text{ brq } A$ inductively as follows. If A_{at} is an atomic formula, then $\mathbf{x} \text{ brq } A_{\text{at}} := A_{\text{at}}$ where $\mathbf{x}$ is the empty tuple. Assuming we already have defined $\mathbf{x} \text{ brq } A$ and $\mathbf{y} \text{ brq } B$, then

$$\begin{aligned} \mathbf{x}, \mathbf{y} \text{ brq } (A \wedge B) &:= (\mathbf{x} \text{ brq } A) \wedge (\mathbf{y} \text{ brq } B) \\ \mathbf{x}, \mathbf{y} \text{ brq } (A \vee B) &:= ((\mathbf{x} \text{ brq } A) \wedge A) \vee ((\mathbf{y} \text{ brq } B) \wedge B) \\ f \text{ brq } (A \rightarrow B) &:= \tilde{\forall} \mathbf{x} ((\mathbf{x} \text{ brq } A) \wedge A \rightarrow (f \mathbf{x} \text{ brq } B)) \\ \mathbf{x} \text{ brq } \forall z \leq^* tA &:= \forall z \leq^* t(\mathbf{x} \text{ brq } A) \\ \mathbf{x} \text{ brq } \exists z \leq^* tA &:= \exists z \leq^* t((\mathbf{x} \text{ brq } A) \wedge A) \\ f \text{ brq } \forall z A &:= \tilde{\forall} u \forall z \leq^* u(fu \text{ brq } A) \\ u, \mathbf{x} \text{ brq } \exists z A &:= \exists z \leq^* u((\mathbf{x} \text{ brq } A) \wedge A). \end{aligned}$$

The q -variant of bounded modified realizability of a formula A is then defined as $A^{\text{brq}} := \tilde{\exists} \mathbf{x} (\mathbf{x} \text{ brq } A)$.

Remark 22. We have $\text{IL}_{\leq^*}^\omega \vdash (\mathbf{x} \text{ brq } A) \rightarrow A$ for all disjunctive and (bounded and unbounded) existential formulas A of $\text{IL}_{\leq^*}^\omega$.

Definition 23 (Bounded modified realizability with truth). To each formula A of $\text{IL}_{\leq^*}^\omega$ let us associate a new formula $\mathbf{x} \text{ brt } A$ inductively as follows. If A_{at} is an atomic formula, then $\mathbf{x} \text{ brt } A_{\text{at}} := A_{\text{at}}$ where $\mathbf{x}$ is the empty tuple. Assuming that we already defined $\mathbf{x} \text{ brt } A$ and

y brt B , then

$$\begin{aligned}
x, y \text{ brt } (A \wedge B) &::= (x \text{ brt } A) \wedge (y \text{ brt } B) \\
x, y \text{ brt } (A \vee B) &::= (x \text{ brt } A) \vee (y \text{ brt } B) \\
f \text{ brt } (A \rightarrow B) &::= \tilde{\forall}x((x \text{ brt } A) \rightarrow (fx \text{ brt } B)) \wedge (A \rightarrow B) \\
x \text{ brt } \forall z \leq^* tA &::= \forall z \leq^* t(x \text{ brt } A) \\
x \text{ brt } \exists z \leq^* tA &::= \exists z \leq^* t(x \text{ brt } A) \\
f \text{ brt } \forall zA &::= \tilde{\forall}u\forall z \leq^* u(fu \text{ brt } A) \wedge \forall zA \\
u, x \text{ brt } \exists zA &::= \exists z \leq^* u(x \text{ brt } A).
\end{aligned}$$

The *bounded modified realizability with truth* of a formula A is then defined as $A^{\text{brt}} ::= \tilde{\exists}x(x \text{ brt } A)$.

Remark 24. We have $\text{IL}_{\leq^*}^\omega \vdash (x \text{ brt } A) \rightarrow A$ for all formulas A of $\text{IL}_{\leq^*}^\omega$.

5.2 Soundness theorems

Theorem 25 (Soundness for q-variant of bounded modified realizability). Let A be an arbitrary formula of $\text{IL}_{\leq^*}^\omega$ and let z be all of its free variables. If $\text{IL}_{\leq^*}^\omega \vdash A$, then we can extract from a proof of A closed monotone (that is, $\text{IL}_{\leq^*}^\omega \vdash t \leq^* t$) terms t of $\text{IL}_{\leq^*}^\omega$ such that $\text{IL}_{\leq^*}^\omega \vdash \tilde{\forall}a\forall z \leq^* a(ta \text{ brq } A)$.

Proof. The proof is by induction on a proof of A . We assume the logical rules and axioms of intuitionistic logic are as described in [1], section 2.1.

$A \vee A \rightarrow A$. Here we use that $\text{IL}_{\leq^*}^\omega \vdash \forall x, y(x \leq^* y \wedge (x \text{ brq } A) \rightarrow (y \text{ brq } A))$. Its interpretation is

$$\tilde{\exists}f\tilde{\forall}x_1, x_2(((x_1 \text{ brq } A) \wedge A) \vee ((x_2 \text{ brq } A) \wedge A)) \wedge (A \vee A) \rightarrow fx_1x_2 \text{ brq } A).$$

The witness for f can be taken as $\lambda a, x_1, x_2. \max(x_1, x_2)$.

$\forall zA \rightarrow A[q/z]$. Here we use $x \text{ brq } (A[q/z]) \equiv (x \text{ brq } A)[q/z]$. Also we use that for all terms $q[z']$, where z' are all the variables in q , there exists a term $q^*[z']$ with exactly the same variables z' such that $\text{IL}_{\leq^*}^\omega \vdash \tilde{\forall}a'\forall z' \leq^* a'(q[z'] \leq^* q^*[a'])$. The realizability of this axioms is

$$\tilde{\exists}F\tilde{\forall}f(\tilde{\forall}u\forall z \leq^* u(fu \text{ brq } A) \wedge \forall zA \rightarrow Ff \text{ brq } A[q/z]).$$

F can be take as $\lambda a, f. f(q^*[a'])$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$).

$A, A \rightarrow B \Rightarrow B$. Here we use the assumption that $\text{IL}_{\leq^*}^\omega \vdash A$. Let z_A, z_{AB} and z_B be all the free variables of $A, A \rightarrow B$ and B , respectively. By induction hypothesis we have witnesses q and r for the realizability of the premises of the rule, that is,

$$\begin{aligned}
\tilde{\forall}a_A\forall z_A \leq^* a_A(qa_A \text{ brq } A) \\
\tilde{\forall}a_{AB}\forall z_{AB} \leq^* a_{AB}\tilde{\forall}x((x \text{ brq } A) \wedge A \rightarrow ra_{AB}x \text{ brq } B).
\end{aligned}$$

A witness for the realizability of the conclusion of the rule can be taken as $\lambda a_B. ra_{AB}(qa_A)$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$).

$\exists z \leq^* qA \rightarrow \exists z(z \leq^* q \wedge A)$. Here we use the fact that the formula $(z \leq^* q)$ is $\tilde{\exists}$ -free (that is, all its unbounded quantifications are universal monotone) and that for all $\tilde{\exists}$ -free formulas A we

have $\mathbb{L}_{\leq^*}^\omega \vdash A \leftrightarrow (\mathbf{x} \text{ brq } A)$, where $\mathbf{x}$ is the empty tuple that may be omitted. The interpretation of this axiom is

$$\exists \tilde{f}, g \tilde{\forall} \mathbf{x} (\exists z \leq^* q(\mathbf{x} \text{ brq } A \wedge A) \wedge \exists z \leq^* qA \rightarrow \exists z \leq^* g\mathbf{x}(z \leq^* q \wedge \mathbf{f}\mathbf{x} \text{ brq } A \wedge A)).$$

The witnesses for $\mathbf{f}$ and g can be taken as $\lambda \mathbf{a}, \mathbf{x}. \mathbf{x}$ and $\lambda \mathbf{a}, \mathbf{x}. q^*[\mathbf{a}']$, respectively. The other cases are treated similarly. $\square$

Proposition 26. For all formulas A of $\mathbb{L}_{\leq}^\omega$ we have

$$\mathbb{L}_{\leq^*}^\omega \vdash (\mathbf{x} \text{ brt } A) \leftrightarrow (\mathbf{x} \text{ brq } A) \wedge A.$$

Proof. Easy induction on the structure of formulas.

$\forall z \leq^* tA$.

$$\begin{aligned} \mathbf{x} \text{ brt } \forall z \leq^* tA &\equiv \forall z \leq^* t(\mathbf{x} \text{ brt } A) \\ &\leftrightarrow \forall z \leq^* t((\mathbf{x} \text{ brq } A) \wedge A) \\ &\leftrightarrow \forall z \leq^* t(\mathbf{x} \text{ brq } A) \wedge \forall z \leq^* tA \\ &\equiv (\mathbf{x} \text{ brq } \forall z \leq^* tA) \wedge \forall z \leq^* tA. \end{aligned}$$

$\forall zA$.

$$\begin{aligned} \mathbf{f} \text{ brt } \forall zA &\equiv \tilde{\forall} u \forall z \leq^* u(\mathbf{f}u \text{ brt } A) \wedge \forall zA \\ &\leftrightarrow \tilde{\forall} u \forall z \leq^* u((\mathbf{f}u \text{ brq } A) \wedge A) \wedge \forall zA \\ &\leftrightarrow \tilde{\forall} u \forall z \leq^* u(\mathbf{f}u \text{ brq } A) \wedge \forall zA \\ &\equiv (\mathbf{f} \text{ brq } \forall zA) \wedge \forall zA. \end{aligned}$$

The other cases are treated similarly. $\square$

Theorem 27 (Soundness for bounded modified realizability with truth). Let A be an arbitrary formula of $\mathbb{L}_{\leq^*}^\omega$ and let $\mathbf{z}$ be all its free variables. If $\mathbb{L}_{\leq^*}^\omega \vdash A$ then we can extract from a proof of A closed monotone terms $\mathbf{t}$ of $\mathbb{L}_{\leq^*}^\omega$ such that $\mathbb{L}_{\leq^*}^\omega \vdash \tilde{\forall} \mathbf{a} \forall \mathbf{z} \leq^* \mathbf{a}(\mathbf{t}\mathbf{a} \text{ brt } A)$.

Proof. Follows from Theorem 25 and Proposition 26. $\square$

5.3 Characterisation theorems

As shown in Section 2.3, the characterisation of the modified realizability with truth uses the same principles for the characterisation of the standard modified realizability. We now show that the same happens with the truth variant of the bounded modified realizability. Let $\mathbb{L}_{\leq^*}^\#$ denote the theory $\mathbb{L}_{\leq^*}^\omega$ extended with the characterisation principles of the bounded modified realizability, i.e. bounded choice principle, bounded independence of premise and the majorizability axioms (cf. [3]).

Theorem 28. For all formulas A of $\mathbb{L}_{\leq^*}^\omega$ we have

- (i) $\mathbb{L}_{\leq^*}^\# \vdash \exists \mathbf{x}(\mathbf{x} \text{ brq } A) \leftrightarrow A$
- (ii) $\mathbb{L}_{\leq^*}^\# \vdash \exists \mathbf{x}(\mathbf{x} \text{ brt } A) \leftrightarrow A$.

Proof. The proof is similar to that of Theorem 6. The characterisation theorem for the bounded modified realizability gives us

$$\mathbb{IL}_{\leq^*}^\# \vdash \tilde{\exists}x(x \text{ br } A) \leftrightarrow A.$$

In order to prove (i) one shows by induction on the structure of A that $\mathbb{IL}_{\leq^*}^\# \vdash \tilde{\forall}x((x \text{ brt } A) \leftrightarrow (x \text{ br } A))$. Point (ii) then follows from (i) and Proposition 26. $\square$

6 Diller-Nahm with Truth

We have seen above how the (bounded) modified realizability with truth and q-modified realizability are closely related (Theorem 5 and Proposition 26), and how they can be seen to come from a common truth interpretation of linear logic (Proposition 13). Following these ideas, in this section we show how a Diller-Nahm with truth interpretation can be obtained from Jørgensen's recent q-variant of the Diller-Nahm interpretation.

6.1 Definitions

Definition 29 (q-variant of Diller-Nahm interpretation, [8]). Each formula A of intuitionistic logic is associated a new formula $A_Q(\mathbf{x}; \mathbf{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\text{at}})_Q(\cdot) := A_{\text{at}}$. Assuming the interpretations of A and B are already given we define:

$$\begin{aligned} (A \wedge B)_Q(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:= A_Q(\mathbf{x}; \mathbf{y}) \wedge B_Q(\mathbf{v}; \mathbf{w}), \\ (A \vee B)_Q(\mathbf{x}, \mathbf{v}, \mathbf{b}; \mathbf{y}, \mathbf{w}) &:= (A_Q(\mathbf{x}; \mathbf{y}) \wedge A) \diamond_b (B_Q(\mathbf{v}; \mathbf{w}) \wedge B), \\ (A \rightarrow B)_Q(\mathbf{f}, \mathbf{g}; \mathbf{x}, \mathbf{w}) &:= \forall \mathbf{y} \in \mathbf{f} \mathbf{x} \mathbf{w} A_Q(\mathbf{x}; \mathbf{y}) \wedge A \rightarrow B_Q(\mathbf{g} \mathbf{x}; \mathbf{w}), \\ (\forall z A)_Q(\mathbf{f}; \mathbf{y}, \mathbf{z}) &:= A_Q(\mathbf{f} \mathbf{z}; \mathbf{y}), \\ (\exists z A)_Q(\mathbf{x}, \mathbf{z}; \mathbf{y}) &:= A_Q(\mathbf{x}; \mathbf{y}) \wedge A. \end{aligned}$$

The *q-variant of the Diller-Nahm interpretation* is then defined as $A^Q := \exists \mathbf{x} \forall \mathbf{y} A_Q(\mathbf{x}; \mathbf{y})$. Note that we are assuming our language contains a new type for non-empty finite sets.

Remark 30. We have $\mathbb{IL}^\omega \vdash A_Q(\mathbf{x}; \mathbf{y}) \rightarrow A$ for all disjunctive and existential formulas A of $\mathbb{IL}^\omega$.

In the same paper, Jørgensen argues that only changing the clause on implication doesn't result in a sound Diller-Nahm with truth. We show that this can be fixed by also changing the clause on universal quantifier.

Definition 31 (Diller-Nahm with truth, for $\mathbb{IL}^\omega$). Each formula A of intuitionistic logic is associated a new formula $A_{\text{Dt}}(\mathbf{x}; \mathbf{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\text{at}})_{\text{Dt}}(\cdot) := A_{\text{at}}$. Assuming the interpretations of A and B are already given we define:

$$\begin{aligned} (A \wedge B)_{\text{Dt}}(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:= A_{\text{Dt}}(\mathbf{x}; \mathbf{y}) \wedge B_{\text{Dt}}(\mathbf{v}; \mathbf{w}), \\ (A \vee B)_{\text{Dt}}(\mathbf{x}, \mathbf{v}, \mathbf{z}; \mathbf{y}, \mathbf{w}) &:= A_{\text{Dt}}(\mathbf{x}; \mathbf{y}) \diamond_z B_{\text{Dt}}(\mathbf{v}; \mathbf{w}), \\ (A \rightarrow B)_{\text{Dt}}(\mathbf{f}, \mathbf{g}; \mathbf{x}, \mathbf{w}) &:= (\forall \mathbf{y} \in \mathbf{f} \mathbf{x} \mathbf{w} A_{\text{Dt}}(\mathbf{x}; \mathbf{y}) \rightarrow B_{\text{Dt}}(\mathbf{g} \mathbf{x}; \mathbf{w})) \wedge (A \rightarrow B), \\ (\forall z A)_{\text{Dt}}(\mathbf{f}; \mathbf{y}, \mathbf{z}) &:= A_{\text{Dt}}(\mathbf{f} \mathbf{z}; \mathbf{y}) \wedge \forall z A, \\ (\exists z A)_{\text{Dt}}(\mathbf{x}, \mathbf{z}; \mathbf{y}) &:= A_{\text{Dt}}(\mathbf{x}; \mathbf{y}). \end{aligned}$$

The *Diller-Nahm interpretation with truth* is then defined as $A^{\text{Dt}} := \exists \mathbf{x} \forall \mathbf{y} A_{\text{Dt}}(\mathbf{x}; \mathbf{y})$.

Remark 32. We have $\mathbb{IL}^\omega \vdash A_{\text{Dt}}(\mathbf{x}; \mathbf{y}) \rightarrow A$ for all formulas A of $\mathbb{IL}^\omega$.

6.2 Soundness

Proposition 33. For all formulas A of IL^ω we have

$$\text{IL}^\omega \vdash A_{\text{Dt}}(\mathbf{x}; \mathbf{y}) \leftrightarrow A_{\text{Q}}(\mathbf{x}; \mathbf{y}) \wedge A.$$

Proof. By induction on the logical structure of A .

$A \rightarrow B$.

$$\begin{aligned} (A \rightarrow B)_{\text{Dt}}(\mathbf{f}, \mathbf{g}; \mathbf{x}, \mathbf{w}) &\equiv (\forall \mathbf{y} \in \mathbf{f} \mathbf{x} \mathbf{w} A_{\text{Dt}}(\mathbf{x}; \mathbf{y}) \rightarrow B_{\text{Dt}}(\mathbf{g} \mathbf{x}; \mathbf{w})) \wedge (A \rightarrow B) \\ &\stackrel{(\text{IH})}{\leftrightarrow} (\forall \mathbf{y} \in \mathbf{f} \mathbf{x} \mathbf{w} A_{\text{Q}}(\mathbf{x}; \mathbf{y}) \wedge A \rightarrow B_{\text{Q}}(\mathbf{g} \mathbf{x}; \mathbf{w}) \wedge B) \wedge (A \rightarrow B) \\ &\leftrightarrow (\forall \mathbf{y} \in \mathbf{f} \mathbf{x} \mathbf{w} A_{\text{Q}}(\mathbf{x}; \mathbf{y}) \wedge A \rightarrow B_{\text{Q}}(\mathbf{g} \mathbf{x}; \mathbf{w})) \wedge (A \rightarrow B) \\ &\equiv (A \rightarrow B)_{\text{Q}}(\mathbf{f}, \mathbf{g}; \mathbf{x}, \mathbf{w}) \wedge (A \rightarrow B). \end{aligned}$$

It should be noted that we must work with non-empty sets, so that $\forall x \in a (A(x) \wedge B) \leftrightarrow \forall x \in a A(x) \wedge B$. The remaining cases can be checked in a similar way. $\square$

Theorem 34 (Soundness for Diller-Nahm with truth). Let A be an arbitrary formula of IL^ω and let $\mathbf{z}$ be all its free variables. If $\text{IL}^\omega \vdash A$ then we can extract from a proof of A closed terms $\mathbf{t}$ of IL^ω such that $\text{IL}^\omega \vdash A_{\text{Dt}}(\mathbf{t} \mathbf{z}; \mathbf{y})$.

Proof. Follows from the soundness of the q-variant of Diller-Nahm (cf. [8]) and Proposition 33. $\square$

6.3 Around Jørgensen's counter-example

Note that in order to obtain a Diller-Nahm with truth we have changed both the clauses on implication and universal quantifier from Diller-Nahm's original definition [2]. Jørgensen [8] showed that by only changing the implication clause one obtains an unsound Diller-Nahm with truth interpretation. The counter-example: take

$$A \equiv \forall z (\forall x \neg B \rightarrow \neg \exists x B).$$

A is clearly intuitionistically valid. We have, however, that for $B \equiv T_{zz}x$, where T is Kleene's famous predicate, there is no term t such that $\forall x, z A_{\text{Dt}}(t; x, z)$, that is

$$\forall x, z (\forall x_0 \in t x z \neg T_{zz}x_0 \rightarrow (\neg T_{zz}x \wedge \neg \exists x_1 T_{zz}x_1))$$

as such t would provide a solution to the halting problem. In our Diller-Nahm interpretation with truth, however, this formula A is interpreted as

$$\exists \mathbf{f} \forall x, z ((\forall x_0 \in \mathbf{f} x z \neg T_{zz}x_0 \wedge \forall x_2 \neg T_{zz}x_2) \rightarrow (\neg T_{zz}x \wedge \neg \exists x_1 T_{zz}x_1)),$$

which is equivalent to the provable statement

$$\forall z (\forall x_2 \neg T_{zz}x_2 \rightarrow (\neg \exists x_1 T_{zz}x_1)).$$

6.4 Diller-Nahm with truth versus realizability

One should observe that for logically simple formulas, the Diller-Nahm with truth in fact coincides with modified realizability with truth. For instance, consider a formula of the kind $\Pi_1 \rightarrow \Pi_2$, that is $\forall x P(x) \rightarrow \forall v \exists w Q(v, w)$ with P and Q quantifier-free. Its Diller-Nahm with truth and modified realizability with truth are both equivalent to

$$\exists g \forall v (\forall x P(x) \rightarrow Q(v, gv)) \wedge (\forall x P(x) \rightarrow \forall v \exists w Q(v, w)).$$

On the other hand, for formulas of higher logical complexity the Diller-Nahm with truth is different from realizability. For instance, the Diller-Nahm with truth interpretation of a proof of $\forall x \exists y P(x, y) \rightarrow \forall v \exists w Q(v, w)$ (that is $\Pi_2 \rightarrow \Pi_2$) gives terms t, s such that in particular

$$\forall f, v (\forall x \in t f v P(x, f x) \wedge \forall x \exists y P(x, y) \rightarrow Q(v, s f v))$$

which is stronger than what is obtained via modified realizability with truth, namely

$$\forall f, v (\forall x P(x, f x) \rightarrow Q(v, s f v))$$

since $\forall x P(x, f x)$ is stronger than $\forall x \in t f v P(x, f x) \wedge \forall x \exists y P(x, y)$.

6.5 Characterisation

It is usual for functional interpretations that a formula A is interpreted as the existence of a term t such that $\forall y |A|_y^t$. Hence, it might seem strange that for the Diller-Nahm with truth we already have that $A_{Dt}(x; y)$ (rather than $\forall y A_{Dt}(x; y)$) implies A . Although we have shown that the resulting modification of Diller-Nahm's interpretation is still sound, one might wonder whether it is "complete", meaning that only true formulas are witnessed. In this section we show that this is the case, albeit at a heavy price: we seem to need full (classical) choice in order to show that A is equivalent to $\exists x \forall y A_{Dt}(x; y)$.

Proposition 35. $PA^\omega + AC \vdash A \leftrightarrow A^{Dt}$.

Proof. By induction on structure of formulas.

$A \rightarrow B$.

$$\begin{aligned} (A \rightarrow B) &\stackrel{(IH)}{\leftrightarrow} \exists x \forall y A_{Dt}(x; y) \rightarrow \exists v \forall w B_{Dt}(v; w) \\ &\leftrightarrow \exists x \forall a \forall y \in a A_{Dt}(x; y) \rightarrow \exists v \forall w B_{Dt}(v; w) \\ &\stackrel{(PA^\omega)}{\leftrightarrow} \forall x \exists v \forall w \exists a (\forall y \in a A_{Dt}(x; y) \rightarrow B_{Dt}(v; w)) \\ &\stackrel{(AC)}{\leftrightarrow} \exists f, g \forall x, w (\forall y \in f x w A_{Dt}(x; y) \rightarrow B_{Dt}(g x; w)) \\ &\equiv \exists f, g \forall x, w (A \rightarrow B)_{Dt}(f, g; x, w). \end{aligned}$$

The other cases are treated similarly. □

Remark 36. One can easily argue that **AC** is necessary for the characterisation of the Diller-Nahm with truth as follows. Assume $A^{Dt} \leftrightarrow A$ for all formulas A . Then, in particular we have

$$\forall x \exists y A(x, y) \rightarrow (\forall x \exists y A(x, y))^{Dt}.$$

AC then follows by observing that $(\forall x \exists y A(x, y))^{\text{Dt}} \rightarrow \exists f \forall x A(x, y)$. It is an open question, however, whether full classical logic is indeed necessary or not for the characterisation of the Diller-Nahm with truth (or its q-variant). A similar phenomena happens with the bounded functional presented in the next section.

7 Bounded Functional Interpretation with Truth

In this section we present q- and a truth variants of the bounded functional interpretation [4]. In the following, $\mathbb{L}_{\leq}^{\omega}$ denotes the formal system described in [4] containing an intensional majorizability relation $\leq$ and primitive bounded quantifications $\forall z \leq tA$ and $\exists z \leq tA$.

7.1 Definitions

Definition 37 (q-variant of bounded functional interpretation). To each formula A of $\mathbb{L}_{\leq}^{\omega}$ is associated a new formula $A_{\text{Bq}}(\mathbf{x}; \mathbf{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\text{at}})_{\text{Bq}}(\cdot) := A_{\text{at}}$. Assuming that we already defined $A_{\text{Bq}}(\mathbf{x}; \mathbf{y})$ and $B_{\text{Bq}}(\mathbf{v}; \mathbf{w})$, then

$$\begin{aligned} (A \wedge B)_{\text{Bq}}(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:= A_{\text{Bq}}(\mathbf{x}; \mathbf{y}) \wedge B_{\text{Bq}}(\mathbf{v}; \mathbf{w}) \\ (A \vee B)_{\text{Bq}}(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:= (\tilde{\forall} \tilde{\mathbf{y}} \leq \mathbf{y} A_{\text{Bq}}(\mathbf{x}; \tilde{\mathbf{y}}) \wedge A) \vee (\tilde{\forall} \tilde{\mathbf{w}} \leq \mathbf{w} B_{\text{Bq}}(\mathbf{v}; \tilde{\mathbf{w}}) \wedge B) \\ (A \rightarrow B)_{\text{Bq}}(\mathbf{f}, \mathbf{g}; \mathbf{x}, \mathbf{w}) &:= \tilde{\forall} \mathbf{y} \leq \mathbf{f} \mathbf{x} \mathbf{w} A_{\text{Bq}}(\mathbf{x}; \mathbf{y}) \wedge A \rightarrow B_{\text{Bq}}(\mathbf{g} \mathbf{x}; \mathbf{w}) \\ (\forall z \leq tA)_{\text{Bq}}(\mathbf{x}; \mathbf{y}) &:= \forall z \leq t A_{\text{Bq}}(\mathbf{x}; \mathbf{y}) \\ (\exists z \leq tA)_{\text{Bq}}(\mathbf{x}; \mathbf{y}) &:= \exists z \leq t (\tilde{\forall} \tilde{\mathbf{y}} \leq \mathbf{y} A_{\text{Bq}}(\mathbf{x}; \tilde{\mathbf{y}}) \wedge A) \\ (\forall z A)_{\text{Bq}}(\mathbf{f}; u, \mathbf{y}) &:= \forall z \leq u A_{\text{Bq}}(\mathbf{f} u; \mathbf{y}) \\ (\exists z A)_{\text{Bq}}(u, \mathbf{x}; \mathbf{y}) &:= \exists z \leq u (\tilde{\forall} \tilde{\mathbf{y}} \leq \mathbf{y} A_{\text{Bq}}(\mathbf{x}; \tilde{\mathbf{y}}) \wedge A). \end{aligned}$$

The *q-variant of bounded functional interpretation* of a formula A is then defined as $A^{\text{Bq}} := \tilde{\exists} \mathbf{x} \tilde{\forall} \mathbf{y} A_{\text{Bq}}(\mathbf{x}; \mathbf{y})$.

Remark 38. We have $\mathbb{L}_{\leq}^{\omega} \vdash A_{\text{Bq}}(\mathbf{x}; \mathbf{y}) \rightarrow A$ for all disjunctive and (bounded and unbounded) existential formulas A of $\mathbb{L}_{\leq}^{\omega}$.

Definition 39 (Bounded functional interpretation with truth). To each formula A of $\mathbb{L}_{\leq}^{\omega}$ is associated a new formula $A_{\text{Bt}}(\mathbf{x}; \mathbf{y})$ inductively as follows. If A_{at} is an atomic formula, then $(A_{\text{at}})_{\text{Bt}}(\cdot) := A_{\text{at}}$. Assuming that we already defined $A_{\text{Bt}}(\mathbf{x}; \mathbf{y})$ and $B_{\text{Bt}}(\mathbf{v}; \mathbf{w})$, then

$$\begin{aligned} (A \wedge B)_{\text{Bt}}(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:= A_{\text{Bt}}(\mathbf{x}; \mathbf{y}) \wedge B_{\text{Bt}}(\mathbf{v}; \mathbf{w}) \\ (A \vee B)_{\text{Bt}}(\mathbf{x}, \mathbf{v}; \mathbf{y}, \mathbf{w}) &:= \tilde{\forall} \tilde{\mathbf{y}} \leq \mathbf{y} A_{\text{Bt}}(\mathbf{x}; \tilde{\mathbf{y}}) \vee \tilde{\forall} \tilde{\mathbf{w}} \leq \mathbf{w} B_{\text{Bt}}(\mathbf{v}; \tilde{\mathbf{w}}) \\ (A \rightarrow B)_{\text{Bt}}(\mathbf{f}, \mathbf{g}; \mathbf{x}, \mathbf{w}) &:= (\tilde{\forall} \mathbf{y} \leq \mathbf{f} \mathbf{x} \mathbf{w} A_{\text{Bt}}(\mathbf{x}; \mathbf{y}) \rightarrow B_{\text{Bt}}(\mathbf{g} \mathbf{x}; \mathbf{w})) \wedge (A \rightarrow B) \\ (\forall z \leq tA)_{\text{Bt}}(\mathbf{x}; \mathbf{y}) &:= \forall z \leq t A_{\text{Bt}}(\mathbf{x}; \mathbf{y}) \\ (\exists z \leq tA)_{\text{Bt}}(\mathbf{x}; \mathbf{y}) &:= \exists z \leq t \tilde{\forall} \tilde{\mathbf{y}} \leq \mathbf{y} A_{\text{Bt}}(\mathbf{x}; \tilde{\mathbf{y}}) \\ (\forall z A)_{\text{Bt}}(\mathbf{f}; u, \mathbf{y}) &:= \forall z \leq u A_{\text{Bt}}(\mathbf{f} u; \mathbf{y}) \wedge \forall z A \\ (\exists z A)_{\text{Bt}}(u, \mathbf{x}; \mathbf{y}) &:= \exists z \leq u \tilde{\forall} \tilde{\mathbf{y}} \leq \mathbf{y} A_{\text{Bt}}(\mathbf{x}; \tilde{\mathbf{y}}). \end{aligned}$$

The *bounded functional interpretation with truth* of a formula A is then defined as $A^{\text{Bt}} := \tilde{\exists} \mathbf{x} \tilde{\forall} \mathbf{y} A_{\text{Bt}}(\mathbf{x}; \mathbf{y})$.

Remark 40. We have $\mathbb{L}_{\leq}^{\omega} \vdash A_{\text{Bt}}(\mathbf{x}; \mathbf{y}) \rightarrow A$ for all formulas A of $\mathbb{L}_{\leq}^{\omega}$.

7.2 Soundness theorems

Theorem 41 (Soundness of q-variant of bounded functional interpretation). Let A be an arbitrary formula of $\mathbb{L}_{\leq}^{\omega}$ and let z be all its free variables. If $\mathbb{L}_{\leq}^{\omega} \vdash A$, then we can extract from a proof of A closed monotone (that is, $\mathbb{L}_{\leq}^{\omega} \vdash t \leq t$) terms t of $\mathbb{L}_{\leq}^{\omega}$ such that $\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall} a \forall z \leq a \tilde{\forall} y A_{Bq}(ta; y)$.

Proof. The proof is by induction on a proof of A .

$A \rightarrow A \wedge A$. The interpretation of this axiom is

$$\tilde{\exists} f, g, h \tilde{\forall} x, y_2, y_3 (\tilde{\forall} y_1 \leq f x y_2 y_3 A_{Bq}(x; y_1) \wedge A \rightarrow A_{Bq}(g x; y_2) \wedge A_{Bq}(h x; y_3)).$$

The witnesses for f , g and h can be taken as $\lambda a, x, y_2, y_3. \max(y_2, y_3)$, $\lambda a, x. x$ and $\lambda a, x. x$, respectively. Here we use that $\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall} y_2, y_3 (y_2 \leq \max(y_2, y_3) \wedge y_3 \leq \max(y_2, y_3))$.

$A \vee A \rightarrow A$. The interpretation of this axiom is

$$\tilde{\exists} f, g, h \tilde{\forall} x_1, x_2, y_3 (\tilde{\forall} y_1, y_2 \leq f x_1 x_2 y_3, g x_1 x_2 y_3 (\bigvee_{i=1}^2 (\tilde{\forall} \tilde{y}_i \leq y_i A_{Bq}(x_i; \tilde{y}_i) \wedge A) \wedge (A \vee A) \rightarrow A_{Bq}(h x_1 x_2; y_3))).$$

The witnesses for f , g and h can be taken as $\lambda a, x_1, x_2, y_3. y_3$, again $\lambda a, x_1, x_2, y_3. y_3$ and $\lambda a, x_1, x_2. \max(x_1, x_2)$, respectively. Here we use that $\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall} x \forall y, z (y \leq z \wedge A_{Bq}(y; x) \rightarrow A_{Bq}(z; x))$.

$A[q/z] \rightarrow \exists z A$. Here we use $A[q/z]_{Bq}(x; y) \equiv A_{Bq}(x; y)[q/z]$. Also we use that for all terms $q[z']$, where z' are all the variables in q , there exists a term $q^*[z']$ with exactly the same variables z' such that $\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall} a' \forall z' \leq a' (q[z'] \leq q^*[a'])$. The interpretation of the axiom is

$$\tilde{\exists} f, g, h \tilde{\forall} x, w (\tilde{\forall} y \leq f x w A[q/z]_{Bq}(x; y) \wedge A[q/z] \rightarrow \exists z \leq g x (\tilde{\forall} \tilde{w} \leq w A_{Bq}(h x; \tilde{w}) \wedge A)).$$

The witnesses for f , g and h can be taken as $\lambda a, x, w. w$, $\lambda a, x. q^*[a']$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$) and $\lambda a, x. x$, respectively.

$A, A \rightarrow B \Rightarrow B$. Here we use the assumption that $\mathbb{L}_{\leq}^{\omega} \vdash A$. Let z_A , z_{AB} and z_B be all the free variables of A , $A \rightarrow B$ and B , respectively. By induction hypothesis we have witnesses q , r and s for the interpretation of the premises of the rule, that is,

$$\begin{aligned} \mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall} a_A \forall z_A \leq a_A \tilde{\forall} y_1 A_{Bq}(q a_A; y_1) \\ \mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall} a_{AB} \forall z_{AB} \leq a_{AB} \tilde{\forall} x_1, y_2 (\tilde{\forall} y_1 \leq r a_{AB} x_1 y_2 A_{Bq}(x_1; y_1) \wedge A \rightarrow B_{Bq}(s a_{AB} x_1; y_2)). \end{aligned}$$

A witness for the interpretation of the conclusion of the rule can be taken as $\lambda a_B. s a_{AB}(q a_A)$ (if it is not closed, we close it by replacing its variables by $\mathbf{0}$).

$A_b \wedge u \leq v \rightarrow t u \leq q v \wedge q u \leq q v \Rightarrow A_b \rightarrow t \leq q$. Here we use the fact that if A_b is a bounded formula (that is, all its quantifications are bounded), then $\mathbb{L}_{\leq}^{\omega} \vdash (A_b)_{Bq}(x; y) \leftrightarrow A_b$, where $(A_b)_{Bq}(x; y)$ is a bounded formula and x and y are empty tuples that may be omitted, and so we can apply the rule with $(A_b)_{Bq}$. By induction hypothesis we have empty witnesses for the interpretation of the premise of the rule, that is,

$$\mathbb{L}^{\omega} \vdash \tilde{\forall} a, b, c \forall z, u, v \leq a, b, c ((A_b)_{Bq} \wedge u \leq v \wedge A_b \wedge u \leq v \rightarrow q u \leq q v).$$

Then we have empty witnesses for the interpretation for the conclusion of the rule, that is,

$$\mathbb{L}^{\omega} \vdash \tilde{\forall} a \forall z \leq a ((A_b)_{Bq} \rightarrow t \leq q).$$

The other cases are treated similarly. □

Proposition 42. For all formulas A of $\mathbb{L}_{\leq}^{\omega}$ we have

$$\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall}x, y(A_{\text{Bt}}(x; y) \leftrightarrow A_{\text{Bq}}(x; y) \wedge A).$$

Proof. Easy induction on the structure of formulas. We use $\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall}y(\tilde{\forall}x \leq y(A \wedge B) \leftrightarrow \tilde{\forall}x \leq yA \wedge B)$, where x is not free in B .

$A \vee B$.

$$\begin{aligned} (A \vee B)_{\text{Bt}}(x, v; y, w) &\equiv \tilde{\forall}\tilde{y} \leq yA_{\text{Bt}}(x; \tilde{y}) \vee \tilde{\forall}\tilde{w} \leq wB_{\text{Bt}}(v; \tilde{w}) \\ &\stackrel{(\text{IH})}{\leftrightarrow} \tilde{\forall}\tilde{y} \leq y(A_{\text{Bq}}(x; \tilde{y}) \wedge A) \vee \tilde{\forall}\tilde{w} \leq w(B_{\text{Bq}}(v; \tilde{w}) \wedge B) \\ &\leftrightarrow (\tilde{\forall}\tilde{y} \leq yA_{\text{Bq}}(x; \tilde{y}) \wedge A) \vee (\tilde{\forall}\tilde{w} \leq wB_{\text{Bq}}(v; \tilde{w}) \wedge B) \\ &\leftrightarrow ((\tilde{\forall}\tilde{y} \leq yA_{\text{Bq}}(x; \tilde{y}) \wedge A) \vee (\tilde{\forall}\tilde{w} \leq wB_{\text{Bq}}(v; \tilde{w}) \wedge B)) \wedge (A \vee B) \\ &\equiv (A \vee B)_{\text{Bq}}(x, v; y, w) \wedge (A \vee B). \end{aligned}$$

$\exists zA$.

$$\begin{aligned} (\exists zA)_{\text{Bt}}(u, x; y) &\equiv \exists z \leq u \tilde{\forall}\tilde{y} \leq yA_{\text{Bt}}(x; \tilde{y}) \\ &\stackrel{(\text{IH})}{\leftrightarrow} \exists z \leq u \tilde{\forall}\tilde{y} \leq y(A_{\text{Bq}}(x; \tilde{y}) \wedge A) \\ &\leftrightarrow \exists z \leq u(\tilde{\forall}\tilde{y} \leq yA_{\text{Bq}}(x; \tilde{y}) \wedge A) \\ &\leftrightarrow \exists z \leq u(\tilde{\forall}\tilde{y} \leq yA_{\text{Bq}}(x; \tilde{y}) \wedge A) \wedge \exists zA \\ &\equiv (\exists zA)_{\text{Bq}}(u, x; y) \wedge \exists zA. \end{aligned}$$

The other cases are treated similarly. □

Theorem 43 (Soundness for bounded functional interpretation with truth). Let A be an arbitrary formula of $\mathbb{L}_{\leq}^{\omega}$ and let z be all its free variables. If $\mathbb{L}_{\leq}^{\omega} \vdash A$, then we can extract from a proof of A closed monotone terms t of $\mathbb{L}_{\leq}^{\omega}$ such that $\mathbb{L}_{\leq}^{\omega} \vdash \tilde{\forall}a \forall z \leq a \tilde{\forall}yA_{\text{Bq}}(ta; y)$.

Proof. Follows from Theorem 41 and Proposition 42. □

8 Conclusion

The second author has recently investigated [5, 13–15] how different functional interpretations of intuitionistic logic can be analysed via functional interpretations of linear logic. In this article we have made use of this analysis in order to show how the q- and truth variants of modified realizability follow from a single modified realizability with truth of linear logic. In particular, we observed that q-variants correspond to the standard embedding $(\cdot)^*$ of intuitionistic logic into linear logic, whereas truth-variants arise from Girard’s alternative embedding $(\cdot)^{\circ}$. This led us to two heuristics that succeeded in obtaining q- and truth variants for bounded modified realizability, the Diller-Nahm interpretation, and the bounded functional interpretation; as well as to explain the relation between Kleene’s and Aczel’s slash translations.

Recently, the analysis of functional interpretations via linear logic has also given rise to a hybrid functional interpretation [7, 16], where multiple interpretations can be applied to a single proof. It seems that also the truth variants of these interpretations can be easily incorporated into the hybrid interpretation. One, however, no longer gets a linear order between the different

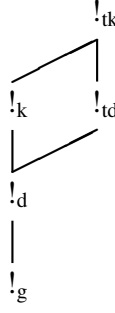


Figure 3: Partial order between multiple modalities (interpretations)

modalities (of the multi-modal linear logic), but rather a lattice, as sketched in Figure 3, where $!_k$, $!_d$ and $!_g$ stand for Kreisel, Diller-Nahm and Gödel modalities, and $!_{t*}$ their respective truth variant (note that there cannot be a Gödel truth interpretation, due to the decidability issue). These multiple modalities are given different interpretations as follows:

$$\begin{aligned}
 \{!_k A\}^x &::= !\forall y \{A\}_y^x && \text{(Kreisel's modified realizability)} \\
 \{!_d A\}_y^x &::= !\forall y' \in y \{A\}_{y'}^x && \text{(Diller-Nahm interpretation)} \\
 \{!_g A\}_y^x &::= !\{A\}_y^x && \text{(Gödel dialectica interpretation)} \\
 \{!_{tk} A\}^x &::= !\forall y \{A\}_y^x \otimes !A && \text{(Kreisel's modified realizability with truth)} \\
 \{!_{td} A\}_y^x &::= !\forall y' \in y \{A\}_{y'}^x \otimes !A && \text{(Diller-Nahm interpretation with truth)}
 \end{aligned}$$

In Figure 3 we write $!_x$ above $!_y$ if the interpretation of $!_x A$ implies the interpretation of $!_y A$. As such, we could say that modified realizability with truth and Gödel's dialectica interpretation are the two “extreme” interpretations.

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$P \vdash P$ (id)	$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$ (cut)
$\Gamma, 0 \vdash A$ (0L)	$\Gamma \vdash \top$ ($\top$ R)
$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$ ($\otimes$ L)	$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$ ($\otimes$ R)
$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}$ ($\oplus$ L)	$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \oplus A_1}$ ($\oplus$ R)
$\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \& A_1 \vdash C}$ ($\&$ L)	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$ ($\&$ R)
$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$ ($\multimap$ L)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$ ($\multimap$ R)
$\frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}$ ($\forall$ L)	$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}$ ($\forall$ R)
$\frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B}$ ($\exists$ L)	$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}$ ($\exists$ R)
$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$ (con)	$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$ (wkn)
$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$ (!L)	$\frac{! \Gamma \vdash A}{! \Gamma \vdash !A}$ (!R)

Table 1: Rules of ILL^ω

$\frac{\Gamma[\gamma_0], A \vdash C[c_0] \quad \Gamma[\gamma_1], B \vdash C[c_1]}{\Gamma[(b)(\gamma_0, \gamma_1)], A \diamond_b B \vdash C[(b)(c_0, c_1)]}$ ($\diamond_b$ L)	$\frac{\Gamma[\gamma_0] \vdash A \quad \Gamma[\gamma_1] \vdash B}{\Gamma[(b)(\gamma_0, \gamma_1)] \vdash A \diamond_b B}$ ($\diamond_b$ R)
$\frac{\Gamma, A \vdash C}{\Gamma, A \diamond_t B \vdash C}$ ($\diamond_t$ L)	$\frac{\Gamma \vdash A}{\Gamma \vdash A \diamond_t B}$ ($\diamond_t$ R)
$\frac{\Gamma, B \vdash C}{\Gamma, A \diamond_f B \vdash C}$ ($\diamond_f$ L)	$\frac{\Gamma \vdash B}{\Gamma \vdash A \diamond_f B}$ ($\diamond_f$ R)

Table 2: Rules of $\diamond$